The smallest common extension of a sequence of models of ZFC

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Abstract. In this note, we show that the model obtained by finite support iteration of a sequence of generic extensions of models of ZFC of length ω is sometimes the smallest common extension of this sequence and very often it is not.

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Iterated forcing is a transfinite sequence of forcing notions together with a commutative system of complete embeddings among corresponding complete Boolean algebras (see e.g. T. Jech [5]). Starting from a model of ZFC, this sequence produces a sequence of models of ZFC. At the limit step there is a freedom in the construction of the forcing notion and the corresponding model. So the natural question arises whether the model constructed at the limit step can be the smallest common extension of the preceding models. We shall partially answer the question when the extension of a countable generic sequence of models of ZFC obtained by the finite support iteration is the smallest common extension of this sequence. We shall essentially use the fact that the finite support iteration construction usually adds a Cohen real. We recommend to compare a similar result for families of extensions constructed by adding a Cohen real obtained by A. Blass [1] and K. Ciesielski and W. Guzicki [4].

Let us recall some terminology (which is almost that of [5]). By a model we shall understand a set M such that M with the true membership relation $\in$ is a model of ZFC. A model N is said to be an extension of the model M, if $M \subseteq N$ and M, N have the same height $On \cap M = On \cap N$. If $N \supseteq M$ is an extension, then $HDf^N(M)$ is the class (in the sense of the model N) of all hereditarily definable elements in N with parameters from M (see e.g. [8], [2]). It is well known that $HDf^N(M)$ is a model (see [8, p. 186]). If P is a separative partially ordered set then r.o.(P) denotes the (up to isomorphism) unique complete Boolean algebra containing P as a dense subset. If B is a Boolean algebra then we denote

$$B_{rig} = \{a \in B; (\forall f)(f \text{ an automorphism of } B \implies f(a) = a\}.$$

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If M is a model then by saying "$B \in M$ is a complete Boolean algebra" we mean that B is a complete Boolean algebra in the sense of the model M. The Boolean algebra B is said to be rigid if $B = B_{rig}$, i.e. if there is no automorphism of B except the identity. If $B \in M$ is a c.B.a. and G is an M-generic ultrafilter over B then by a slight modification of the proofs on pages 304 and 320 in [8] (compare also [2], [5, pp.269-270]) one can easily show that

$$HDf^{M[G]}(M) = M[G \cap B_{rig}]. \tag{1}$$

Assume now that $B \in M$ is a complete Boolean algebra. Let G, H be M-generic ultrafilters over B such that $M[G] = M[H]$. P. Vopěnka and P. Hájek [8] have shown that

(2) there is an automorphism $f \in M$ of B such that $f(G) = H$.

Let $B \in M$ be a c.B.a., $C \in M^B$ be such that $\|C \text{ is a c.B.a.}\|_B = 1$. As usually we denote by $B * C$ the (up to isomorphism) unique c.B.a. such that the Boolean-valued model M^{B*C} is isomorphic to the model $(M^B)^C$ (constructed inside the model M^B). Moreover, if G is an M-generic ultrafilter over B and H is an $M[G]$-generic ultrafilter over the G-interpretation of C then there is an M-generic ultrafilter $G*H$ over M (defined by a formula as e.g. in [5, p. 234,(23.10)]) for which $M[G][H] = M[G*H]$. If the c.B.a. C is of the form r.o.$(\check{E})$, where E is a c.B.a. in the model M then $B*C$ is the direct sum $B \oplus E$ and $G*H = G \times H$. On the other hand, if B is a complete Boolean subalgebra of a c.B.a. D, everything in the model M, then there exists (in a certain sense unique) $C \in M^B$ such that $\|C \text{ is a c.B.a.}\|_B = 1$ and D is isomorphic to $B * C$. The unique c.B.a. C is denoted by $D : B$ (compare [5, p. 237]). If G is an M-generic ultrafilter over B then the G-interpretation of $D : B$ is the quotient algebra $D/\bar{G}$ where $\bar{G} = \{a \in D; a \geq b \text{ for some } b \in G\}$ (see [6]). If J is an M-generic ultrafilter over D, $G = J \cap B$ then there is an $M[G]$-generic ultrafilter H over the G-interpretation of $D : B$ such that $J = G*H$. Moreover, if D is of the form $B \oplus E$ then $J = G \times H$.

If $a \in B$ then we denote $B|a = \{x \in B; x \leq a\}$. If B is a complete Boolean algebra then $B|a$ is a complete Boolean algebra, too. The c.B.a. B is called a direct product of the complete Boolean algebras A_1, A_2 if there are non-zero disjoint elements $a_1, a_2 \in B, a_1 \vee a_2 = 1$ such that A_i is isomorphic to $B|a_i$ for $i = 1, 2$. We shall write $B = A_1 \otimes A_2$. If G is an M-generic ultrafilter over B then either $a_1 \in G$ or $a_2 \in G$. If $a_i \in G$ we shall say that G is concentrated on the algebra A_i.

A generic sequence of models

$$\{M_i\}_{i\in\omega}, \{B_i\}_{i\in\omega} \tag{3}$$

is a sequence

$$\{M_i\}_{i\in\omega} \tag{4}$$

of models together with a sequence $\{B_i\}_{i\in\omega} \in M_0$ of complete Boolean algebras such that for every $i \in \omega$

$$B_i \text{ is a complete subalgebra of } B_{i+1}, \tag{5}$$

$$\text{there is an } M_0\text{-generic ultrafilter } G_i \text{ over } B_i \text{ such that } M_i = M_0[G_i], \tag{6}$$

and

$$M_{i+1} \text{ is an extension of } M_i. \tag{7}$$

According to (6) we shall always suppose that $B_0 = \{0, 1\}$.

A model N is a common extension of the sequence (4) iff

$$N \text{ is an extension of } M_i \text{ for every } i \in \omega. \tag{8}$$

However, the common extension of the generic sequence (3) should be something more - the sequence (4) must be definable in the common extension. Thus we define: a model N is a common extension of the generic sequence (3) iff (8) and the following condition (9) hold true

$$\begin{gathered}\text{there exists a sequence } \{A_i\}_{i\in\omega} \in N \text{ such that}\\ (\forall i)(A_i \text{ is an } M_0\text{-generic ultrafilter over } B_i \text{ and } M_i = M_0[A_i]).\end{gathered} \tag{9}$$

The condition (9) is in a certain sense the weakest formulation of the definability of the sequence $\{M_i\}_{i\in\omega}$ in N.

If $B = \text{r.o.}(\bigcup_{i\in\omega} B_i)$, G is an M_0-generic ultrafilter over B and $M_i = M_0[G \cap B_i]$ for all $i \in \omega$, then one can easily see that $M_0[G]$ is a common extension of the generic sequence (3). This common extension $M_0[G]$ is called a finite support iteration of the generic sequence (3).

The following simple construction shows that a generic sequence of models need not have a common extension.

Example 1. Let M be a countable model. Let $a \subseteq \omega$ code the countable ordinal $On \cap M$ (the height of M). Let $D_i = C_i \otimes R_i$, where C_i is the Cohen algebra r.o.$(\bigcup_{n\in\omega} {}^n2)$ and R_i is the random algebra (Borel sets modulo sets of measure zero), both constructed in the model M_i. Let $M_0 = M, B_0 = \{0, 1\}, G_0 = \{1\}$. By induction we set $B_{i+1} = B_i * D_i, G_{i+1} = G_i * H_i, M_{i+1} = M[G_{i+1}]$, where H_i is an M_i-generic ultrafilter over D_i such that H_i is concentrated on C_i if and only if $i \in a$. Evidently

$$i \in a \text{ if and only if } M_{i+1} \text{ contains a Cohen real over } M_i.$$

So, if a model N satisfies (9) then $a \in N$, which contradicts to (8). Hence there exists no common extension of the generic sequence $\{M_i\}_{i\in\omega}$.

We shall partially answer the natural question: *is a finite support iteration the smallest common extension of a generic sequence of models*?

We start with a simple result.

Theorem 1. *Let $\{M_i\}_{i\in\omega}, \{B_i\}_{i\in\omega}$ be a generic sequence of models. If for all but finitely many i*

$$\|B_{i+1} : B_i \text{ is rigid}\|_{B_i} = 1 \tag{10}$$

then any finite support iteration of (3) is the smallest common extension of the generic sequence (3).

PROOF: One can easily see that for any $k \le i \in \omega$ the c.B.a. $B_{i+1} : B_i$ is isomorphic (in the model M_i) to the c.B.a. $(B_{i+1} : B_k) : (B_i : B_k)$. On the other hand, for any natural number k, a model is a common extension of the generic sequence of models (3) if and only if it is a common extension of the generic sequence $\{M_i\}_{k\le i\in\omega}$, $\{B_i : B_k\}_{k\le i\in\omega}$. Thus we can assume that (10) holds true for every i.

Let G be an M_0-generic ultrafilter over $B = \text{r.o.}(\bigcup_{i\in\omega} B_i)$, $M_i = M_0[G\cap B_i]$ and N be a common extension of (3). We show that $M_0[G] \subseteq N$. Actually, it suffices to show that $G \in N$.

Set $G_i = G\cap B_i$. We denote by C_i the complete Boolean algebra $C_i = B_{i+1}/\bar{G}_i$, i.e. C_i is the G_i-interpretation of the algebra $B_{i+1} : B_i$. By (2) and (10), for every i there exists the unique M_i-generic ultrafilter H_i over C_i such that $M_{i+1} = M_i[H_i]$. If E is an M_0-generic ultrafilter over B_{i+1} such that $M_{i+1} = M_0[E]$ and $E \supseteq G_i$, then

$$E = G_i * H_i = G_{i+1}. \tag{11}$$

Now, let $\{A_i\}_{i\in\omega}$ be the sequence of (9). Since $M_0[G_i] = M_0[A_i]$ we obtain that every G_i is in N. Using (11) by induction one can easily show that for every $i \in \omega$ there exists unique M_0-generic ultrafilter $E \in N$, $E \subseteq B$ such that

$$(\forall j \le i) M_0[A_j] = M_0[E \cap B_j].$$

Thus the sequence $\{G_i\}_{i\in\omega}$ is an element of N.

Since $\bigcup_{i\in\omega} B_i$ is dense in B, $a \in G$ if and only if there are an integer $i \in \omega$ and an element $b \in G_i$ such that $b \le a$. Hence $G \in N$. □

Corollary. *If for all but finitely many i the condition* (10) *holds true then there exists at most one finite support iteration of the generic sequence* (3).

We shall use the following simple fact:

(12) If c.B.a. B is not rigid then there are non-zero elements $a, b \in B$ and an automorphism f of B such that $a \wedge b = 0$ and $f(a) = b$.

Actually, since B is not rigid there is an automorphism f of B which is not the identity, i.e. $f(c) \ne c$ for some $c \ne 0$. If $f(c) - c \ne 0$ we set $a = c - f^{-1}(c)$ and $b = f(c) - c$. If $f(c) - c = 0$, set $a = f^{-1}(c) - c$ and $b = c - f(c)$.

Now we can prove the second promised result.

Theorem 2. *Let $\{C_i\}_{i\in\omega} \in M_0$ be a sequence of complete Boolean algebras. Let $B_{i+1} = B_i \oplus C_i$ for every $i \in \omega$. Assume that for infinitely many i, the c.B.a. C_i is not rigid. If G is an M_0-generic ultrafilter over $B = r.o.(\bigcup_{i\in\omega} B_i)$, $M_i = M[G \cap B_i]$ for every i, then the finite support iteration $M_0[G]$ is not the smallest common extension of* (3)*. Actually it is not even a minimal common extension of* (3).

Proof: Let

$$a = \{i \in \omega;\ C_i \text{ is not rigid}\}.$$

By (12), for every $i \in a$ there are non-zero elements $a_i, b_i \in C_i$, $a_i \wedge b_i = 0$ and an automorphism f_i of C_i such that $f_i(a_i) = b_i$. If $i \notin a$ we set $a_i = b_i = 0$.

Let C be the set of all sequences $p = \{p_i\}_{i\in\omega}$ for which p_i is a non-zero element of C_i for every $i \in \omega$ and the set $supp(p) = \{i \in \omega; p_i \neq 1\}$ is finite. It is well known that C ordered co-ordinatewise is a dense subset of the c.B.a. B. Let $length(p)$ denote the first natural number greater than every element of $supp(p)$.

We denote

$$Q_i = C_i| - b_i.$$

Since $C_i|b_i$ is isomorphic to $C_i|a_i$, the forcing notion Q_i gives the same informations as the forcing notion C_i.

We denote by Q the set of all sequences $q = \{q_i\}_{i\in\omega} \in C$ for which $q_i \leq a_i$ or $q_i \wedge (a_i \vee b_i) = 0$ for every $i < length(q)$. By P we denote the Cohen forcing, i.e. the set $\bigcup_{n\in\omega} {}^n2$ of all finite sequences of $0, 1$ ordered by the extension. Let

$$T = \{[q, s] \in Q \times P; dom(s) = |\{i; q_i \leq a_i\}|\}.$$

One can easily show that

(13) $$T \text{ is dense in } Q \times P.$$

We define an embedding $f : T \longrightarrow C$ as follows:

$$f([q, s])_i = \begin{cases} f_i(q_i), & \text{if } q_i \leq a_i, s_{|\{j<i; q_j \leq a_j\}|} = 1, \\ q_i, & \text{otherwise.} \end{cases}$$

Let $p = \{p_i\}_{i\in\omega}$ be a non-zero element of C and let

$$dom(s) = |\{i < length(p); p_i \wedge (a_i \vee b_i) \neq 0\}|.$$

For $i < length(p)$ we set $q_i = p_i \wedge a_i$ and $s_{|\{j<i; p_j \wedge (a_j \vee b_j) \neq 0\}|} = 0$ if $p_i \wedge a_i \neq 0$ and we set $q_i = f_i^{-1}(p_i \wedge b_i)$ and $s_{|\{j<i; p_j \wedge (a_j \vee b_j) \neq 0\}|} = 1$ if $p_i \wedge a_i = 0$ and $p_i \wedge b_i \neq 0$. If $p_i \wedge (a_i \vee b_i) = 0$ or $i \geq length(p)$ we set $q_i = p_i$. Then $f([q, s]) \leq p$. Thus

(14) $$f(T) \text{ is dense in } C.$$

Moreover, one can easily see that

(15) $\quad f$ is an isomorphism of the partially ordered set T onto $f(T)$.

Now by (13), (14) and (15) we obtain that r.o.$(C) = B$ is isomorphic to r.o.$(Q \times P) =$ r.o.$(Q) \oplus$ r.o.(P). Thus, there are an M_0-generic ultrafilter H over r.o.(Q) and an $M_0[H]$-generic ultrafilter J over r.o.(P) such that $H \times J$ is isomorphic (via f) to G, more precisely

$$[q, s] \in H \times J \equiv f([q, s]) \in G$$

for any $[q, s] \in T$. Since P is a non-trivial forcing notion we have

$$M_0[G] = M_0[H \times J] \supseteq M_0[H], M_0[G] \neq M_0[H].$$

We show that $M_0[H]$ is a common extension of (3).

It follows immediately from the definition that

$$\text{r.o.}(Q) = \text{r.o.}(\bigcup_{i \in \omega} \text{r.o.}(Q_0^+ \times \cdots \times Q_i^+)).$$

We shall consider r.o.$(Q_0^+ \times \cdots \times Q_i^+)$ as a subalgebra of r.o.(Q). Since

$$B_i = \text{r.o.}(C_0^+ \times \cdots \times C_{i-1}^+),$$

we have

$$\text{r.o.}(Q_0^+ \times \cdots \times Q_{i-1}^+) = B_i|[-b_0, \ldots, -b_{i-1}].$$

For any i we set

$$G_i = G \cap B_i,\ H_i = H \cap \text{r.o.}(Q_0^+ \times \cdots \times Q_{i-1}^+).$$

Let E_i be such an $M_0[G_i]$-generic ultrafilter over C_i that $G_{i+1} = G_i * E_i$. Similarly, let F_i be such an $M_0[H_i]$-generic ultrafilter over Q_i that $H_{i+1} = H_i * F_i$. We show by induction that

$$M_0[G_i] = M_0[H_i]. \tag{16}$$

Assume that (16) holds true for i. By a simple computation we obtain

$$\begin{aligned} F_i &= E_i| - b_i, && \text{if } -b_i \in E_i, \\ F_i|a_i &= f_i^{-1}(E_i|b_i), && \text{otherwise.} \end{aligned}$$

So by the induction hypothesis we obtain

$$M_0[H_{i+1}] = M_0[H_i][F_i] = M_0[G_i][E_i] = M_0[G_{i+1}].$$

Since by (16) for every i, $M_0[G_i] \subseteq M_0[H]$, the model $M_0[H]$ satisfies the condition (8).

Denote by A_i the extension of H_i to the algebra B_i. Then $\{A_i\}_{i \in \omega} \in M_0[H]$ and

$$M_i = M_0[H_i] = M_0[A_i].$$

Therefore, the condition (9) is also fulfilled. □

In Theorem 2 we have assumed that the interpretations of Boolean algebras $B_{i+1} : B_i$ are in M_0. We show that this assumption cannot be omitted.

Example 2. We sketch a construction of a generic sequence of the models (3) such that $\|B_{i+1} : B_i \text{ is not rigid}\|_{B_i} = 1$ for every $i \in \omega$ and the finite support iteration of (3) is the smallest common extension.

Let K, L denote terms of ZFC such that the interpretations of them are complete rigid atomless Boolean algebras not collapsing cardinals, the algebra K adds a new real and the algebra L does not (for the existence of such terms see e.g. P. Štěpánek [7]). We shall denote the interpretations of K, L by the same letters. From the context one can always understand the model in which those terms are interpreted.

Let M_0 be a model. By induction inside the model M_0 one can construct the sequences

$$\{B_i\}_{i\in\omega}, \{C_i\}_{i\in\omega}, \{a_i^j\}_{i\in\omega}, j = 1, 2$$

such that for every $i \in \omega$ and every $j = 1, 2$ (we consider C_i as a subalgebra of B_{i+1}) the following holds true:

$$B_i \text{ is c.B.a.}, C_i \in M^{B_i}, \|C_i \text{ is c.B.a.}\|_{B_i} = 1;$$
$$B_{i+1} = B_i * C_i, B_0 = \{0, 1\};$$
$$\|a_i^1, a_i^2 \in C_i^+, a_i^1 \wedge a_i^2 = 0, a_i^1 \vee a_i^2 = 1\|_{B_i} = 1;$$
$$\|C_{i+1}|a_{i+1}^j = K\|_{B_{i+1}} = a_i^1;$$
$$\|C_{i+1}|a_{i+1}^j = L\|_{B_{i+1}} = a_i^2;$$
$$C_0|a_0^j = K.$$

Let G be an M_0-generic ultrafilter over $B = \text{r.o.}(\bigcup_{i\in\omega} B_i), G_i = G \cap B_i, M_i = M_0[G_i]$. Let H_i be the M_i-generic ultrafilter over C_i such that $G_{i+1} = G_i * H_i$.

There are exactly two M_i-generic ultrafilters A_i over C_i such that $M_{i+1} = M_i[A_i]$; one is concentrated on $C_i|a_i^1$ and the other one on $C_i|a_i^2$. Because H_i is concentrated on $C_i|a_i^1$ if and only if $P(\omega) \cap M_{i+1} \neq P(\omega) \cap M_{i+2}$, we can decide in the model M_{i+2} whether H_i is concentrated on $C_i|a_i^1$ or $C_i|a_i^2$, i.e. we can decide which one of the two generic ultrafilters is the ultrafilter H_i.

Thus, if N is a common extension of the generic sequence (3) then $\{H_i\}_{i\in\omega} \in N$ and therefore (as in the proof of Theorem 1) we obtain $M_0[G] \subseteq N$.

Generally, the question whether there exists a minimal common extension of a generic sequence of models is still open.

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