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Free ℓ-groups and free products of ℓ-groups

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Abstract. In this paper we have given the construction of free ℓ-groups generated by a po-group and the construction of free products in any sub-product class $\mathcal{U}$ of ℓ-groups. We have proved that the $\mathcal{U}$-free products satisfy the weak subalgebra property.

Keywords: lattice-ordered group (ℓ-group), free ℓ-group, free product of ℓ-groups, sub-product class of ℓ-groups

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1. Introduction

We use the standard terminologies and notations of [1], [2], [5]. The group operation of an ℓ-group is written by additive notation. A po-group is a partially ordered group $[G, P]$ where $P = \{x \in G \mid x \geq 0\}$ is the positive semigroup of G. Let G and H be two po-groups. A map $\varphi : G \to H$ is called a po-group homomorphism, if φ is a group homomorphism and $x \geq y$ implies $\varphi(x) \geq \varphi(y)$ for any $x, y \in G$. A po-group homomorphism φ is called a po-group isomorphism, if φ is an injection and φ^{-1} is also a po-group homomorphism.

A partial ℓ-group G is a set with partial operations corresponding to the usual ℓ-group operations $\cdot$, $^{-1}$, $|$, $\vee$ and $\wedge$ such that whenever the operations are defined for elements of G, the the ℓ-group laws are satisfied. Suppose $[G, P]$ is a po-group. Then G has implicit partial operations $\vee$ and $\wedge$ as determined by the partial order. That is,

$$x \vee y = y \vee x = y \quad \text{if and only if} \quad x \leq y$$

and

$$x \wedge y = y \wedge x = x \quad \text{if and only if} \quad x \leq y.$$

Using these partial lattice operations together with the full group operations, G can be considered as a partial ℓ-group. Then we have the following definition as a special case of the $\mathcal{U}$-free algebra generated by a partial algebra.

Definition 1.1. Let $\mathcal{U}$ be a class of ℓ-groups and $[G, P]$ be a po-group. The ℓ-group $\mathcal{F}_{\mathcal{U}}[G, P]$ is called the $\mathcal{U}$-free ℓ-group generated by $[G, P]$ (or $\mathcal{U}$-free ℓ-group over $[G, P]$) if the following conditions are satisfied:

(1) $\mathcal{F}_{\mathcal{U}}([G,P]) \in \mathcal{U}$;
(2) there exists a po-group isomorphism $\alpha : G \to \mathcal{F}_{\mathcal{U}}([G,P])$ such that $\alpha(G)$ generates $\mathcal{F}_{\mathcal{U}}([G,P])$ as an ℓ-group;
(3) if $K \in \mathcal{U}$ and $\beta : G \to K$ is a po-group homomorphism, then there exists an ℓ-homomorphism $\gamma : \mathcal{F}_{\mathcal{U}}([G,P]) \to K$ such that $\gamma\alpha = \beta$.

$$\begin{tikzcd} {[G,P]} \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & \mathcal{F}_{\mathcal{U}}([G,P]) \arrow[d, "\gamma"] \\ & K \end{tikzcd}$$

Definition 1.2. Let $\mathcal{U}$ be a class of ℓ-groups and $\{G_\lambda \mid \lambda \in \Lambda\}$ be a family of ℓ-groups in $\mathcal{U}$. The $\mathcal{U}$-free product of G_λ is an ℓ-group G, denoted by ${}^{\mathcal{U}}\coprod_{\lambda\in\Lambda} G$, together with a family of injective ℓ-homomorphisms $\alpha_\lambda : G_\lambda \to G$ (called coprojections) such that

(1) ${}^{\mathcal{U}}\coprod_{\lambda\in\Lambda} G_\lambda \in \mathcal{U}$;
(2) $\bigcup_{\lambda\in\Lambda} \alpha_\lambda(G_\lambda)$ generates ${}^{\mathcal{U}}\coprod_{\lambda\in\Lambda} G_\lambda$ as an ℓ-group;
(3) if $K \in \mathcal{U}$ and $\{\beta_\lambda : G_\lambda \to K \mid \lambda \in \Lambda\}$ is a family of ℓ-homomorphisms, then there exists a (necessarily) unique ℓ-homomorphism $\gamma : G \to K$ satisfying $\beta_\lambda = \gamma\alpha_\lambda$ for all $\lambda \in \Lambda$.

A family $\mathcal{U}$ of ℓ-groups is called a sub-product class, if it is closed under taking (1) ℓ-groups and (2) direct products. All our sub-product classes of ℓ-groups are always assumed to contain along with a given ℓ-group all its ℓ-isomorphic copies. Clearly, all varieties of ℓ-groups are sub-product classes of ℓ-groups. Let $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{A}$ be the varieties of all ℓ-groups, representable ℓ-groups and abelian ℓ-groups, respectively.

In this paper we will discuss the existence and constructions of free ℓ-groups generated by a po-group and free products in any sub-product classes of ℓ-groups. In what follows, $\mathcal{U}$ is always denoted a sub-product class of ℓ-groups.

2. Construction for a $\mathcal{U}$-free ℓ-group generated by a po-group

In 1963 and 1965, E.C. Weinberg initially considered the $\mathcal{A}$-free ℓ-group generated by a po-group $[G,P]$. He has given a necessary and sufficient condition for existence and a simple description of $\mathcal{F}_{\mathcal{A}}([G,P])$ in [17], [18].

In 1970, P. Conrad generalized Weinberg's result as follows.

Lemma 2.1 ([3]).

(1) *There exists an $\mathcal{L}$-free ℓ-group $\mathcal{F}_{\mathcal{L}}([G,P])$ generated by $[G,P]$, if and only if there exists a po-group isomorphism of $[G,P]$ into an ℓ-group, if and only if P is the intersection of right order on G.*

(2) *Suppose that* $P = \bigcap_{\lambda\in\Lambda} P_\lambda$ *where* $\{P_\lambda \mid \lambda \in \Lambda\}$ *is the set of all right orders of* G *such that* $P_\lambda \supseteq P$*. If* G_λ *is* G *with one such right order, then denote by* $A(G_\lambda)$ *the* ℓ*-group of order preserving permutations of* G_λ*. Each* $x \in G$ *corresponds to an element* ρ_x *of* $A(G_\lambda)$ *defined by* $\rho_x g = g + x$*. Then* $\mathcal{F}_{\mathcal{L}}([G,P])$ *is the sublattice of* $\prod_{\lambda\in\Lambda} A(G_\lambda)$ *generated by the long constants* $\langle g\rangle$ $(g \in G)$*.*

By Grätzer existence theorem on a free algebra generated by a partial algebra (Theorem 28.2 of [6]) we have

Theorem 2.2. *There exists a* $\mathcal{U}$*-free* ℓ*-group* $\mathcal{F}_{\mathcal{U}}([G,P])$ *generated by a po-group* $[G,P]$ *if and only if* $[G,P]$ *is po-group isomorphic to an* ℓ*-group in* $\mathcal{U}$*.*

Lemma 2.3 (Lemma 11.3.1 of [5])**.** *Let* L *and* L' *be* ℓ*-groups and* M *be a subgroup of* L *which generates* L *as a lattice. Let* $\varphi : M \to L'$ *be a group homomorphism such that for each finite subset* $\{x_{jk} \mid j \in J, k \in K\}$ *of* M*,* $\bigvee_{j\in J}\bigwedge_{k\in K} x_{jk} = 0$ *implies* $\bigvee_{j\in J}\bigwedge_{k\in K}\varphi(x_{jk}) = 0$*. Then* φ *can be uniquely extended to an* ℓ*-homomorphism* $\varphi' : L \to L'$*.*

Let $\mathcal{U}$ be a sub-product class of ℓ-groups. An ℓ-homomorphic image H of an ℓ-group G is said to be a $\mathcal{U}$-homomorphic image, if $H \in \mathcal{U}$. Suppose that a po-group $[G,P]$ is po-group isomorphic into an ℓ-group $F_0 \in \mathcal{U}$ with the po-group isomorphism δ. By Lemma 2.1 (1) there exists the $\mathcal{L}$-free ℓ-group $\mathcal{F}_{\mathcal{L}}([G,P])$ generated by $[G,P]$ with the po-group isomorphism $\alpha : [G,P] \to \mathcal{F}_{\mathcal{L}}([G,P])$. By Definition 1.1 there exists an ℓ-homomorphism $\gamma : \mathcal{F}_{\mathcal{L}}([G,P]) \to F_0$ such that $\gamma\alpha = \delta$. Let $D = \{F_\lambda \mid \lambda \in \Lambda\}$ be the set of all $\mathcal{U}$-homomorphic images of $\mathcal{F}_{\mathcal{L}}([G,P])$ with the ℓ-homomorphisms γ_λ $(\lambda \in \Lambda)$. Thus $\gamma(\mathcal{F}_{\mathcal{L}}([G,P])) \in D$ and D is not empty.

$$\begin{tikzcd}
F_0 & & \mathcal{F}_{\mathcal{L}}([G,P]) \arrow[ll, "\gamma"'] \arrow[dl, "\gamma_o"] \arrow[dd, "\gamma_\lambda"] \\
& L & \\
{[G,P]} \arrow[uu, "\delta"] \arrow[uurr, "\alpha"] \arrow[ur, "\beta"] \arrow[r, "\pi"] & G'\subseteq F \arrow[u, bend left, "\beta'"] \arrow[u, "\beta^*"'] \arrow[r, hook] & \prod_{\lambda\in\Lambda} F_\lambda
\end{tikzcd}$$

For each $\lambda \in \Lambda$, $\gamma_\lambda\alpha$ is a po-group homomorphism of $[G,P]$ into F_λ. Then the direct product $\prod_{\lambda\in\Lambda} F_\lambda$ is an ℓ-group in $\mathcal{U}$. Let π be the natural map of G onto the subgroup G' of long constants of $\prod_{\lambda\in\Lambda} F_\lambda$. That is,

$$\pi(g) = (\dots, \gamma_\lambda\alpha(g), \dots)$$

for $g \in G$. Since $\gamma\alpha = \delta$ is a po-group isomorphism, π is a po-group isomorphism. Let F be the sublattice of $\prod_{\lambda\in\Lambda} F_\lambda$ generated by G'. Then F is an ℓ-subgroup of $\prod_{\lambda\in\Lambda}$, and so $F \in \mathcal{U}$.

Theorem 2.4. *Suppose that a po-group $[G,P]$ is po-group isomorphic into an ℓ-group in a sub-product class of ℓ-groups. Then the $\mathcal{U}$-free ℓ-group $\mathcal{F}_{\mathcal{U}}([G,P])$ generated by $[G,P]$ is the sublattice F of the direct product $\prod_{\lambda\in\Lambda} F_\lambda$ generated by the po-group isomorphic image G' of G where $\{F_\lambda \mid \lambda\in\Lambda\}$ are all ℓ-homomorphic images of the $\mathcal{L}$-free ℓ-group $\mathcal{F}_{\mathcal{L}}([G,P])$ generated by $[G,P]$.*

Proof: We have already known that $F\in\mathcal{U}$ and $[G,P]$ is po-group isomorphic into F. Suppose that β is a po-group homomorphism of $[G,P]$ into an ℓ-group $L\in\mathcal{U}$. Then there exists an ℓ-homomorphism $\gamma_o : \mathcal{F}_{\mathcal{L}}([G,P]) \to L$ such that $\gamma_o\alpha=\beta$. So $\gamma'(\mathcal{F}_{\mathcal{L}}([G,P]))\in D$. For $g'=\pi(g)\in G'$ $(g\in G)$, put

$$\beta'(g')=\beta(g).$$

Then β' is a group homomorphism of G' into L and $\beta'\pi=\beta$. By Lemma 2.3 we only need to show that for each finite subset $\{g_{jk}\mid j\in J, k\in K\}\subseteq G$, $\bigvee_{j\in J}\bigwedge_{k\in K}\beta'\pi(g_{jk})\neq 0$ implies $\bigvee_{j\in J}\bigwedge_{k\in K}\pi(g_{jk})\neq 0$. In fact,

$$\bigvee_{j\in J}\bigwedge_{k\in K}\gamma_o\alpha(g_{jk})=\bigvee_{j\in J}\bigwedge_{k\in K}\beta(g_{jk})=\bigvee_{j\in J}\bigwedge_{k\in K}\beta'\pi(g_{jk})\neq 0.$$

Hence

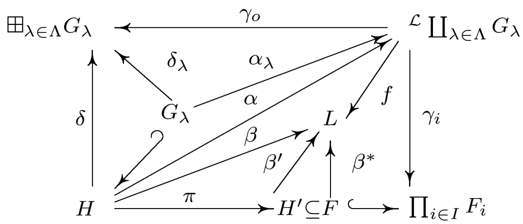
$$\begin{aligned}\bigvee_{j\in J}\bigwedge_{k\in K}\pi(g_{jk}) &= \bigvee_{j\in J}\bigwedge_{k\in K}(\dots,\gamma_o\alpha(g_{jk}),\dots)\\ &=(\dots,\bigvee_{j\in J}\bigwedge_{k\in K}\gamma_o\alpha(g_{jk}),\dots)\neq 0.\end{aligned}$$

Therefore β' can be uniquely extended to an ℓ-homomorphism $\beta^*: F\to L$. □

3. Construction of $\mathcal{U}$-free products

Let $\mathcal{U}$ be a sub-product class of ℓ-groups and $\{G_\lambda\mid\lambda\in\Lambda\}$ be a family of ℓ-groups in $\mathcal{U}$. By Corollary 2 of Theorem 2 of [6] the $\mathcal{U}$-free product ${}^{\mathcal{U}}\coprod_{\lambda\in\Lambda} G_\lambda$ always exists. Specifically, there exists an $\mathcal{L}$-free product ${}^{\mathcal{L}}\coprod_{\lambda\in\Lambda} G_\lambda$ with the coprojection α_λ. In [7]–[14] J. Martinez, W. Powell and C. Tsinakis have given several descriptions and some properties for the free products in the varieties $\mathcal{R}$ and $\mathcal{A}$. W.C. Holland and E. Scrimger have given a description for $\mathcal{L}$-free product. Let H be the group free product of $\{G_\lambda\mid\lambda\in\Lambda\}$. Let $P=\{h\in H\mid h$ be a sum of conjugates in H of elements of $\bigcup_{\lambda\in\Lambda} G_\lambda^+\}$ and $P'=\{Q\mid Q$ is the positive cone of a right order on H with $P\subseteq Q\}$. Then $[H,P']$ is a po-group and its $\mathcal{L}$-free extension $\mathcal{F}_{\mathcal{L}}([H,P'])$ by the ℓ-ideal generated by $\{g^-\wedge g^+\mid g\in\bigcup_{\lambda\in\Lambda} G_\lambda\}$ (Theorem 3.7 of [4]). There exists a group homomorphism $\alpha: H\to {}^{\mathcal{L}}\coprod_{\lambda\in\Lambda} G_\lambda$ which extends every α_λ $(\lambda\in\Lambda)$.

It is clear that the cardinal sum $\boxplus G_\lambda$ is an ℓ-group in $\mathcal{U}$ and every G_λ $(\lambda\in\Lambda)$ can naturally be embedded into $\boxplus_{\lambda\in\Lambda} G_\lambda$ as an ℓ-group with embedding δ_λ. Hence there exists a group homomorphism $\delta: H\to\boxplus_{\lambda\in\Lambda} G_\lambda$ which extends each δ_λ



$(\lambda \in \Lambda)$ and there exists an ℓ-homomorphism $\gamma_o : {}^{\mathcal{L}}\coprod_{\lambda\in\Lambda} G_\lambda \to \boxplus_{\lambda\in\Lambda} G_\lambda$ such that $\gamma_o\alpha_\lambda = \delta_\lambda$ for each $\lambda \in \Lambda$. Let $D = \{F_i \mid i \in I\}$ be the set of all $\mathcal{U}$-homomorphic images of ${}^{\mathcal{L}}\coprod_{\lambda\in\Lambda} G_\lambda$ with the ℓ-homomorphisms γ_i $(i \in I)$. Thus, $\boxplus_{\lambda\in\Lambda} G_\lambda \in D$ and D is not empty. For each $\lambda \in \Lambda$ and each $i \in I$, $\gamma_i\alpha_\lambda$ is an ℓ-homomorphism of G_λ into F_i. The direct product $\prod_{i\in I} F_i$ is an ℓ-group in $\mathcal{U}$. For each $\lambda \in \Lambda$, let π_λ be the natural ℓ-homomorphism of G_λ onto the ℓ-subgroup G'_λ of $\prod_{i\in I} F_i$. That is,

$$\pi_\lambda(g_\lambda) = (\dots, \gamma_i\alpha_\lambda(g_\lambda), \dots)$$

for $g_\lambda \in G_\lambda$. Let H' be the subgroup of $\prod_{i\in I} F_i$ generated by $\bigcup_{\lambda\in\Lambda} G'_\lambda$. Let π be the group homomorphism of H onto H' which extends every π_λ $(\lambda \in \Lambda)$. That is,

$$\pi(h) = (\dots, \gamma_i\alpha(h), \dots)$$

for $h \in H$. Since $\boxplus_{\lambda\in\Lambda} G_\lambda \in D$ and every δ_λ $(\lambda \in \Lambda)$ is an ℓ-isomorphism, π_λ is an ℓ-isomorphism for each $\lambda \in \Lambda$. Let F be the sublattice of $\prod_{i\in I} F_i$ generated by H'. For each $h \in H$, put $h' = \pi(h)$. Since $\prod_{i\in I} F_i$ is a distributive lattice,

$$F = \left\{ \bigvee_{j\in J} \bigwedge_{k\in K} h'_{jk} \mid h_{jk} \in H, \quad J \text{ and } K \text{ finite} \right\}.$$

Then we have the following construction theorem for ${}^{\mathcal{U}}\coprod_{\lambda\in\Lambda} G_\lambda$.

Theorem 3.1. *Suppose that $\{G_\lambda \mid \lambda \in \Lambda\}$ is a family of ℓ-groups in a subproduct class of ℓ-groups. Then the $\mathcal{U}$-free product ${}^{\mathcal{U}}\coprod_{\lambda\in\Lambda} G_\lambda$ is the sublattice F of the direct product $\prod_{i\in I} F_i$ generated by the group homomorphic image H' of the group free product H of G_λ, where $\{F_i \mid i \in I\}$ are all $\mathcal{U}$-homomorphic images of the $\mathcal{L}$-free product ${}^{\mathcal{L}}\coprod_{\lambda\in\Lambda} G_\lambda$.*

Proof: We have seen that $F \in \mathcal{U}$ and each G_λ $(\lambda \in \Lambda)$ can be embedded into F as an ℓ-group. Suppose that $L \in \mathcal{U}$ and $\{\beta_\lambda : G_\lambda \to L \mid \lambda \in \Lambda\}$ is a family of ℓ-homomorphisms. We must show that there exists a unique ℓ-homomorphism $\beta^* : F \to L$ such that $\beta^*\pi_\lambda = \beta_\lambda$. By the universal property of group free product, there exists a group homomorphism $\beta : H \to L$ which extends every β_λ $(\lambda \in \Lambda)$. For any $h' = \pi(h) \in H'$, put

$$\beta'(h') = \beta(h).$$

By the universal property of an $\mathcal{L}$-free product, there exists a unique ℓ-homomorphism $f : {}^{\mathcal{L}}\coprod_{\lambda\in\Lambda} G_\lambda \to L$ such that $\beta_\lambda = f\alpha_\lambda$ for each $\lambda \in \Lambda$. Then

$$f\alpha = \beta'\pi = \beta.$$

By Lemma 2.3 we only need to show that for each finite subset $\{h_{jk} \mid j \in J, k \in K\} \subseteq H$, $\bigvee_{j\in J}\bigwedge_{k\in K} \beta'\pi(h_{jk}) \neq 0$ implies $\bigvee_{j\in J}\bigwedge_{k\in K} \pi(h_{jk}) \neq 0$. In fact,

$$\bigvee_{j\in J}\bigwedge_{k\in K} f\alpha(h_{jk}) = \bigvee_{j\in J}\bigwedge_{k\in K} \beta'\pi(h_{jk}) \neq 0.$$

Because $f({}^{\mathcal{L}}\coprod_{\lambda\in\Lambda} G_\lambda) \in D$, $\bigvee_{j\in J}\bigwedge_{k\in K} \gamma_i\alpha(h_{jk}) \neq 0$ for some $i \in I$. So

$$\begin{aligned}\bigvee_{j\in J}\bigwedge_{k\in K} \pi(h_{jk}) &= \bigvee_{j\in J}\bigwedge_{k\in K} (\dots, \gamma_i\alpha(h_{jk}), \dots)\\ &= (\dots, \bigvee_{j\in J}\bigwedge_{k\in K} \gamma_i\alpha(h_{jk}), \dots) \neq 0.\end{aligned}$$

Therefore β' can be uniquely extended to an ℓ-homomorphism $\beta^* : F \to L$. □

By using the similar proof as the one for Theorem 3.1 we can get the following result.

Theorem 3.2. *Suppose that $\mathcal{U}$ is a sub-product class of ℓ-groups which is contained in $\mathcal{A}$ and $\{G_\lambda \mid \lambda \in \Lambda\}$ is a family in $\mathcal{U}$. Then the $\mathcal{U}$-free product ${}^{\mathcal{U}}\coprod_{\lambda\in\Lambda} G_\lambda$ is the sublattice of $\prod_{i\in I} F_i$ generated by the group homomorphic image H' of the group free product H of G_λ, where $\{F_i \mid i \in I\}$ are all ℓ-homomorphic images of the $\mathcal{A}$-free product ${}^{\mathcal{A}}\coprod_{\lambda\in\Lambda} G_\lambda$.*

4. The weak subalgebra property

Let $\mathcal{U}$ be a sub-product class of ℓ-groups. $\mathcal{U}$-free products are said to have the subalgebra property if for any family $\{G_\lambda \mid \lambda \in \Lambda\}$ in $\mathcal{U}$ with ℓ-subgroups $H_\lambda \in G_\lambda$, ${}^{\mathcal{U}}\coprod_{\lambda\in\Lambda} H_\lambda$ is simply the ℓ-subgroup of ${}^{\mathcal{U}}\coprod_{\lambda\in\Lambda} G_\lambda$ generated by $\bigcup_{\lambda\in\Lambda} H_\lambda$. It is well known that $\mathcal{A}$-free products satisfy the subalgebra property (Theorem 3.2 of [11]). $\mathcal{U}$-free products are said to have the weak subalgebra property if $\{G_\lambda \mid \lambda \in \Lambda\}$ is a family in $\mathcal{U}$ with ℓ-subgroups $H_\lambda \subseteq G_\lambda$ and any family of ℓ-homomorphisms $\sigma_\lambda : H_\lambda \to L \in \mathcal{U}$ can be extended to a family of ℓ-homomorphisms $\sigma'_\lambda : G_\lambda \to L' \in \mathcal{U}$ such that L is an ℓ-subgroup of L' and $\sigma'_\lambda|_{H_\lambda} = \sigma_\lambda$, then ${}^{\mathcal{U}}\coprod_{\lambda\in\Lambda} H_\lambda$ is the ℓ-subgroup of ${}^{\mathcal{U}}\coprod_{\lambda\in\Lambda} G_\lambda$ generated by $\bigcup_{\lambda\in\Lambda} H_\lambda$.

Theorem 4.1. *Suppose that $\mathcal{U}$ is a sub-product class of ℓ-groups which is contained in $\mathcal{A}$. Then $\mathcal{U}$-free products satisfy the weak subalgebra property.*

Proof: Suppose that $\{G_\lambda \mid \lambda \in \Lambda\}$ is a family in $\mathcal{U}$ with ℓ-subgroups $H_\lambda \subseteq G_\lambda$ and any family of ℓ-homomorphisms $\sigma_\lambda : H_\lambda \to L \in \mathcal{U}$ can be extended to a family of ℓ-homomorphisms $\sigma'_\lambda : G_\lambda \to L' \in \mathcal{U}$ such that L is an ℓ-subgroup of L' and $\sigma'_\lambda|_{H_\lambda} = \sigma_\lambda$. We see that ${}^{\mathcal{A}}\coprod_{\lambda\in\Lambda} H_\lambda$ is the ℓ-subgroup of ${}^{\mathcal{A}}\coprod_{\lambda\in\Lambda} G_\lambda$ generated by $\bigcup_{\lambda\in\Lambda} H_\lambda$.

(1) First we show that any ℓ-homomorphism $\gamma : {}^{\mathcal{A}}\coprod_{\lambda\in\Lambda} H_\lambda \to L \in \mathcal{U}$ can be extended to an ℓ-homomorphism $\gamma' : {}^{\mathcal{A}}\coprod_{\lambda\in\Lambda} G_\lambda \to L' \in \mathcal{U}$ such that L is an ℓ-subgroup of L' and $\gamma'|_{{}^{\mathcal{A}}\coprod_{\lambda\in\Lambda} H_\lambda} = \gamma$. In fact, any ℓ-homomorphism $\gamma : {}^{\mathcal{A}}\coprod_{\lambda\in\Lambda} H_\lambda \to L \in \mathcal{U}$ induces a family of ℓ-homomorphisms $\sigma_\lambda : H_\lambda \to L \in \mathcal{U}$ such that $\gamma\alpha_\lambda = \sigma_\lambda$ for each $\lambda \in \Lambda$ where α_λ is the inclusion map. Thus σ_λ can

$$\begin{tikzcd}
H_\lambda \arrow[rrr, hook] \arrow[dr, "\sigma_\lambda"] \arrow[dd, "\alpha_\lambda"'] & & & G_\lambda \arrow[dl, "\sigma'_\lambda"'] \arrow[dd, "\alpha'_\lambda"] \\
& L \arrow[r, hook] & L' & \\
{}^{\mathcal{A}}\coprod_{\lambda\in\Lambda} H_\lambda \arrow[rrr, hook] \arrow[ur, "\gamma"'] & & & {}^{\mathcal{A}}\coprod_{\lambda\in\Lambda} G_\lambda \arrow[ul, "\gamma'"]
\end{tikzcd}$$

be extended to a family of ℓ-homomorphisms $\sigma'_\lambda : G_\lambda \to L' \in \mathcal{U}$ such that L is an ℓ-subgroup of L' and $\sigma'_\lambda|_{H_\lambda} = \sigma_\lambda$. By the universal property there exists an ℓ-homomorphism $\gamma' : {}^{\mathcal{A}}\coprod_{\lambda\in\Lambda} G_\lambda \to L'$ such that $\gamma'\alpha'_\lambda = \sigma'_\lambda$ for each $\lambda \in \Lambda$ where α'_λ is the inclusion map. Hence

$$\sigma_\lambda = \sigma'_\lambda|_{H_\lambda} = (\gamma'\alpha'_\lambda)|_{H_\lambda} = \gamma'|_{H_\lambda}$$

for each $\lambda \in \Lambda$. By the uniqueness $\gamma'|_{{}^{\mathcal{A}}\coprod_{\lambda\in\Lambda} H_\lambda} = \gamma$.

(2) Now we show that ${}^{\mathcal{U}}\coprod_{\lambda\in\Lambda} H_\lambda$ is the ℓ-subgroup of ${}^{\mathcal{U}}\coprod_{\lambda\in\Lambda} G_\lambda$ generated by the $\bigcup_{\lambda\in\Lambda} H_\lambda$. Let $G_0 = \oplus_{\lambda\in\Lambda} G_\lambda$, $H_0 = \oplus_{\lambda\in\Lambda} H_\lambda$, $G = {}^{\mathcal{A}}\coprod_{\lambda\in\Lambda} G_\lambda$ and $H = {}^{\mathcal{A}}\coprod_{\lambda\in\Lambda} H_\lambda$. Then G_0 and H_0 are subgroups of G and H, respectively, and H_0 is a subgroup of G_0, H is a subgroup of G. Let $D = \{F_i \mid i \in I\}$ be the set of all $\mathcal{U}$-homomorphic images of G with the ℓ-homomorphisms γ'_i $(i \in I)$. For each

$$\begin{tikzcd}
\prod_{i\in I} E_i \arrow[rrrrr, hook] & & & & & \prod_{i\in I} F_i \\
& {}^{\mathcal{U}}\coprod_{\lambda\in\Lambda} H_\lambda \arrow[ul, hook] \arrow[rrr, hook] & & & {}^{\mathcal{U}}\coprod_{\lambda\in\Lambda} G_\lambda \arrow[ur, hook] & \\
& & H'_0 \arrow[ul, hook] \arrow[r, hook] & G'_0 \arrow[ur, hook] & & \\
& & H_0 \arrow[u, "\pi"] \arrow[r, hook] \arrow[dll, hook] & G_0 \arrow[u, "\pi'"'] \arrow[drr, hook] & & \\
H \arrow[uuuu] \arrow[rrrrr, hook] & & & & & G \arrow[uuuu]
\end{tikzcd}$$

$i \in I$, $\gamma'_i|_H(H)$ is a $\mathcal{U}$-homomorphic image of H. Conversely, if E is a $\mathcal{U}$-homomorphic image of H with the ℓ-homomorphism γ. It follows from (1) that γ can be extended to an ℓ-homomorphism $\gamma' : G \to F \in \mathcal{U}$ such that E is an

ℓ-subgroup of F and $\gamma'|_H = \gamma$. Hence the set of all $\mathcal{U}$-homomorphic images of H is $C = \{E_i \mid i \in I\}$ where E_i is an ℓ-subgroup of F_i and $\gamma_i'|_H(H) = E_i$ for each $i \in I$. By Theorem 3.2 we see that $\mathcal{U}$-free product ${}^{\mathcal{U}}\coprod_{\lambda\in\Lambda} G_\lambda$ is the sublattice of the direct product $\prod_{i\in I} F_i$ generated by the group homomorphic image G_0' of G_0 with the group homomorphism π', and the $\mathcal{U}$-free product ${}^{\mathcal{U}}\coprod_{\lambda\in\Lambda} H_\lambda$ is the sublattice of the direct product $\prod_{i\in I} E_i$ generated by the group homomorphic image H_0' of H_0 with the group homomorphism π. $\pi'|_{G_\lambda}$ and $\pi|_{H_\lambda}$ are all ℓ-homomorphisms for $\lambda \in \Lambda$. Hence ${}^{\mathcal{U}}\coprod_{\lambda\in\Lambda} G_\lambda$ is the ℓ-subgroup of $\prod_{i\in I} F_i$ generated by $\bigcup_{\lambda\in\Lambda} G_\lambda'$ where $G_\lambda' = \pi'(G_\lambda)$ and ${}^{\mathcal{U}}\coprod_{\lambda\in\Lambda} H_\lambda$ is the ℓ-subgroup of $\prod_{i\in I} E_i$ generated by $\bigcup_{\lambda\in\Lambda} H_\lambda'$ where $H_\lambda' = \pi(H_\lambda)$. From the above we see that $\prod_{\in I} E_i$ is an ℓ-subgroup of $\prod_{i\in I} F_i$ and

$$\pi'|_{H_0} = \pi.$$

Therefore ${}^{\mathcal{U}}\coprod_{\lambda\in\Lambda} H_\lambda$ is the ℓ-subgroup of $\prod_{i\in I} F_i$ generated by $\bigcup_{\lambda\in\Lambda} H_\lambda'$, and so ${}^{\mathcal{U}}\coprod_{\lambda\in\Lambda} H_\lambda$ is also the ℓ-subgroup of ${}^{\mathcal{U}}\coprod_{\lambda\in\Lambda} G_\lambda$ generated by $\bigcup_{\lambda\in\Lambda} H_\lambda'$. □

5. An example

Theorem 2.4 and Theorem 2.1 are applicable to all varieties of ℓ-groups. But here we give an example of a class of ℓ-groups which is not a variety.

An ℓ-group G is said to be weak Hamiltonian if each closed convex ℓ-subgroup of G is normal. Let WH be the set of all weak Hamiltonian ℓ-groups. It is easy to show that WH is a sub-product class of ℓ-groups (see [16]) and

$$\mathcal{A} \subseteq \mathrm{WH} \subseteq \mathcal{R} \subseteq \mathcal{L}.$$

So we have the construction theorem for the WH-free product.

Theorem 5.1. *Suppose that* $\{G_\lambda \mid \lambda \in \Lambda\}$ *is a family in* WH. *Then* ${}^{\mathrm{WH}}\coprod_{\lambda\in\Lambda} G_\lambda$ *is the sublattice of* $\prod_{i\in I} F_i$ *generated by the group homomorphic image* H' *of the group free product* H *of* G_λ, *where* $\{F_i \mid i \in I\}$ *are all weak Hamiltonian* ℓ*-homomorphic images of* ${}^{\mathcal{L}}\coprod_{\lambda\in\Lambda} G_\lambda$.

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