Remarks on special ideals in lattices

Ladislav Beran

Abstract. The author studies some characteristic properties of semiprime ideals. The semiprimeness is also used to characterize distributive and modular lattices. Prime ideals are described as the meet-irreducible semiprime ideals. In relatively complemented lattices they are characterized as the maximal semiprime ideals. D-radicals of ideals are introduced and investigated. In particular, the prime radicals are determined by means of $\hat{C}$-radicals. In addition, a necessary and sufficient condition for the equality of prime radicals is obtained.

Keywords: semiprime ideal, prime ideal, congruence of a lattice, allele, lattice polynomial, meet-irreducible element, kernel, forbidden exterior quotients, D-radical, prime radical

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1. Introduction

The notion of a semiprime ideal was introduced by Rav in [8] in the following way: An ideal I of a lattice L is said to be *semiprime* if the implication

$$(a \wedge b \in I \;\;\&\;\; a \wedge c \in I) \Rightarrow a \wedge (b \vee c) \in I$$

is true for every $a, b, c \in L$.

In a recent paper, a new method was used to characterize the semiprime ideals by means of lattice quotients. For a detailed description of the method see [3], whereas for a comparative study of this technique against a classical background see [1]. The semiprime ideals in lattices have been studied in [6], [2] and [4].

For completeness we include some definitions here.

Let a, b be elements of a lattice L. If $a \leq b$, we say that these elements form a *quotient* b/a of L. We write $b/a \sim_w d/c$ if either

$$b = a \vee d \;\;\&\;\; a \wedge d \geq c$$

or

$$a = b \wedge c \;\;\&\;\; b \vee c \leq d.$$

If there exist quotients y_i/x_i such that

$$b/a = y_0/x_0 \sim_w y_1/x_1 \sim_w \cdots \sim_w y_n/x_n = d/c,$$

we write $b/a \approx_w d/c$.

A quotient b/a is called an *allele* if there exists a quotient d/c satisfying $b/a \approx_w \approx_w d/c$ and such that either $b \leq c$ or $d \leq a$. The set of all the alleles of L will be denoted by $\mathbf{A}(L)$.

Let $\hat{C}(L)$ denote the smallest congruence θ of L for which the quotient lattice L/θ is distributive. It can be shown [1] that $(a, b) \in \hat{C}(L)$ if and only if there exist $a_i \in L$ satisfying

$$a_0 = a \wedge b \leq a_1 \leq a_2 \cdots \leq a_m = a \vee b \tag{1}$$

and such that $a_{i+1}/a_i \in \mathbf{A}(L)$ for every $i = 0, 1, \ldots, m-1$.

Proposition 1. *Let I be an ideal of a lattice L. Then the following conditions are equivalent:*

(i) *the ideal I is semiprime;*
(ii) *for any $a, \tilde{a}, b$ of L,*

$$(b \wedge a \in I \ \ \& \ \ b \wedge \tilde{a} \in I \ \ \& \ \ a \vee \tilde{a} \geq b) \Rightarrow b \in I;$$

(iii) *there is no allele b/a of L with $a \in I$ and $b \notin I$;*
(iv) *for any x, y of L,*

$$(x \in I \ \ \& \ \ x \leq y \ \ \& \ \ (x, y) \in \hat{C}(L)) \Rightarrow y \in I;$$

(v) *for any x, y of L,*

$$(x \in I \ \ \& \ \ (x, y) \in \hat{C}(L)) \Rightarrow y \in I;$$

(vi) *the ideal $(I]_{Id(L)}$ generated by I in the ideal lattice $Id(L)$ is semiprime.*

Proof: (i) $\Leftrightarrow$ (ii). Clearly, any semiprime ideal satisfies (ii).

Suppose now that $x \wedge y \in I$ and $x \wedge z \in I$. Put $a = y$, $\tilde{a} = z$ and $b = x \wedge (y \vee z)$. From (ii) it follows that $x \wedge (y \vee z) \in I$.

(i) $\Leftrightarrow$ (iii). This is Main Theorem of [3].

(iii) $\Leftrightarrow$ (iv) and (iv) $\Leftrightarrow$ (v). Immediate.

(i) $\Leftrightarrow$ (vi). This has been proved by Rav [8]. $\square$

Corollary 2. (i) *Let $x \in L$. Then the principal ideal $(x]$ is semiprime if and only if there is no allele y/x with $y > x$.*

(ii) *An ideal X of L is semiprime if and only if there is no ideal Y satisfying $X \subsetneqq Y$ and $Y/X \in \mathbf{A}(Id(L))$.*

Proof: (i) Suppose that $(x]$ satisfies the condition and let q/i be an allele with $i \in (x]$. Since $(i, q) \in \hat{C}(L)$, $(x, x \vee q) \in \hat{C}(L)$. By the assumption and (1), $x \vee q \in (x]$ and so $q \in (x]$. Thus $(x]$ is semiprime.

The remainder follows from Proposition 1 (i).

(ii) Use (i) and Proposition 1 (v). $\square$

2. Properties characterizing semiprime ideals

First we need some notation.

Let I be an ideal of L and let $M \subset L$. By M_I^* we mean the set of all $a \in L$ such that $a \wedge m \in I$ for every $m \in M$. We write m_I^* (or simply m^*) instead of $\{m\}_I^*$.

Note that the ideal I is semiprime if and only if m_I^* is an ideal of L for every $m \in L$.

Given an ideal I of L, let ψ and θ be relations defined on L in the following way:

$$(a,b) \in \psi \Leftrightarrow a_I^* = b_I^*;\ (a,b) \in \theta \Leftrightarrow (a \wedge b)_I^* = (a \vee b)_I^*.$$

The relation ψ was used by Rav in the proof of his Main Theorem in [8]. Note that $\theta \subset \psi$. However, the converse inclusion need not be true.

Theorem 3. *The following conditions are equivalent for any ideal I of a lattice L:*

(i) *The ideal I is semiprime.*
(ii) *The relation ψ satisfies $\psi \supset \hat{C}(L)$.*
(iii) *The relation θ satisfies $\theta \supset \hat{C}(L)$.*
(iv) *The relations θ and ψ satisfy $\theta = \psi \supset \hat{C}(L)$.*

PROOF: (i) $\Rightarrow$ (iv). Let $a^* = b^*$ and let $z \in (a \wedge b)^*$. Then $z \wedge a \wedge b \in I$, which gives $z \wedge a \in b^* = a^*$. Hence $z \wedge a \in I$ and, similarly, $z \wedge b \in I$. Since I is semiprime, it follows that $z \wedge (a \vee b) \in I$. Consequently, $z \in (a \vee b)^*$ and this implies $(a \wedge b)^* = (a \vee b)^*$. Thus $\theta = \psi$. By [8, p. 109], L/θ is distributive and so $\theta \supset \hat{C}(L)$.

(iv) $\Rightarrow$ (iii). Trivial.

(iii) $\Rightarrow$ (ii). Use $\theta \subset \psi$.

(ii) $\Rightarrow$ (i). Let $q/i \in \mathbf{A}(L)$ be such that $i \in I$. Then $(i,q) \in \hat{C}(L) \subset \psi$, and, therefore, $q^* = i^* = L$. This yields $q \in I$. □

Theorem 4. *An ideal I of a lattice L is semiprime if and only if*

$$[(a \vee b) \wedge c]_I^* \supset [a \vee (b \wedge c)]_I^* \tag{2}$$

for every $a, b, c \in I$.

PROOF: Suppose I is semiprime and let $x \in [a \vee (b \wedge c)]^*$. Then $x \wedge [a \vee (b \wedge c)] \in I$, and, a fortiori,

$$x \wedge c \wedge a \in I \ \ \&\ \ x \wedge c \wedge b \in I.$$

Since I is semiprime, $x \wedge c \wedge (a \vee b) \in I$. Therefore, $x \in [(a \vee b) \wedge c]^*$.

Suppose that (2) is valid and let $a \wedge c \in I$ and $b \wedge c \in I$. Replace a in (2) by $a \wedge c$. Then

$$\{[(a \wedge c) \vee b] \wedge c\}^* \supset [(a \wedge c) \vee (b \wedge c)]^*. \tag{3}$$

Since $(a \wedge c) \vee (b \wedge c) \in I$, it is readily seen that $\{[(a \wedge c) \vee b] \wedge c\}^* = L$. Accordingly, $[(a \wedge c) \vee b] \wedge c \in I$, and, by (2), $c \in [b \vee (a \wedge c)]^* \subset [(b \vee a) \wedge c]^*$. Hence $(a \vee b) \wedge c \in I$. □

Theorem 5. *An ideal I of a lattice L is semiprime if and only if the following implication holds for every $a, b, c \in L$:*

$$[(c \wedge a)^*_I \supset (c \wedge b)^*_I \ \ \& \ \ (c \vee a)^*_I \supset (c \vee b)^*_I] \Rightarrow a^*_I \supset b^*_I. \tag{4}$$

Proof: First we shall suppose that I is semiprime. Then we can consider the quotient lattice L/ψ where ψ was defined above. If $x/\psi,\ y/\psi \in L/\psi$, then $x/\psi \leq y/\psi$ if and only if $x^*_I \supset y^*_I$. Hence the antecedent of (4) can be rewritten as

$$c/\psi \wedge a/\psi \leq c/\psi \wedge b/\psi \ \ \& \ \ c/\psi \vee a/\psi \leq c/\psi \vee b/\psi.$$

This, together with a result of M. Molinaro [7, p. 75], implies that $a/\psi \leq b/\psi$. Thus $a^* \supset b^*$.

Finally, let (4) be valid and let x, y and z be arbitrary elements of L. Let $a = (x \vee y) \wedge z$, $b = x \vee (y \wedge z)$ and $c = y$. Then

$$c \wedge a = y \wedge z \leq c \wedge b = y \wedge [x \vee (y \wedge z)]$$

and

$$c \vee a = y \vee [(x \vee y) \wedge z] \leq c \vee b = x \vee y.$$

Consequently we have

$$(c \wedge a)^* \supset (c \wedge b)^* \ \ \& \ \ (c \vee a)^* \supset (c \vee b)^*.$$

By assumption, $a^* \supset b^*$. From Theorem 4 we see that I is semiprime. □

Theorem 6. *An ideal I of a lattice L is semiprime if and only if for any lattice polynomial $p(x_1, x_2, \ldots, x_n)$ and any choice of elements $a_1, a_2, \ldots, a_n \in L$ the relations*

$$p(a_1, a_2, \ldots, a_n) \in I \ \ \& \ \ a_1 \hat{C}(L) a_2 \hat{C}(L) \ldots \hat{C}(L) a_n$$

imply $a_1, a_2, \ldots, a_n \in I$.

Proof: Let I be semiprime and let $p(a_1, a_2, \ldots, a_n) \in I$. Then

$$\begin{aligned} I &= p(a_1, a_2, \ldots, a_n)/\psi = p(a_1/\psi, a_2/\psi, \ldots, a_n/\psi) \\ &= p(a_1/\psi, a_1/\psi, \ldots, a_1/\psi) = a_1/\psi. \end{aligned}$$

Thus $a_1 \in I$ and the same is true for the other a_i.

Now suppose that the stated implication is true and let $p(x_1, x_2) = x_1 \wedge x_2$. If $a \leq b$ are such that $a \in I$ and $(a, b) \in \hat{C}(L)$, then $p(a, b) = a \in I$. We therefore have from Proposition 1 (iv) that I is semiprime. □

3. Semiprimeness as a descriptive tool

Theorem 7. *A lattice L is distributive if and only if every principal ideal $(a]$ $(a \in L)$ is semiprime.*

Proof: Let $I = ((a \wedge b) \vee (a \wedge c)]$ be semiprime. Since $a \wedge b$ and $a \wedge c$ belong to I, we get $a \wedge (b \vee c) \in I$. Thus $a \wedge (b \vee c) \leq (a \wedge b) \vee (a \wedge c)$ and we conclude that $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

Evidently, every ideal of a distributive lattice is semiprime. □

Theorem 8. *A lattice L is modular if and only if for any $a, b, c \in L$, the ideal $(a \vee [b \wedge (a \vee c)]]$ is a semiprime ideal of the sublattice generated by a, b, c in L.*

Proof: Suppose that L is modular and let M denote the sublattice generated by a, b, c. Then, by modularity, $I = (a \vee [b \wedge (a \vee c)]] = ((a \vee b) \wedge (a \vee c)]$. Now M is isomorphic to a quotient lattice of the free modular lattice M_{28} (see [5, p. 64]) with three generators x, y, z. However, a closer inspection of the quotient lattices of M_{28} shows that in any of these quotient lattices the ideal corresponding to $((x \vee y) \wedge (x \vee z)]$ is semiprime. Hence also our ideal I is semiprime.

Conversely, suppose the ideal $I = (a \vee [b \wedge (a \vee c)]]$ is semiprime. Note that $a \wedge (a \vee c) \in I$ and $b \wedge (a \vee c) \in I$. Consequently, $(a \vee b) \wedge (a \vee c) \in I$. Thus $(a \vee b) \wedge (a \vee c) = a \vee [b \wedge (a \vee c)]$ and L is modular. □

Theorem 9. *Let I be a semiprime ideal of a lattice L. Then I is prime if and only if I is a meet-irreducible element of the ideal lattice $Id(L)$.*

Proof: One easily shows that each prime ideal is a meet-irreducible element in $Id(L)$.

It remains to show that every semiprime ideal I which is meet-irreducible in $Id(L)$ is also prime. To do this, consider $b, c \in L$ satisfying $b \wedge c \in I$.

We first note that the inclusion in $I \subset (I \vee (b]) \cap (I \vee (c])$ can be replaced by the equality sign. Indeed, let $x \in (I \vee (b]) \cap (I \vee (c])$. Then there exist $i, j \in I$ and $b_1 \leq b$, $c_1 \leq c$ such that $x \leq (i \vee b_1) \wedge (j \vee c_1)$. Hence $x \leq (h \vee b_1) \wedge (h \vee c_1)$ where $h = i \vee j \in I$. But $b_1 \wedge c_1 \leq b \wedge c \in I$. Therefore, $h \vee (b_1 \wedge c_1) \in I$.

Now $L/\hat{C}(L)$ is distributive, and so $(h \vee (b_1 \wedge c_1), (h \vee b_1) \wedge (h \vee c_1)) \in \hat{C}(L)$. Since I is semiprime, we have, by Proposition 1 (iv), $(h \vee b_1) \wedge (h \vee c_1) \in I$. Consequently, $x \in I$. Combining this with the meet-irreducibility of I we can derive easily that either $b \in I \vee (b] = I$ or $c \in I \vee (c] = I$. □

Corollary 10. *Let $(a]$ be a semiprime ideal of a lattice L. Then $(a]$ is prime if and only if a is a meet-irreducible element of the lattice L.*

Proof: Use the fact that a is a meet-irreducible element of L if and only if $(a]$ is a meet-irreducible element of $Id(L)$. □

By [8, p. 108], any semiprime ideal of L is the kernel of a congruence of L. Hence the following lemma can be applied to semiprime ideals.

Lemma 11. *Let I be an ideal of a lattice L which is the kernel of a congruence θ of L.*

Then

$$(I \wedge J \supset K \wedge J \ \ \& \ \ I \vee J \supset K \vee J) \Rightarrow I \supset K$$

for any ideals J, K of L.

PROOF: Let $k \in K$. Since $K \subset I \vee J$, there exist $i \in I$ and $j \in J$ such that $k \leq i \vee j$. At the same time, $j \wedge k \in J \wedge K \subset I$. Hence $(i, j \wedge k) \in \theta$ and, consequently, $(j, i \vee j) \in \theta$. From $j \leq j \vee k \leq i \vee j$ it follows that $(j, j \vee k) \in \theta$. But then $(j \wedge k, k) \in \theta$. Since I is the kernel of θ and $j \wedge k \in I$, we get $k \in I$. $\square$

Lemma 12. *Let I be a semiprime ideal of a lattice L and let $a, b \in L$ be such that $a \wedge b \in I$.*

Then either $(a] \vee I \neq L$ or

$$(a] \vee I = L \ \ \& \ \ b \in I.$$

PROOF: Suppose that $(a] \vee I = L$. Put $J = (a]$, $K = (b]$ and use Lemma 11. It follows that $b \in K \subset I$. $\square$

The following theorem generalizes a result of Chevalier [6, p. 383] stated for orthomodular lattices.

Theorem 13. *Let L be a relatively complemented lattice. Then a proper ideal I of L is prime if and only if it is a maximal semiprime ideal of L.*

PROOF: It is well-known that in a relatively complemented lattice every proper prime ideal is maximal.

What remains to be shown is that any maximal semiprime ideal $I \neq L$ is prime. Let I be an ideal having these properties and let $a \wedge b \in I$ for some $a, b \in I$.

Suppose first that

$$(a] \vee I \neq L \ \ \& \ \ a \notin I. \tag{5}$$

Then $(a] \vee I$ is not semiprime and, by Proposition 1 (iv), there exist $p \in (a] \vee I$ and $q \notin (a] \vee I$ such that $(p, q) \in \hat{C}(L)$ with $p \leq q$. But $p \in (a] \vee I$ means that $p \leq a \vee i$ for a suitable $i \in I$. Now

$$p \leq q \wedge (a \vee i) \leq q \ \ \& \ \ (p, q) \in \hat{C}(L).$$

Hence $(q \wedge (a \vee i), q) \in \hat{C}(L)$ and, therefore,

$$(a \vee i, q \vee a \vee i) \in \hat{C}(L). \tag{6}$$

Let r^+ be a relative complement of $a \vee i$ in the interval $[i, a \vee i \vee q]$. From (6) we can see that $(i, r^+) \in \hat{C}(L)$. If r^+ belonged to I, then $r^+ \vee a \vee i$ would belong to $(a] \vee I$. But then

$$q \leq a \vee i \vee q = r^+ \vee a \vee i \in (a] \vee I,$$

a contradiction.

Thus $r^+ \notin I$, $i \in I$ and, moreover, $(i, r^+) \in \hat{C}(L)$. But this contradicts Proposition 1 (iv).

We may therefore assume that (5) and a similar statement for b are not true.

However, if $(a] \vee I = L$ or $(b] \vee I = L$, then we can use Lemma 12. Thus either $a \in I$ or $b \in I$ and we are done. □

We now turn our attention to the prime radicals. Recall [8, p. 111] that the *prime radical* $\operatorname{rad}(I)$ of an ideal I in a lattice L is the intersection of all the semiprime ideals of L which contain I.

There is a simple way how to generalize this notion [4]: Given any lattice L, let $D(L)$ denote a congruence of L and let D be the class of all these congruences. We shall say that an ideal I of L is an *ideal with forbidden exterior quotients* in D, if the implication

$$(a \leq b \ \ \& \ \ (a, b) \in D(L) \ \ \& \ \ a \in I) \Rightarrow b \in I$$

is true for any choice of a and b in L.

From Proposition 1 (iv) we conclude that an ideal I is semiprime if and only if it is an ideal with forbidden exterior quotients in $\hat{C}$ where $\hat{C}$ denotes the class of all congruences $\hat{C}(L)$.

If I is an ideal of L, we put

$$\Gamma_D(I) = \{x \in L; (\exists i) i \in I \ \ \& \ \ (i, x) \in D(L)\}$$

calling it the *D-radical* of I.

Proposition 14. *The D-radical of an ideal I is equal to the intersection of all the ideals with forbidden exterior quotients in D containing I.*

Proof: Straightforward. □

Corollary 15. *The $\hat{C}$-radical of any ideal I in a lattice L is equal to the prime radical of I.* □

Let I and J be ideals of a lattice L. If $\Gamma_D(I) \subset \Gamma_D(J)$, then it is clear that for any $i \in I$ there exists $j \in J$ such that $(i, j) \in D(L)$. From this remark we could deduce directly a simple characterization of the case where $\Gamma_D(I) = \Gamma_D(J)$. However, there is another approach which seems to be more fruitful:

Theorem 16. *The following two conditions on ideals I, J of a lattice L are equivalent:*

(i) $\Gamma_D(I) = \Gamma_D(J)$.

(ii) *For any $i \in I$ and any $j \in J$ there exist $i_1 \in I$ and $j_1 \in J$ such that*

$$i \leq i_1 \ \ \& \ \ j \leq j_1 \ \ \& \ \ (i_1, j_1) \in D(L).$$

Proof: Suppose first that $\Gamma_D(I) = \Gamma_D(J)$ and let $i \in I$, $j \in J$.

Since $i \in \Gamma_D(I) \subset \Gamma_D(J)$, there exists $j_2 \in J$ such that (i, j_2) belongs to $D(L)$. Then $(i \vee j, j_2 \vee j) \in D(L)$. It follows from $j_2 \vee j \in \Gamma_D(J) \subset \Gamma_D(I)$ that there exists $i_2 \in I$ such that $(i_2, j \vee j_2) \in D(L)$. Hence

$$(i \vee i_2, i \vee j \vee j_2) \in D(L) \ \ \& \ \ (i \vee j \vee i_2, i \vee j \vee j_2) \in D(L). \tag{7}$$

Now $i \vee i_2 \in \Gamma_D(I) \subset \Gamma_D(J)$ and so there is $j_3 \in J$ with $(i \vee i_2, j_3) \in D(L)$. Therefore,

$$(i \vee i_2 \vee j, j_3 \vee j) \in D(L). \tag{8}$$

Put $i_1 = i \vee i_2$, $j_1 = j \vee j_3$. Then using (7) and (8), we get $(i_1, j_1) \in D(L)$ and it is evident that $i \leq i_1$ and $j \leq j_1$.

Next suppose conversely that I and J satisfy the condition (ii). By symmetry, it is sufficient to prove that $\Gamma_D(I) \subset \Gamma_D(J)$.

Let $x \in \Gamma_D(I)$. Then there exists $i \in I$ with $(x, i) \in D(L)$. Let j be an element of J. By the assumption, there are $i_1 \geq i$, $j_1 \geq j$ such that $(i_1, j_1) \in D(L)$. However, from $(x, i) \in D(L)$ we obtain $(x \vee i_1 \vee j_1, i_1 \vee j_1) \in D(L)$. Similarly, $(i_1, j_1) \in D(L)$ implies that $(i_1 \vee j_1, j_1) \in D(L)$. Therefore, $(x \vee i_1 \vee j_1, j_1) \in D(L)$ and, consequently, $x \vee i_1 \vee j_1 \in \Gamma_D(J)$. Since $\Gamma_D(J)$ is an ideal, we have $x \in \Gamma_D(J)$. □

Corollary 17. *Let a, b be elements of a lattice L.*

Then

(i) *the D-radical $\Gamma_D((a])$ is equal to the D-radical $\Gamma_D((b])$ if and only if $(a, b) \in D(L)$;*

(ii) *the prime radical* $\mathrm{rad}((a])$ *is equal to the prime radical* $\mathrm{rad}((b])$ *if and only if $(a, b) \in \hat{C}(L)$.*

Proof: (i) Suppose $\Gamma_D((a]) = \Gamma_D((b])$. By Theorem 16, there are a_1, b_1 such that

$$a \leq a_1 \ \ \& \ \ b \leq b_1 \ \ \& \ \ a_1 \in (a] \ \ \& \ \ b_1 \in (b] \ \ \& \ \ (a_1, b_1) \in D(L).$$

Hence $(a, b) \in D(L)$.

Conversely, suppose $(a, b) \in D(L)$. For any $i \in (a]$ and $j \in (b]$ we then can put $i_1 = a$, $j_1 = b$ and use Theorem 16.

(ii) Now immediate. □

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Department of Algebra, Charles University, Sokolovská 83, 186 00 Prague 8, Czech Republic

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