Systems of nonlinear delay integral equations modelling population growth in a periodic environment

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Abstract. In this paper we study the existence and uniqueness of positive and periodic solutions of nonlinear delay integral systems of the type

$$\begin{aligned} x(t) &= \int_{t-\tau_1}^{t} f(s, x(s), y(s))\, ds \\ y(t) &= \int_{t-\tau_2}^{t} g(s, x(s), y(s))\, ds \end{aligned}$$

which model population growth in a periodic environment when there is an interaction between two species. For the proofs, we develop an adequate method of sub-supersolutions which provides, in some cases, an iterative scheme converging to the solution.

Keywords: nonlinear integral equations, monotone methods, population dynamics, positive solutions

Classification: 45G15, 92D25, 45M15

1. Introduction

In [5], Cooke and Kaplan formulated a model to explain the observed periodic outbreaks of certain infectious diseases. This model can also be interpreted as a growth equation for population when the birth rate varies seasonally. In fact, let $x(t)$ be the number of individuals present in a single species population at the time t (they assumed that the population is uniformly distributed in a given geographical area) and suppose that $f(t, x(t))$ is the number of new births per time unit. If each individual lives to the age τ $(\tau > 0)$ exactly, and then dies, it is reasonable (under some technical additional assumptions, see [5], [9]) to state that

$$x'(t) = f(t, (x(t)) - f(t-\tau, x(t-\tau)) \tag{1.1}$$

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The integrated form of (1.1) is

$$x(t) \;=\; \int_{t-\tau}^{t} f(s,x(s))\,ds + c$$

where c is a constant. Usually one chooses $c = 0$ in the previous equation (see [5], [9]) obtaining the equation

$$x(t) \;=\; \int_{t-\tau}^{t} f(s,x(s))\,ds \tag{1.2}$$

Because of seasonal factors, $f(t,x)$ may, in many situations, be a periodic function of t and, in these cases, one is interested in the existence, uniqueness, stability, etc. of periodic and positive solutions of (1.2).

From the work by Cooke and Kaplan [5], different authors have studied the equation (1.2) (see [2], [3], [4], [6], [8], [9], [10]). However, to the best of our knowledge, we do not know any result concerning systems of equations whose scalar version is (1.2). The interest of such problem is already shown in the work by Cooke and Kaplan [5]: these authors pointed out that it corresponds to models formulating the interaction of several species. This leads to systems like (1.2) where $x(t)$ is a vector function with n components.

In this paper we develop a sub-supersolutions method to study this kind of systems. The main difference with respect to the scalar case (see [2], [3]) is that it is not possible, in general, to define (by using a known sub-supersolution) an iterative scheme converging to the solution. However, in some particular cases (such as the cooperative and competition ones) you may do it and this is shown in Section 2, where, moreover, properties about the minimality and maximality of the found solutions are studied. In Section 3 we prove a result about uniqueness of positive solutions for the case where the system is of cooperative type.

For the sake of simplicity we treat only the case of systems of two equations but it is clear that you may use the ideas contained in this paper to study systems with more number of equations. Also, you could consider more general types of nonlinear delay integral equations such as it is done, in the scalar case, in [3] and [10].

2. Existence of positive solutions

In this section we will study the existence of solutions of the system of nonlinear integral equations

$$\begin{aligned} x(t) &= \int_{t-\tau_1}^{t} f(s,x(s),y(s))\,ds \\ y(t) &= \int_{t-\tau_2}^{t} g(s,x(s),y(s))\,ds. \end{aligned} \tag{2.1}$$

From now on, we assume the following hypothesis:

(H) $f, g : \mathbb{R} \times I_1 \times I_2 \to \mathbb{R}$, are continuous functions where I_1 and I_2 are subintervals of $[0, +\infty)$ such that f and g are nonnegative and ω-periodic $(\omega > 0)$ with respect to the first variable. Also τ_1 and τ_2 are two strictly positive constants. Moreover, due to the interpretation of problem (2.1), we assume that $f(t, 0, y) = g(t, x, 0) = 0$, $\forall (t, x, y) \in \mathbb{R} \times I_1 \times I_2$, (this is completely coherent because if the number of individuals of the species x (or y) is zero at some time t, then the number of new births of this species must be zero). In particular this implies that $(0, 0)$ is always a solution of the system (2.1).

Taking into account the origin of (2.1) we will be interested in the existence of nontrivial, nonnegative, continuous and ω-periodic solutions, i.e. continuous functions $(x, y) : \mathbb{R} \to I_1 \times I_2$ such that $x(t + \omega) = x(t),\ y(t + \omega) = y(t), \forall t \in \mathbb{R}$. Especially we will be interested in the existence of **coexistence states** for (2.1), i.e. solutions of (2.1) with both components nonnegative and nontrivial. At some other times, it may be of interest the existence of **semitrivial solutions** of (2.1), i.e. solutions (x, y) of (2.1) with exactly one nontrivial component (x or y) and its possible influence in the existence of coexistence states of (2.1) (see Remarks 2.2,2 and 3.2).

Next theorem shows a general method for finding nontrivial solutions of (2.1), based on the notion of upper and lower solutions. To this end, E will be the real Banach space of all real and continuous ω-periodic functions defined on $\mathbb{R}$ with the norm

$$\|x\| = \max_{0 \le t \le \omega} |x(t)|,\ \forall x \in E.$$

Also, if $x, y \in E$, with $x(t) \le y(t)$, $\forall t \in \mathbb{R}$, $[x, y]_E$ will denote the following set

$$[x, y]_E = \{z \in E :\ x(t) \le z(t) \le y(t), \forall t \in \mathbb{R}\}.$$

If $a, b \in \mathbb{R}$, $a \le b$, $[a, b]$ will denote the usual interval of $\mathbb{R}$.

Theorem 2.1. *Assume:*

(i) *There exists a pair* $(x_0, y_0) - (x^0, y^0)$ *of sub-supersolutions of* (2.1), *i.e.* $x_0, x^0 :\ \mathbb{R} \to I_1$, $y_0, y^0 :\ \mathbb{R} \to I_2$, *are continuous and* ω*-periodic functions such that*

$$x_0(t) \le x^0(t),\ y_0(t) \le y^0(t),\ \forall t \in \mathbb{R}$$

and

$$\begin{aligned} x_0(t) &\le \int_{t-\tau_1}^{t} f(s, x_0(s), y(s))\, ds \\ &\le \int_{t-\tau_1}^{t} f(s, x^0(s), y(s))\, ds \le x^0(t), \forall t \in \mathbb{R}, \forall y \in [y_0, y^0]_E, \\ y_0(t) &\le \int_{t-\tau_2}^{t} g(s, x(s), y_0(s))\, ds \\ &\le \int_{t-\tau_2}^{t} g(s, x(s), y^0(s))\, ds \le y^0(t), \forall t \in \mathbb{R}, \forall x \in [x_0, x^0]_E. \end{aligned}$$

(ii) *f is nondecreasing with respect to $x \in [\min_{\mathbb{R}} x_0(t), \max_{\mathbb{R}} x^0(t)]$ for fixed $(t,y) \in \mathbb{R} \times I_2$ and g is nondecreasing with respect to $y \in [\min_{\mathbb{R}} y_0(t), \max_{\mathbb{R}} y^0(t)]$ for fixed $(t,x) \in \mathbb{R} \times I_1$.*

Then (2.1) has at least one solution $(x,y) \in [x_0, x^0]_E \times [y_0, y^0]_E$.

Proof: Consider the Banach space $E \times E$ and the subset D of it defined by

$$D = \{(x,y) \in E \times E,\ x_0(t) \le x(t) \le x^0(t),\ y_0(t) \le y(t) \le y^0(t), \forall t \in \mathbb{R}\}$$

Clearly D is convex, closed and bounded.
If $F : D \to E \times E$ is defined by

$$F(x,y)(t) = \\ = \Big(\int_{t-\tau_1}^{t} f(s,x(s),y(s))\,ds, \int_{t-\tau_2}^{t} g(s,x(s),y(s))\,ds\Big), \forall (x,y) \in D, \forall t \in \mathbb{R},$$

it is easily checked that F is compact. Also, from (i) and (ii) we deduce that $F(D) \subset D$, so that Schauder's fixed point theorem guarantees the existence of a solution (x,y) of (2.1) belonging to $D = [x_0, x^0]_E \times [y_0, y^0]_E$. □

Remarks 2.1.
1.- The previous theorem is an adequate version for systems of equations of the method of upper and lower solutions for scalar equations (see [2], [3], [6]).

2.- If f and g satisfy, moreover of hypotheses of Theorem 2.1, additional monotone properties, one may obtain an iterative scheme which provides a monotone convergent sequence to solutions of (2.1); also, these solutions present a property related to the minimality or maximality of solutions of (2.1) in $[x_0, x^0]_E \times [y_0, y^0]_E$. This will be done in next two theorems.

3.- The previous theorem is very general but it does not show how to obtain sub-supersolutions in concrete situations. The following result indicates a way to get the easiest sub-supersolutions, i.e. constant functions. The basic hypotheses are of two types: first, "a convenient boundedness behaviour of f and g with respect to x and y respectively, for x and y large", and second, "an appropriate behaviour of the functions $\frac{f(t,x,y)}{x}$ and $\frac{g(t,x,y)}{y}$ when x and y are small, respectively".

Corollary 2.1. *Let us suppose:*

(i) *In hypothesis (H), $I_1 = [0,M]$, $I_2 = [0,N]$ (M and N are positive constants) with*

$$f(t,M,y) \le \frac{M}{\tau_1},\ g(t,x,N) \le \frac{N}{\tau_2},\ \forall (t,x,y) \in \mathbb{R} \times [0,M] \times [0,N].$$

(ii)

$$\liminf_{x \to 0^+} \frac{f(t,x,y)}{x} = a(t,y)\ uniformly\ in\ (t,y) \in [0,\omega] \times [0,N]$$

(*and therefore in* $\mathbb{R} \times [0, N]$),

$$\liminf_{y \to 0^+} \frac{g(t,x,y)}{y} = b(t,x) \ uniformly\ in\ (t,x) \in [0,\omega] \times [0,M]$$

(*and therefore in* $\mathbb{R} \times [0, M]$)
with a *and* b *continuous functions such that*

$$\min_{t \in [0,\omega]} \int_{t-\tau_1}^{t} a(s, y(s))\, ds \geq m_1 > 1, \forall y \in [0, N]_E$$

and

$$\min_{t \in [0,\omega]} \int_{t-\tau_2}^{t} b(s, x(s))\, ds \geq m_2 > 1, \forall x \in [0, M]_E$$

where m_1 *and* m_2 *are independent from* y *and* x *respectively.*

(iii) $f(t,x,y)$ *is increasing with respect to* $x \in [0,M]$ *for fixed* $(t,y) \in \mathbb{R} \times [0,N]$ *and* $g(t,x,y)$ *is increasing with respect to* $y \in [0,N]$ *for fixed* $(t,x) \in \mathbb{R} \times [0,M]$.

Then, (2.1) *has at least one solution with both components strictly positive for all values of* t.

Proof: Let $\varepsilon > 0$ such that $m_i - \varepsilon\tau_i \geq 1,\ i = 1,2$. Then, there exist $\delta_1 (= \delta_1(\varepsilon))$, $\delta_2 (= \delta_2(\varepsilon))$ such that $0 < \delta_1 < M,\ 0 < \delta_2 < N$ and

$$\begin{aligned} f(t,x,y) &\geq (a(t,y) - \varepsilon)x,\ \ \forall x \in [0,\delta_1], \forall y \in [0,N], \forall t \in \mathbb{R}, \\ g(t,x,y) &\geq (b(t,x) - \varepsilon)y,\ \ \forall y \in [0,\delta_2], \forall x \in [0,M], \forall t \in \mathbb{R}. \end{aligned}$$

Take in Theorem 2.1, $x_0 = \delta_1$, $y_0 = \delta_2$, $x^0 = M$, $y^0 = N$.

Then if $y \in [y_0, y^0]_E$ and $t \in \mathbb{R}$, we have:

$$\begin{aligned} x_0 \leq (m_1 - \varepsilon\tau_1)x_0 &\leq \int_{t-\tau_1}^{t} (a(s,y(s)) - \varepsilon)x_0\, ds \leq \\ &\leq \int_{t-\tau_1}^{t} f(s, x_0, y(s))\, ds \leq \int_{t-\tau_1}^{t} f(s, x^0, y(s))\, ds \leq x^0 \end{aligned}$$

Analogously, if $x \in [x_0, x^0]_E$ and $t \in \mathbb{R}$, we have:

$$\begin{aligned} y_0 \leq (m_2 - \varepsilon\tau_2)y_0 &\leq \int_{t-\tau_2}^{t} (b(s,x(s)) - \varepsilon)y_0\, ds \leq \\ &\leq \int_{t-\tau_2}^{t} g(s, x(s), y_0)\, ds \leq \int_{t-\tau_2}^{t} g(s, x(s), y^0)\, ds \leq y^0 \end{aligned}$$

Therefore, all the hypotheses of Theorem 2.1 are verified and we obtain the desired conclusion. □

Remarks 2.2.

1.- The hypothesis (i) of the previous corollary is trivially satisfied in the case where $f, g : \mathbb{R} \times [0, +\infty) \times [0, +\infty) \to \mathbb{R}$ are bounded. On the other hand, (ii) means that the interaction between species x and y is sufficiently large to produce the existence of (periodic) coexistence states (see [5]).

2.- Under the hypotheses of the previous corollary, the system (2.1) possesses semitrivial solutions of the type $(X, 0)$ and $(0, Y)$ with $X(t) > 0,\ Y(t) > 0, \forall t \in \mathbb{R}$ (i.e. each species may survive in the absence of the other one).

In fact, to prove the existence of $(X, 0)$, think that $y(t) \equiv 0$ is always a solution of the second equation of the system (2.1). Moreover, if you define $h : \mathbb{R} \times I_1 \to [0, +\infty)$ by $h(s, x) \equiv f(s, x, 0)$, then the equation

$$x(t) \ = \ \int_{t-\tau_1}^{t} h(s, x(s))\, ds$$

satisfies all the hypotheses of Theorem 1 in [3] and consequently it has a strictly positive solution X. Now, the pair $(X, 0)$ is a semitrivial solution of (2.1).

In the same way you may prove the existence of $(0, Y)$. Note that not in all situations you have the existence of semitrivial solutions of (2.1) (see Example 2).

3.- The previous remark points out the utility of the use of the method of upper and lower solutions to study (2.1). Think that, from Corollary 2.1, you obtain directly the existence of a coexistence state for (2.1). If, instead of it, you think in applying topological methods (such as the fixed point index, [1]), perhaps you could obtain the existence of a nonnegative and nontrivial solution (x, y) of (2.1) but you would be due to decide after this, doing a careful analysis of the particular situation, if both components are nonnegative and nontrivial (this is, in general, a difficult question if the system admits semitrivial solutions as it happens in our case).

Example 1. Let us consider (2.1) with

$$\begin{aligned} f(t, x, y) \ &= \ a_1(t)x(P - x)c(y) \\ g(t, x, y) \ &= \ b_1(t)y(Q - y)d(x) \end{aligned}$$

where $a_1, b_1 : \mathbb{R} \to (0, +\infty)$ are continuous and ω -periodic functions and $P > 0$, $Q > 0$.

Also, $c : [0, \frac{Q}{2}] \to (0, +\infty),\ d : [0, \frac{P}{2}] \to (0, +\infty)$ are continuous functions.

Then if

$$\frac{1}{P\tau_1} < a_1(t)c(y) \leq \frac{2}{P\tau_1},\ \forall (t, y) \in \mathbb{R} \times [0, \frac{Q}{2}]$$

and

$$\frac{1}{Q\tau_2} < b_1(t)d(x) \leq \frac{2}{Q\tau_2},\ \forall (t, x) \in \mathbb{R} \times [0, \frac{P}{2}],$$

the equation (2.1) has at least a coexistence state $(x, y) \in (0, \frac{P}{2}] \times (0, \frac{Q}{2}]$. To see this, it is sufficient to take in Corollary 2.1, $M = \frac{P}{2}$, $N = \frac{Q}{2}$, $a(t, y) = Pa_1(t)c(y)$ and $b(t, x) = Qb_1(t)d(x)$.

Remarks 2.3.
1.- Note that in this example, $a(t,y) = \frac{\partial f(t,x,y)}{\partial x}|_{x=0}$, $b(t,x) = \frac{\partial g(t,x,y)}{\partial y}|_{y=0}$. This may be a good indication to calculate the functions a and b in Corollary 2.1, in general, if these derivatives exist. As we have said, the hypothesis (i) of this corollary is always verified if $I_1 = I_2 = [0,+\infty)$ and f and g are bounded functions. Also, in order to get x^0 and y^0, you could give other kinds of conditions involving the quantities $\limsup\limits_{x\to+\infty} \dfrac{f(t,x,y)}{x}$ and $\limsup\limits_{y\to+\infty} \dfrac{g(t,x,y)}{y}$. This allow to consider unbounded nonlinearities.

2.- The functions f and g of the above type were considered by Cooke and Kaplan [5] in the scalar case. In the case of systems, our condition on the functions $a_1(t)c(y)$ and $b_1(t)d(x)$ may be interpreted as a control on the interaction of the species x and y in order to have solutions (x,y) such that $0 < x(t) \leq \frac{P}{2}$, $0 < y(t) \leq \frac{Q}{2}$, $\forall t \in \mathbb{R}$.

Next we are going to treat the case when f and g satisfy some additional monotone properties moreover of those imposed in Theorem 2.1; in this case we will construct iterative monotone sequences converging to nontrivial solutions of (2.1). This is another advantage of the use of monotone methods with respect to the topological ones in the study of (2.1).

Remember that (2.1) is a model to study the evolution on the time of a population constituted by two species x and y where $f(t,x(t),y(t))$ and $g(t,x(t),y(t))$ are, respectively, the number of new births per a time unit of the species x and y. Based on this biological interpretation we are going to give the following definition. Previously to it we need something more about the notation: If $h(t,x,y)$ is a given function of three variables $(t,x,y) \in \mathbb{R} \times I_1 \times I_2$, $h(t,x,y) \nearrow x$ $(h(t,x,y) \searrow x)$ means that the function $h(t,x,y)$ is nondecreasing (nonincreasing) with respect to $x \in I_1$ for fixed $(t,y) \in \mathbb{R} \times I_2$ (and analogous definition for the variable y).

Definition. One says that (2.1) is
(a) of a cooperative type if

$$f(t,x,y) \nearrow x,\ f(t,x,y) \nearrow y;\ g(t,x,y) \nearrow x,\ g(t,x,y) \nearrow y.$$

(b) of a competition type if

$$f(t,x,y) \nearrow x,\ f(t,x,y) \searrow y;\ g(t,x,y) \searrow x,\ g(t,x,y) \nearrow y.$$

(c) of a prey-predator type if

$$f(t,x,y) \nearrow x,\ f(t,x,y) \searrow y;\ g(t,x,y) \nearrow x,\ g(t,x,y) \nearrow y.$$

Theorem 2.2. *Assume that, moreover of the hypotheses of Theorem 2.1, the system* (2.1) *is of a cooperative type. Then, if*

$$
\begin{aligned}
x_{n+1}(t) &\equiv \int_{t-\tau_1}^{t} f(s, x_n(s), y_n(s))\, ds, \\
y_{n+1}(t) &\equiv \int_{t-\tau_2}^{t} g(s, x_n(s), y_n(s))\, ds, \\
x^{n+1}(t) &\equiv \int_{t-\tau_1}^{t} f(s, x^n(s), y^n(s))\, ds \\
y^{n+1}(t) &\equiv \int_{t-\tau_2}^{t} g(s, x^n(s), y^n(s))\, ds,
\end{aligned}
\tag{2.2}
$$

$\forall t \in \mathbb{R}$, $\forall n \in \mathbb{N}$,
one has

$$
\begin{aligned}
x_0(t) \le x_n(t) \le x_{n+1}(t) \le x^{n+1}(t) \le x^n(t) \le x^0(t) \\
y_0(t) \le y_n(t) \le y_{n+1}(t) \le y^{n+1}(t) \le y^n(t) \le y^0(t)
\end{aligned}
\tag{2.3}
$$

$\forall t \in \mathbb{R}$, $\forall n \in \mathbb{N}$
and $\{(x_n, y_n)\}$, $\{(x^n, y^n)\}$ *converge respectively, to* (x_*, y_*), (x^*, y^*) *which are solutions of* (2.1) *(possibly equal) belonging to* $[x_0, x^0]_E \times [y_0, y^0]_E$. *Moreover, if* (x, y) *is any solution of* (2.1) *in* $[x_0, x^0]_E \times [y_0, y^0]_E$, *then*

$$
\begin{aligned}
x_*(t) \le x(t) \le x^*(t) \\
y_*(t) \le y(t) \le y^*(t), \ \forall t \in \mathbb{R}
\end{aligned}
\tag{2.4}
$$

(i.e. (x_*, y_*) *and* (x^*, y^*) *are, respectively, the minimal and maximal solutions of* (2.1) *in* $[x_0, x^0]_E \times [y_0, y^0]_E$*).*

Proof: (2.3) is easily proved by induction. Hence, $\{(x_n, y_n)\}$, $\{(x^n, y^n)\}$ converge, respectively, to (x_*, y_*) and (x^*, y^*) belonging to $[x_0, x^0]_E \times [y_0, y^0]_E$. From (2.2), one deduces that (x_*, y_*) and (x^*, y^*) are solutions of (2.1) such that $x_* \le x^*$, $y_* \le y^*$. Moreover, if (x, y) is any solution of (2.1) in $[x_0, x^0]_E \times [y_0, y^0]_E$, then

$$x_n(t) \le x(t) \le x^n(t), \ y_n(t) \le y(t) \le y^n(t), \ \forall t \in \mathbb{R}, \forall n \in \mathbb{N}$$

which implies

$$x_*(t) \le x(t) \le x^*(t), \ y_*(t) \le y(t) \le y^*(t), \ \forall t \in \mathbb{R}.$$

□

Remark 2.4. If one has a pair of sub-supersolutions for (2.1), previous theorem gives an iterative scheme which provides monotone convergent sequences to solutions of (2.1). This is particularly interesting if you know that (2.1) has at most one solution in $[x_0, x^0]_E \times [y_0, y^0]_E$ because in this case such sequences converge to the unique solution of (2.1) in $[x_0, x^0]_E \times [y_0, y^0]_E$ (see Example 2 below).

A similar theorem to previous one may be proved if (2.1) is of a competition type.

Theorem 2.3. *Assume that, moreover of the hypotheses of Theorem* 2.1, *the system* (2.1) *is of a competition type. Then, if*

$$
\begin{aligned}
x_{n+1}(t) &\equiv \int_{t-\tau_1}^{t} f(s, x_n(s), y^n(s))\, ds \\
y^{n+1}(t) &\equiv \int_{t-\tau_2}^{t} g(s, x_n(s), y^n(s))\, ds, \\
x^{n+1}(t) &\equiv \int_{t-\tau_1}^{t} f(s, x^n(s), y_n(s))\, ds \\
y_{n+1}(t) &\equiv \int_{t-\tau_2}^{t} g(s, x^n(s), y_n(s))\, ds,
\end{aligned}
\tag{2.5}
$$

$\forall t \in \mathbb{R}$, $\forall n \in \mathbb{N}$,
one has that

$$
\begin{aligned}
x_0(t) \le x_n(t) \le x_{n+1}(t) \le x^{n+1}(t) \le x^n(t) \le x^0(t) \\
y_0(t) \le y_n(t) \le y_{n+1}(t) \le y^{n+1}(t) \le y^n(t) \le y^0(t)
\end{aligned}
\tag{2.6}
$$

$\forall t \in \mathbb{R}$, $\forall n \in \mathbb{N}$
and $\{(x_n, y^n)\}$, $\{(x^n, y_n)\}$ *converge, respectively, to* (x_*, y^*), (x^*, y_*) *which are solutions of* (2.1) (*possibly equal*) *belonging to* $[x_0, x^0]_E \times [y_0, y^0]_E$. *Moreover, if* (x, y) *is any solution of* (2.1) *in* $[x_0, x^0]_E \times [y_0, y^0]_E$, *then*

$$
\begin{aligned}
&x_*(t) \le x(t) \le x^*(t) \\
&y_*(t) \le y(t) \le y^*(t), \ \forall t \in \mathbb{R}.
\end{aligned}
$$

Remark 2.5. It does not seem possible to prove an analogue of Theorem 2.2 and Theorem 2.3 for the prey-predator case.

3. Uniqueness of positive solutions

In this section we prove a uniqueness result for coexistence states of (2.1). To this end, we need that (2.1) be of cooperative type and that the nonlinearities f and g satisfy a kind of concavity condition as those appeared in [7].

Theorem 3.1. *Suppose that* $I_1 = I_2 = [0, +\infty)$ *in the hypothesis* (H) *and that:*

(i) (2.1) *is of a cooperative type.*

(ii) $f(s, \tau x, \tau y) > \tau f(s, x, y)$, $g(s, \tau x, \tau y) > \tau g(s, x, y)$, $\forall \tau \in (0, 1)$, $\forall x, y \in (0, +\infty)$.

Then (2.1) *has at most one* ω *-periodic solution* (x, y) *satisfying* $x(t) > 0$, $y(t) > 0$, $\forall t \in \mathbb{R}$.

Proof: Let $(x_1, y_1), (x_2, y_2)$ be two distinct solutions of (2.1) with

$$x_i(t) > 0, \;\; y_i(t) > 0, \;\; \forall t \in \mathbb{R}, \;\; i = 1, 2.$$

Clearly, it is not restrictive to assume that there exists $t_1 \in \mathbb{R}$ such that $x_1(t_1) > x_2(t_1)$.
Define

$$\mu \; = \min \left\{ \frac{x_2(t)}{x_1(t)}, \frac{y_2(t)}{y_1(t)}, t \in \mathbb{R} \right\}$$

Then $0 < \mu < 1$, $x_2(t) \geq \mu x_1(t)$, $y_2(t) \geq \mu y_1(t)$, $\forall t \in \mathbb{R}$ and there is $t_2 \in \mathbb{R}$ such that $x_2(t_2) = \mu x_1(t_2)$ or $y_2(t_2) = \mu y_1(t_2)$. Assume $x_2(t_2) = \mu x_1(t_2)$. Hence, $\forall t \in \mathbb{R}$, we have

$$\begin{aligned} x_2(t) &= \int_{t-\tau_1}^{t} f(s, x_2(s), y_2(s))\, ds \\ &\geq \int_{t-\tau_1}^{t} f(s, \mu x_1(s), \mu y_1(s))\, ds \\ &> \mu \int_{t-\tau_1}^{t} f(s, x_1(s), y_1(s))\, ds \\ &= \mu x_1(t) \end{aligned}$$

which contradicts the existence of t_2.
If $y_2(t_2) = \mu y_1(t_2)$ we also arrive to a contradiction taking $y_2(t)$ instead of $x_2(t)$. □

Remarks 3.1.
1.- If (2.1) is of a cooperative type and we have a pair of sub-supersolutions $(x_0, y_0) - (x^0, y^0)$ for it with $x_0(t) > 0, y_0(t) > 0 \;\; \forall t \in \mathbb{R}$, then it may be easily deduced that if the hypothesis (ii) of the previous theorem is checked for all $\tau \in (0, 1)$, for all $x \in [\min_{t \in \mathbb{R}} x_0(t), \max_{t \in \mathbb{R}} x^0(t)]$ and for all $y \in [\min_{t \in \mathbb{R}} y_0(t), \max_{t \in \mathbb{R}} y^0(t)]$, then (2.1) has a unique solution (x, y) in the set $[x_0, x^0]_E \times [y_0, y^0]_E$. Also, it is easy to see that for any pair of continuous and ω-periodic functions (u, v) in $[x_0, x^0]_E \times [y_0, y^0]_E$ we have:

$$\|u_n - x\| \to 0, \;\; \|v_n - y\| \to 0$$

as $n \to +\infty$, where

$$u_{n+1}(t) = \int_{t-\tau_1}^{t} f(s, u_n(s), v_n(s))\, ds, u_0(t) = u(t), \forall t \in \mathbb{R}, \forall n \in \mathbb{N}$$

$$v_{n+1}(t) = \int_{t-\tau_2}^{t} g(s, u_n(s), v_n(s))\, ds, v_0(t) = v(t), \forall t \in \mathbb{R}, \forall n \in \mathbb{N}.$$

2.- A hypothesis of type (ii) appears in the book by Krasnoselskii [7] in the study of other types of nonlinear integral equations. In the scalar case, they have also been used by Smith [9] to prove uniqueness of positive solutions of (1.2).

3.- The same conclusion of the previous theorem is obtained if (ii) is replaced by
(ii′) There exists $\alpha \in (0,1)$, $\beta \in (0,1)$ such that

$$f(s, \tau x, \tau y) \geq \tau^{\alpha} f(s,x,y)$$
$$g(s, \tau x, \tau y) \geq \tau^{\beta} g(s,x,y)$$
$$\forall \tau \in (0,1),\ \forall (x,y) \in (0,+\infty) \times (0,+\infty)$$

This type of conditions may be seen (in the scalar case) in [6].

Example 2. Consider the system (2.1), where

$$\begin{aligned} f(t,x,y) &= a(t)\frac{\sqrt{x}}{m+x}\frac{\sqrt{y}}{n+y} \\ g(t,x,y) &= b(t)\frac{\sqrt{y}}{n+y}\frac{\sqrt{x}}{m+x} \end{aligned} \tag{3.1}$$

$a, b : \mathbb{R} \to [0,+\infty)$ are continuous and ω-periodic functions, m, n are positive constants and $x \geq 0$, $y \geq 0$.
Then, if

$$\begin{aligned} mn &< \int_{t-\tau_1}^{t} a(s)\, ds \leq (m + \min\{n,m\})(n + \min\{n,m\}) \\ mn &< \int_{t-\tau_2}^{t} b(s)\, ds \leq (m + \min\{n,m\})(n + \min\{n,m\}) \end{aligned} \tag{3.2}$$

for all $t \in \mathbb{R}$,
system (2.1) has a unique solution (x,y) such that $x(t) > 0$, $y(t) > 0$, $\forall t \in \mathbb{R}$.

In fact the system (2.1) with f and g defined by (3.1), has a pair of sub-supersolutions given by $x_0 = y_0 \in \mathbb{R}^+$, both sufficiently small and $x^0 = y^0 = \min\{n,m\}$. So that, from Theorem 2.1. and Theorem 3.1. we obtain the desired conclusion.

Remark 3.2. Observe that in this case the system (2.1) has no semitrivial solutions so that the cooperation between the species x and y is crucial to have a coexistence state. Also, (3.2) says how must be such cooperation in order to have a coexistence state (x,y) with

$$0 < x(t) \leq \min\{n,m\},\ 0 < y(t) \leq \min\{n,m\},\ \forall t \in \mathbb{R}.$$

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