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p-sequential like properties in function spaces

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Abstract. We introduce the properties of a space to be strictly WFU(M) or strictly SFU(M), where $\emptyset \neq M \subset \omega^*$, and we analyze them and other generalizations of p-sequentiality ($p \in \omega^*$) in Function Spaces, such as Kombarov's weakly and strongly M-sequentiality, and Kocinac's WFU(M) and SFU(M)-properties. We characterize these in $C_\pi(X)$ in terms of cover-properties in X; and we prove that weak M-sequentiality is equivalent to WFU($L(M)$)-property, where $L(M) = \{{}^\lambda p : \lambda < \omega_1 \text{ and } p \in M\}$, in the class of spaces which are p-compact for every $p \in M \subset \omega^*$; and that $C_\pi(X)$ is a WFU($L(M)$)-space iff X satisfies the M-version δ_M of Gerlitz and Nagy's property δ. We also prove that if $C_\pi(X)$ is a strictly WFU(M)-space (resp., WFU(M)-space and every RK-predecessor of $p \in M$ is rapid), then X satisfies C'' (resp., X is zero-dimensional), and, if in addition, $X \subset \mathbb{R}$, then X has strong measure zero (resp., X has measure zero), and we conclude that $C_\pi(\mathbb{R})$ is not p-sequential if $p \in \omega^*$ is selective. Furthermore, we show: (a) if $p \in \omega^*$ is selective, then $C_\pi(X)$ is an FU(p)-space iff $C_\pi(X)$ is a strictly WFU($T(p)$)-space, where $T(p)$ is the set of RK-equivalent ultrafilters of p; and (b) $p \in \omega^*$ is semiselective iff the subspace $\omega \cup \{p\}$ of $\beta\omega$ is a strictly WFU($T(P)$)-space. Finally, we study these properties in $C_\pi(Z)$ when Z is a topological product of spaces.

Keywords: selective, semiselective and rapid ultrafilter; Rudin-Keisler order; weakly M-sequential, strongly M-sequential, WFU(M)-space, SFU(M)-space, strictly WFU(M)-space, strictly SFU(M)-space; countable strong fan tightness, Id-fan tightness, property C'', measure zero

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0. Introduction and preliminaries

In this paper, space will mean a Tychonoff topological space, and $C_\pi(X)$ will denote the space of all the continuous real valued functions defined on X and endowed with the topology of pointwise convergence. For $f \in C_\pi(X)$ and $\delta > 0$, we put $\operatorname{coz}_\delta f = \{x \in X : |f(x)| < \delta\}$. ω is the set of natural numbers equipped with the discrete topology and $\beta\omega$ is the Stone-Čech compactification of ω which can be viewed as the set of all ultrafilters on ω, where $\widehat{A} = \{q \in \beta\omega : A \in q\}$ is a basic open neighborhood of p for each $A \in p$. The remainder $\omega^* = \beta\omega \setminus \omega$ coincides with the set of free ultrafilters on ω and if $f : \omega \to \beta\omega$ is a function, we will denote by $\overline{f} : \beta\omega \to \beta\omega$ the Stone extension of f.

We say that $p \in \omega^*$ is *selective* if for every collection $\{A_n\}_{n<\omega}$ of disjoint infinite subsets of ω, with $A_n \notin p$ for every $n < \omega$, there exists $A \in p$ such that $|A \cap A_n| \leq 1$ for every $n < \omega$; $p \in \omega^*$ is *semiselective* if for every sequence $\{A_n\}_{n<\omega} \subset p$ there is $a_n \in A_n$ for each $n < \omega$ such that $\{a_n : n < \omega\} \in p$; and

$p \in \omega^*$ is *rapid* if for every $f \in {}^\omega\omega$ we can find $A \in p$ such that $|A \cap f(n)| \leq n$ for each $n < \omega$. It is not difficult to prove that every selective ultrafilter is semiselective, and every semiselective is a rapid ultrafilter. The existence of these sorts of ultrafilters is independent from the axioms of ZFC; the reader is referred to [Bo], [Ku], [L] and [M].

The *Rudin-Keisler (pre)-order* in ω^* is defined as follows: for $p, q \in \omega^*$, $p \leq_{\mathrm{RK}} q$ if there is $f : \omega \to \omega$ such that $\overline{f}(q) = p$. If $p \leq_{\mathrm{RK}} q$ and $q \leq_{\mathrm{RK}} p$, then we say that p and q are RK-equivalent (in symbols, $p \simeq_{\mathrm{RK}} q$). It is not difficult to verify that $p \simeq_{\mathrm{RK}} q$ iff there is a permutation σ of ω such that $\overline{\sigma}(q) = p$. The *type* of $p \in \omega^*$ is the set $T(p)$ of all RK-equivalent ultrafilters of p. Observe that the Rudin-Keisler pre-order in ω^* is an order in $\{T(p) : p \in \omega^*\}$. Kunen showed (see [CN, 9.6]) that selective ultrafilters on ω^* are precisely the RK-minimal points of ω^*.

For $p \in \omega^*$, the *p-sum* of a sequence $\{p_n : n < \omega\}$ of free ultrafilters on ω, studied by Frolík ([F]) and, in a more general context, by Vopěnka ([V]) and Katětov ([K]), is the ultrafilter

$$\Sigma_p p_n = \{A \subset \omega \times \omega : \{n < \omega : \{m < \omega : (n, m) \in A\} \in p_n\} \in p\}.$$

Throughout this paper, $\Sigma_p p_n$ will be viewed either as an ultrafilter on ω via a bijection between $\omega \times \omega$ and ω, or as an ultrafilter on $\omega \times \omega$. If $p, q \in \omega^*$ and $p_n = q$ for every $n < \omega$, then $\Sigma_p p_n$ is the usual tensor product $p \otimes q$. Booth [Bo] showed that the induced product $T(p) \otimes T(q) = T(p \otimes q)$, for $p, q \in \omega^*$, produces a semigroup structure in the set of types in ω^*.

For each $\nu \in \omega_1$, we choose an increasing sequence $(\nu(n))_{n<\omega}$ of ordinals in ω_1 such that

(a) if $2 \leq \nu < \omega$, $\nu(n) = \nu - 1$;
(b) $\omega(n) = n$ for every $n < \omega$;
(c) if ν is a limit ordinal, then $\nu(n) \nearrow \nu$;
(d) if $\nu = \mu + m$ where μ is a limit ordinal and $m < \omega$, then $\nu(n) = \mu(n) + m$ for each $n < \omega$.

For each $p \in \omega^*$, we can define the right powers and the left powers of $T(p)$ as follows (see [Bo] and [GT 2]):

$$T(p)^2 = T(p) \otimes T(p) = {}^2T(p);$$

If $T(p)^\lambda$ and ${}^\lambda T(p)$ have already been defined for every $\lambda < \nu < \omega_1$, then

$$T(p)^\nu = T(p)^\mu \otimes T(p) \quad \text{and} \quad {}^\mu T(p) = T(p) \otimes {}^\mu T(p)$$

for $\nu = \mu + 1$; and, if ν is a limit ordinal,

$$T(p)^\nu = T(\overline{f}(p)) \text{ and } {}^\nu T(p) = T(\overline{g}(p)),$$

where $f, g \in \omega \to \omega^*$ are embeddings defined in such a way that $f(n) \in T(p)^{\nu(n)}$ and $g(n) \in {}^{\nu(n)}T(p)$ for $n < \omega$.

Observe that, by the associativity of $\otimes$, ${}^nT(p) = T(p)^n$ for every $n < \omega$, and therefore ${}^\omega T(p) = T(p)^\omega$. On the other hand, it is proved in [Bo, Corollary 2.23] that $T(p)^{\omega+1} <_{\mathrm{RK}} {}^{\omega+1}T(p)$.

If $0 < \nu < \omega_1$ and $p \in \omega^*$, then p^ν and ${}^\nu p$ stand for arbitrary points in $T(p)^\nu$ and ${}^\nu T(p)$, respectively.

The basic properties of these products are the following:

0.1. (1) If $p \in \omega^*$ and $o < \mu < \nu < \omega_1$, then $p^\mu <_{\mathrm{RK}} p^\nu$ ([Bo]) and ${}^\mu p <_{\mathrm{RK}} {}^\nu p$ ([GT 2]).

(2) For each $0 < \mu < \omega_1$ there are $\theta, \tau < \omega_1$ such that $p^\mu \leq_{\mathrm{RK}} {}^\theta p$ and ${}^\mu p \leq_{\mathrm{RK}} p^\tau$ ([GT 2]).

0.2 Notation. For $\emptyset \neq M \subset \omega$, we set $L(M) = \{{}^\lambda p : \lambda < \omega_1,\ p \in M\}$ and $R(M) = \{p^\lambda : \lambda < \omega_1,\ p \in M\}$.

Bernstein introduced in [B] the notion of p-limit of a sequence for $p \in \omega^*$: Let $(x_n)_{n<\omega}$ be a sequence in X. Then $x \in X$ is a *p-limit* point of $(x_n)_{n<\omega}$ (in symbols, $x = p\text{-}\lim x_n$ or $x = p\text{-}\lim_{n\to\infty} x_n$ if it is necessary to emphasize what the indexes of the sequence are being considered) if for each $V \in \mathcal{N}(x)$, $\{n < \omega : x_n \in V\} \in p$. This notion suggests the following definitions which are natural generalizations of the concepts of sequentiality and Fréchet-Urysohn.

0.3 Definitions. Let $\emptyset \neq M \subset \omega^*$ and let X be a space.

(1) (Kombarov [Km]) X is said to be *weakly M-sequential* if for each non-closed subset A of X there are a sequence $(x_n)_{n<\omega}$ in A, $p \in M$ and $x \in X \setminus A$ such that $x = p\text{-}\lim x_n$.

(2) (Kombarov [Km]) X is said to be *strongly M-sequential* if for each non-closed subset A of X there are a sequence $(x_n)_{n<\omega}$ in A and $x \in X \setminus A$ such that $x = p\text{-}\lim x_n$ for all $p \in M$.

(3) (Kocinac [Ko]) X is a WFU(M)-*space* if for every $A \subset X$ and $x \in \operatorname{cl} A$, there are $p \in M$ and a sequence $(x_n)_{n<\omega}$ in A such that $x = p\text{-}\lim x_n$.

(4) (Kocinac [Ko]) X is a SFU(M)-*space* if for $A \subset X$ and $x \in \operatorname{cl} A$, there is a sequence $(x_n)_{n<\omega}$ in A such that $x = p\text{-}\lim x_n$ for every $p \in M$.

Observe that, for $p \in \omega^*$, weakly $\{p\}$-sequential $=$ strongly $\{p\}$-sequential and WFU$(\{p\})$-space $=$ SFU$(\{p\})$-space; in this case, we simply say p-sequential and FU(p)-space, respectively (the concept of FU(p)-space was discovered by Comfort and Savchenko independently). We remark that for a space X we have: (a) X is strongly ω^*-sequential if and only if X is sequential; (b) X is weakly ω^*-sequential if and only if X has countable tightness if and only if X is a WFU(ω^*)-space; and (c) X is a SFU(ω^*)-space if and only if X is Fréchet-Urysohn.

Moreover, if $p \in M \subset \omega^*$, then

$$\begin{array}{ccccc}
\mathrm{SFU}(M)\text{-space} & \Longrightarrow & \mathrm{FU}(p)\text{-space} & \Longrightarrow & \mathrm{WFU}(M)\text{-space} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\text{strong } M\text{-sequentiality} & \Longrightarrow & p\text{-sequentiality} & \Longrightarrow & \text{weak } M\text{-sequentiality}
\end{array}$$

Next we give examples to show that the arrows cannot be reversed.

Let $p, q \in \omega^*$ be such that $p <_{\mathrm{RK}} q$, then $\xi(q)$ is a $\mathrm{FU}(q)$-space and is not p-sequential (see [GF 2]); the sequential space S_2 (see [AF]) is not Fréchet-Urysohn, and its p-version $S_2(p)$ (see [GF 2]) is p-sequential and is not a $\mathrm{FU}(p)$-space.

If $p \leq_{\mathrm{RK}} q$, then every p-limit point is also a q-limit point, as it is stated in the next lemma.

0.4 Lemma. *Let $(x_n)_{n<\omega}$ be a sequence in a space X such that $p\text{-}\lim x_n = x \in X$. If $f : \omega \to \omega$ is a function such that $\overline{f}(q) = p$, then $x = q - \lim_{n\to\infty} x_{f(n)}$.*

As a consequence of the previous lemma and 0.1, we have that if $\emptyset \neq M \subset \omega^*$, then

(a) weak $R(M)$-sequentiality $\Leftrightarrow$ weak $L(M)$-sequentiality;
(b) strong $R(M)$-sequentiality $\Leftrightarrow$ strong $L(M)$-sequentiality;
(c) $\mathrm{WFU}(R(M))$-space $\Leftrightarrow$ $\mathrm{WFU}(L(M))$-space;
(d) $\mathrm{SFU}(R(M))$-space $\Leftrightarrow$ $\mathrm{SFU}(L(M))$-space.

We observe next that strong M-sequentiality and weak M-sequentiality imply $\mathrm{SFU}(N)$-space and $\mathrm{WFU}(N)$-space for a suitable $N \subset \omega^*$, respectively. The proof of our result resembles the one given in Theorems 3.6 and 3.8 in [GT 2]. We recall that a space X is *p-compact*, where $p \in \omega^*$, if every sequence in X has a p-limit point in X.

0.5 Theorem. *For $\emptyset \neq M \subset \omega^*$, we have that every weakly M-sequential space is a* $\mathrm{WFU}(L(M))$*-space. In addition, if X is p-compact for every $p \in M$, then X is a* $\mathrm{WFU}(L(M))$*-space if and only if X is weakly M-sequential.*

If $p \in \omega^*$, then $\xi(p \otimes p)$ is a $\mathrm{WFU}(L(p))$-space, but it is not p-sequential. Also, as pointed out by the referee, $\xi(p \otimes p)$ is an example of a p^2-sequential non $\mathrm{SFU}(L(p^2))$-space.

Theorem 0.5 leads us to ask:

0.6 Problem. For $\emptyset \neq M \subset \omega^*$ and X a space, if $C_\pi(X)$ is a $\mathrm{WFU}(L(M))$-space (resp. $\mathrm{SFU}(L(M))$-space) must $C_\pi(X)$ be weakly (resp. strongly) M-sequential?

0.7 Definitions. Let $\emptyset \neq M \subset \omega^*$.

(a) A space Y is a *strictly* $\mathrm{WFU}(M)$*-space* if for every sequence $(F_n)_{n<\omega}$ of subsets of Y and every $y \in \bigcap_{n<\omega} \mathrm{cl}_Y F_n$, there exist $y_n \in F_n$ for each $n < \omega$, and $p \in M$ such that $y = p\text{-}\lim y_n$.

(b) A space Y is a *strictly* $\mathrm{SFU}(M)$*-space* if for every sequence $(F_n)_{n<\omega}$ of subsets of Y and every $y \in \bigcap_{n<\omega} \mathrm{cl}_Y F_n$, there exists $y_n \in F_n$ for each $n < \omega$ such that $y = p\text{-}\lim y_n$ for every $p \in M$.

(c) ([S]) A space Y has *countable strong fan tightness* if for every $y \in Y$ and every sequence $(A_n)_{n<\omega}$ of subsets of Y such that $y \in \bigcap_{n<\omega} \operatorname{cl} A_n$, there exists $y_n \in A_n$ for each $n < \omega$, such that $y \in \operatorname{cl}\{y_n : n < \omega\}$.

Note that, for $p \in \omega^*$, a space Y is a strictly WFU($\{p\}$)-space $\Leftrightarrow$ Y is a strictly SFU($\{p\}$)-space; so, in this case, we say that Y is a *strictly* FU(p)-*space*.

Observe also that: (a) Y is a strictly SFU(ω^*)-space $\Leftrightarrow$ Y is strictly Fréchet-Urysohn; (b) Y is strictly WFU(ω^*)-space $\Leftrightarrow$ Y has countable strong fan tightness; and (c) for $p \in M \subseteq \omega^*$, Y is a strictly SFU(M)-space $\Rightarrow$ Y is a strictly FU(p)-space $\Rightarrow$ Y is a strictly WFU(M)-space $\Rightarrow$ Y has countable strong fan tightness.

A collection $\mathcal{G}$ of subsets of a space X is an ω-*cover* of X if for every finite subset F of X there exists $G \in \mathcal{G}$ such that $F \subset G$.

The following definition was introduced in [GN 2].

A space X is said to have property γ if for every open ω-cover $\mathcal{G}$ of X, there is a sequence $(G_n)_{n<\omega}$ in $\mathcal{G}$ such that

$$X = \lim G_n = \bigcup_{n<\omega} \bigcap_{n<m} G_m.$$

It is shown in [GN 2] and [G] that the following statements are equivalent:

(a) $C_\pi(X)$ is a Fréchet-Urysohn space.
(b) $C_\pi(X)$ is a strictly Fréchet-Urysohn space .
(c) $C_\pi(X)$ is sequential.
(d) X satisfies γ.

The equivalence (a) $\Leftrightarrow$ (b) was proved by Pytkeev ([Py]) as well, and Nyikos ([N]) showed that every Fréchet-Urysohn topological group is strictly Fréchet-Urysohn.

In this paper, we principally study the p-sequentiality and the FU(p)-property on function spaces, and their effects on the base space. This leads us to consider the following generalization of property γ.

0.8 Definition ([GT 1]). X satisfies property γ_p if for each open ω-cover $\mathcal{G}$ of X there is a sequence $(G_n)_{n<\omega}$ in $\mathcal{G}$ such that $X = \lim_p G_n$, where $\lim_p G_n = \bigcup_{A\in p} \bigcap_{n\in A} G_n$.

It is natural to ask whether the p-limit version of the statements (a), (b), (c) and (d), quoted above, are also equivalent for each $p \in \omega^*$. The authors proved in [GT 1] the following.

0.9 Theorem. *Let X be a space and let $p \in \omega^*$. $C_\pi(X)$ is a* FU(p)*-space if and only if X satisfies γ_p.*

Unfortunately, the next problem remains unsolved.

0.10 Problem ([GT 1]). If $C_\pi(X)$ is p-sequential, must X satisfy γ_p?

The equivalence between the p-Fréchet-Urysohn property and strictly p-Fréchet-Urysohn property does not hold, in general, on function spaces. In fact, we showed in [GT 1] that $C_\pi(\mathbb{R})$ is a FU(q)-space for some $q \in \omega^*$, meanwhile $C_\pi(\mathbb{R})$ cannot be a strictly FU(p)-space for all $p \in \omega^*$ (see Corollary 2.5 below).

Before finishing this section we will give some results concerning limit of sets, sum of ultrafilters and RK-order which will be fundamental for our purposes; their proofs are not difficult, so we have omitted them. If q is an ultrafilter on $\omega \times \omega$ and $(G_{n,m})_{n,m}$ is a bisequence of sets, then the symbol $\lim_q G_{n,m}$ has a clear meaning (see 0.8).

0.11 Lemma. *Let $p \in \omega^*$, $\{q_n\}_{n<\omega} \subset \omega^*$ and let $(G_{n,m})_{n,m<\omega}$ be a bisequence of subsets of a space X. Then*

$$\lim_{\Sigma_p q_n} G_{n,m} = \lim_{n\to\infty} {}_p \, (\lim_{m\to\infty} {}_{q_n} G_{n,m}).$$

(Observe that the symbol $\lim_{\Sigma_p q_n} G_{n,m}$ makes sense because $\Sigma_p q_n$ can be considered as an ultrafilter on $\omega \times \omega$.)

0.12 Lemma. *Let $p, q \in \omega^*$ and let $(G_n)_{n<\omega}$ be a sequence of subsets of a space X. If $f : \omega \to \omega$ is a function satisfying $\overline{f}(q) = p$, then $\lim_p G_n = \lim_{n\to\infty} {}_q G_{f(n)}$.*

In the next section of this article we give some necessary and sufficient conditions on a space X in order that its function space $C_\pi(X)$ be either WFU(M)-spaces or SFU(M)-spaces, for $\emptyset \neq M \subset \omega^*$. We also prove that $C_\pi(\mathbb{R})$ is not p-sequential if $p \in \omega^*$ is selective. In Section 2 we study the strictly WFU(M) and strictly SFU(M)- function spaces, and we characterize the semiselective ultrafilters in terms of these properties. In the last section we study the product of spaces having one of the cover-properties analyzed in this work.

1. Weakly and strongly Fréchet-Urysohn function spaces and rapid ultrafilters

We are going to characterize the WFU(M) and SFU(M)-properties on a function space $C_\pi(X)$ in terms of cover-properties in X.

1.1 Definitions. Let $\emptyset \neq M \subset \omega^*$.

(1) A space X satisfies property $W\gamma_M$ if for each open ω-cover $\mathcal{G}$ of X, there exist a sequence $(G_n)_{n<\omega}$ in $\mathcal{G}$ and $p \in M$ such that $\lim_p G_n = X$.

(2) A space X satisfies property $S\gamma_M$ if for each open ω-cover $\mathcal{G}$ of X, there is a sequence $(G_n)_{n<\omega}$ such that $\lim_p G_n = X$ for every $p \in M$.

(3) A space X satisfies property $W\Gamma_M$ if X has ε and for each sequence $(\mathcal{G}_n)_{n<\omega}$, where $\mathcal{G}_n = \{G^n_k : k < \omega\}$ for $n < \omega$, of open ω-covers of X satisfying $G^{n+1}_k \subset G^n_k$ for each $n, k < \omega$, there exists $p \in M$ and a sequence $(k_m)_{m<\omega}$ of positive integers such that $X = \lim_{m\to\infty} {}_p G^n_{k_m}$ for each $n < \omega$.

(4) A space X satisfies property $S\Gamma_M$ if X has ε and for each sequence $(\mathcal{G}_n)_{n<\omega}$, where $\mathcal{G}_n = \{G_k^n : k < \omega\}$ for $n < \omega$, of open ω-covers of X satisfying $G_k^{n+1} \subset G_k^n$ for each $n, k < \omega$, there is a sequence $(k_m)_{m<\omega}$ of positive integers such that, for every $p \in M$, $X = \lim_{m\to\infty} {}_p G_{k_m}^n$ for each $n < \omega$.

If $p \in \omega^*$, then we denote by Γ_p the equivalent properties $W\Gamma_{\{p\}}$ and $S\Gamma_{\{p\}}$.

We recall that a space X satisfies property ε if for each open ω-cover $\mathcal{G}$ of X we can find a countable ω-subcover. It is shown in [GN 2] and [Ar 3] that X has $\varepsilon \Leftrightarrow X^n$ is Lindelöf for every $n < \omega \Leftrightarrow C_\pi(X)$ has countable tightness.

Observe that, if $p \in M \subset \omega^*$, then $S\gamma_M \Rightarrow \gamma_p \Rightarrow W\gamma_M \Rightarrow \varepsilon$ and $S\gamma_M \Rightarrow \Gamma_p \Rightarrow W\Gamma_M \Rightarrow \varepsilon$.

In [GT 1] we proved the following four lemmas which will be useful.

1.2 Lemma. *Let $\emptyset \neq \Phi \subset C_\pi(X)$ and $f \in \operatorname{cl}\Phi$, and let $\varrho > 0$. Then $\mathcal{G} = \{\operatorname{coz}_\varrho(g - f) : g \in \phi\}$ is an open ω-cover of X.*

1.3 Lemma. *Let $f \in C_\pi(X)$, let $\varrho > 0$ and let $\mathcal{H}$ be an open ω-cover of X with $X \notin \mathcal{H}$. If $\Phi = \{g \in C_\pi(X) : \operatorname{coz}_\varrho(g-f) \subset H$ for some $H \in \mathcal{H}\}$, then $f \in \operatorname{cl}\Phi \setminus \Phi$.*

1.4 Lemma. *Let $p \in \omega^*$, let X be a space and let $f, f_0, \dots, f_n, \dots \in C_\pi(X)$. Then $f = p\text{-}\lim f_n$ if and only if $\lim_p \operatorname{coz}_\varrho(f_n - f) = X$ for every $\varrho > 0$.*

1.5 Lemma. *Let p, X, $f, f_0, \dots, f_n, \dots$ as in the previous lemma. Let $(\varrho_n)_{n<\omega}$ be a strictly decreasing sequence of positive real numbers converging to 0. If $X = \lim_p \operatorname{coz}_{\varrho_n}(f_n - f)$, then $f = p\text{-}\lim f_n$.*

1.6 Theorem. *Let X be a space and let $\emptyset \neq M \subset \omega^*$. Then the following statements are equivalent.*

(a) *$C_\pi(X)$ is a WFU(M)-space.*
(b) *X satisfies $W\gamma_M$.*
(c) *X satisfies $W\Gamma_M$.*

Proof: (a) $\Rightarrow$ (b). Let $\mathcal{G}$ be an open ω-cover of X. If $X \in \mathcal{G}$, we take $G_n = X$ for every $n < \omega$ and, obviously, $\lim_p G_n = X$ for every $p \in M$. Now we assume that $X \notin \mathcal{G}$, so $0 \in \operatorname{cl}\Phi \setminus \Phi$ where $\Phi = \{f \in C_\pi(X) : \operatorname{coz}_1 f \subset G$ for some $G \in \mathcal{G}\}$ (Lemma 1.3). By hypothesis, there is $p \in M$ and $(f_n)_{n<\omega} \subset \Phi$ such that $\lim_p f_n = 0$. It follows from Lemma 1.4 that $\lim_p \operatorname{coz}_1 f_n = X$. If, for each $n < \omega$, $G_n \in \mathcal{G}$ satisfies that $\operatorname{coz}_1 f_n \subset G_n$, then $\lim_p G_n = X$.

(b) $\Rightarrow$ (c). We know that $W\gamma_M$ implies ε. Let $\mathcal{G}_n = \{G_k^n : k < \omega\}$ be an open ω-cover of X, for each $n < \omega$, such that $G_k^{n+1} \subset G_k^n$ for each $n, k < \omega$. We may assume that X is infinite and hence choose $\{x_n : n < \omega\}$ an infinite subset of X. For each $n < \omega$, define $\mathcal{F}_n = \{G_k^n \setminus \{x_n\} : k < \omega\}$ and put $\mathcal{F} = \bigcup_{n<\omega} \mathcal{F}_n$. It is not hard to prove that $\mathcal{F}$ is an open ω-cover of X as well. By assumption, there is a sequence $(F_j)_{j<\omega}$ in $\mathcal{F}$ and $p \in M$ for which $X = \lim_p F_j$. For each $j < \omega$, we have that $F_j = G_{k_j}^{n_j} \setminus \{x_{n_j}\}$ for some $n_j < \omega$. We claim that $X = \lim_{j\to\infty} {}_p G_{k_j}^n$ for each $n < \omega$. In fact, fix $n < \omega$. First, observe that $\{j < \omega : n \leq n_j\} \in p$.

Otherwise, there would be $s < n$ for which $B = \{j < \omega : n_j = s\} \in p$ and so $X = \bigcup_{j\in B} F_j \subset \bigcup \mathcal{F}_s$, a contradiction. If $x \in X$, then $\{j < \omega : x \in F_j\} \in p$ and hence $\{j < \omega : n \leq n_j, x \in G^{n_j}_{k_j}\} \in p$. Since $G^{n_j}_{k_j} \subset G^{n}_{k_j}$ for $n \leq n_j$, $\{j < \omega : x \in G^{n}_{k_j}\} \in p$. This proves our claim.

(c) $\Rightarrow$ (a). Let $\Phi \subset C_\pi(X)$ such that $0 \in \operatorname{cl}\Phi \setminus \Phi$. Since X satisfies ε, $C_\pi(X)$ has countable tightness, so we can find $\Psi = \{f_n : n < \omega\} \subset \Phi$ such that $0 \in \operatorname{cl}\Psi$. We take, for each $k < \omega$, $\mathcal{G}_k = \{\operatorname{coz}_{1/2^k} f_n : n < \omega\}$. According to Lemma 1.2, each $\mathcal{G}_k$ is an open ω-cover of X. Besides, $\operatorname{coz}_{1/2^{k+1}} f_n \subset \operatorname{coz}_{1/2^k} f_n$ for every $k, n < \omega$. Thus, by hypothesis, there is $p \in M$ and a sequence $(n_s)_{s<\omega}$ in ω such that $\lim_{s\to\infty} {}_p \operatorname{coz}_{1/2^k} f_{n_s} = X$ for every $k < \omega$. Therefore, $0 = p\text{-}\lim_{s\to\infty} f_{n_s}$ (by Lemma 1.4). $\square$

Analogously, we can prove the strong version of the previous theorem:

1.7 Theorem. *Let X be a space and let $\emptyset \neq M \subset \omega^*$. Then the following statements are equivalent.*

(a) *$C_\pi(X)$ is a $\mathrm{SFU}(M)$-space.*
(b) *X satisfies $S\gamma_M$.*
(c) *X satisfies $S\Gamma_M$.*

A natural cover-property that is closely related to p-sequentiality in function spaces is the p-version δ_p of property δ (δ was introduced and studied in [GN$_2$]). In order to define δ_p we need some notation:

Let X be a space, $\mathcal{G} \subset \mathcal{P}(X)$ and $p \in \omega^*$. We proceed by induction: Set $S(p, \mathcal{G}, 0) = \mathcal{G}$ and if $S(p, \mathcal{G}, \lambda)$ has been defined for all $\lambda < \mu < \omega_1$, we put $S(p, \mathcal{G}, \mu) = \{\lim_p G_n : (G_n)_{n<\omega}$ is a sequence in $\bigcup_{\lambda<\mu} S(p, \mathcal{G}, \lambda)\}$. Finally, we define $S_p(\mathcal{G}) = \bigcup_{\lambda<\omega_1} S(p, \mathcal{G}, \lambda)$.

1.8 Definition. Let $\emptyset \neq M \subset \omega^*$. A space X satisfies property δ_M if for every open ω-cover $\mathcal{G}$, X belongs to $S_p(\mathcal{G})$ for some $p \in M$.

We will write δ_p instead of $\delta_{\{p\}}$ for each $p \in \omega^*$.

We will prove that δ_M is the translation in X of $\mathrm{WFU}(L(M))$-property in $C_\pi(X)$, where $L(M)$ is the set defined in 0.2. First we will give some lemmas.

1.9 Lemma. (a) *If $\lambda < \mu < \omega_1$, then $S(p, \mathcal{G}, \lambda) \subset S(p, \mathcal{G}, \mu)$;*
(b) *if $p \leq_{\mathrm{RK}} q$, then $S(p, \mathcal{G}, \lambda) \subset S(q, \mathcal{G}, \lambda)$ for every $\lambda < \omega_1$;*
(c) *if $\gamma < \mu < \omega_1$, then $S({}^\gamma p, \mathcal{G}, \lambda) \subset S({}^\mu p, \mathcal{G}, \lambda)$.*

Proof: (a) is trivial and (c) is a consequence of (b) and 0.1 (2).

The proof of (b) is by induction. The containment $S(p, \mathcal{G}, 1) \subset S(q, \mathcal{G}, 1)$ follows from Lemma 0.12. Suppose now that $S(p, \mathcal{G}, \lambda) \subset S(q, \mathcal{G}, \lambda)$ for all $\lambda < \mu$. Thus, $S(p, \mathcal{G}, \mu) = \{\lim_p G_n : (G_n)_{n<\omega} \subset \bigcup_{\lambda<\mu} S(p, \mathcal{G}, \lambda)\} \subset \{\lim_p G_n : (G_n)_{n<\omega} \subset \bigcup_{\lambda<\omega} S(q, \mathcal{G}, \lambda)\}$. It follows from Lemma 0.12 that $S(p, \mathcal{G}, \mu) \subset \{\lim_q G_n : (G_n)_{n<\omega} \subset \bigcup_{\lambda<\mu} S(q, \mathcal{G}, \lambda)\} = S(q, \mathcal{G}, \mu)$. $\square$

1.10 Lemma. *Let X be a space, $\mathcal{A} \subset \mathcal{P}(X)$ and $p \in \omega^*$. Then*

(a) *for every $\nu < \omega_1$, $S(p, \mathcal{A}, \nu) \subset S({}^{\nu+1}p, \mathcal{A}, 1)$;*
(b) *for every $\nu < \omega_1$, $S({}^{\nu}p, \mathcal{A}, 1) \subset S(p, \mathcal{A}, \nu)$.*

Proof: We proceed by transfinite recursion.

(a) The statement is true when $\nu = 1$ because of Lemma 1.9. Suppose that, for every $\nu < \mu$, $S(p, \mathcal{A}, \nu) \subset S({}^{\nu+1}p, \mathcal{A}, 1)$. Let $G \in S(p, \mathcal{A}, \mu)$; so $G = \lim_p G_n$ with $(G_n)_{n<\omega}$ a sequence in $\bigcup_{\nu<\mu} S(p, \mathcal{A}, \nu)$. Thus, we obtain that $\{G_n : n < \omega\} \subset S({}^{\mu}p, \mathcal{A}, 1)$ (by Lemma 1.9). So, for each $n < \omega$, there is a sequence $(G_{n,m})_{n,m<\omega}$ in $\mathcal{A}$ such that $G_n = \lim\limits_{m\to\infty} {}_{{}^{\mu}p}(G_{n,m})$. Because of Lemma 0.11, we must have $G = \lim\limits_{n\to\infty} {}_p(\lim\limits_{m\to\infty} {}_{{}^{\mu}p} G_{n,m}) = \lim_{{}^{\mu+1}p} G_{n,m}$. Thus, $G \in S({}^{\mu+1}p, \mathcal{A}, 1)$.

(b) This is evidently true when $\nu = 1$. Suppose that, for every $\nu < \mu$, $S({}^{\nu}p, \mathcal{A}, 1) \subset S(p, \mathcal{A}, \nu)$. Let $G \in S({}^{\mu}p, \mathcal{A}, 1)$, so $G = \lim_{{}^{\mu}p} G_n$ for some sequence $(G_n)_{n<\omega}$ in $\mathcal{A}$. If $\mu = \lambda + 1$, then there is a bisequence $(G_{n,m})_{n,m<\omega} \subset \mathcal{A}$ such that $G = \lim_{{}^{\lambda+1}p} G_n = \lim\limits_{n\to\infty} {}_p(\lim\limits_{m\to\infty} {}_{{}^{\lambda}p} G_{n,m})$ (Lemmas 0.11 and 0.12). By induction hypothesis $\lim\limits_{m\to\infty} {}_{{}^{\lambda}p} G_{n,m} \in S(p, \mathcal{A}, \lambda)$ for each $n < \omega$. Therefore, $G \in S(p, \mathcal{A}, \mu)$.

If μ is a limit ordinal, then ${}^{\mu}p \simeq_{\mathrm{RK}} \Sigma_p {}^{\mu(n)}p$ and so, using the results in 0.11 and 0.12 again, we obtain that $G = \lim_{{}^{\mu}p} G_n = \lim_{\Sigma_p {}^{\mu(n)}p} G_n = \lim\limits_{n\to\infty} {}_p \lim\limits_{m\to\infty} {}_{{}^{\mu(n)}p} G_{n,m}$ for some bisequence $(G_{n,m})_{n,m<\omega}$ in $\mathcal{A}$. But $\lim\limits_{m\to\infty} {}_{{}^{\mu(n)}p} G_{n,m} \in \bigcup_{\nu<\mu} S(p, \mathcal{A}, \nu)$ for every $n < \omega$. Hence, $G \in S(p, \mathcal{A}, \mu)$. □

To prove the following theorem we only need to observe that a space X satisfies $W\gamma_{L(M)}$, for $\emptyset \neq M \subset \omega^*$, if and only if for every open ω-cover $\mathcal{G}$ of X, $X \in \bigcup_{\nu<\omega_1} S({}^{\nu}p, 1, \mathcal{G})$ for some $p \in M$, and apply the previous lemma.

1.11 Theorem. *Let $\emptyset \neq M \subset \omega^*$. A space X satisfies δ_M if and only if X satisfies $W\gamma_{L(M)}$ (iff $C_\pi(X)$ is a* WFU$(L(M))$*-space).*

The following lemma follows from Lemma 0.12.

1.12 Lemma. *Let $\emptyset \neq M$, $N \subset \omega^*$. If for each $p \in M$ there is $q \in N$ such that $p \leq_{\mathrm{RK}} q$, then every space having δ_M has δ_N, and every space having $W\gamma_M$ satisfies $W\gamma_N$.*

From 0.1, 1.11 and 1.12 it follows that:

1.13 Corollary. *Let $p \in \omega^*$ and $\mu, \nu < \omega_1$. All the properties δ_p, δ_{p^ν}, $\delta_{{}^{\mu}p}$, $W\gamma_{L(p)}$, $W\gamma_{L(p^\nu)}$, $W\gamma_{L({}^{\mu}p)}$, $W\gamma_{R(p)}$, $W\gamma_{R(p^\nu)}$ and $W\gamma_{R({}^{\mu}p)}$ are equivalent.*

Now we are going to study the properties $W\gamma_M$ on $\mathbb{R}$. In particular we will give some results related to question 0.10.

1.14 Definition. Let Y be a p-sequential space. $\sigma_p(Y)$ will denote *the degree of p-sequentiality of Y* which is defined to be the least ordinal $\mu \leq \omega_1$ such that for every $A \subset Y$, $\mathrm{cl}_Y A = \bigcup_{\lambda<\mu} A(p,\lambda)$, where

$$A(p,0) = A, \quad \text{and}$$

$$A(p,\lambda) = \{y \in Y : \exists\, (y_n)_{n<\omega} \subset \bigcup_{\eta<\lambda} A(p,\eta)\ (y = p\text{-}\lim y_n)\},$$

for $\lambda \leq \omega_1$.

Observe that $\sigma_p(Y)$ is defined iff Y is p-sequential.

As a consequence of Theorem 3.6 in [GT 2] we obtain:

1.15 Theorem. *Let $p \in \omega^*$. If $\sigma_p(C_\pi(X)) = \mu < \omega_1$, then $C_\pi(X)$ is a $\mathrm{FU}({}^{\mu+1}p)$-space.*

1.16 Corollary. *Let $p \in \omega^*$. If $\sigma_p(C_\pi(X)) = \mu < \omega_1$, then X satisfies $\gamma_{{}^{\mu+1}p}$.*

The next result is a generalization of, and its proof is similar to, Lemma 3.15 in [GT 1].

1.17 Lemma. *Let $p \in \omega^*$ be selective. Then, for $\lambda < \omega_1$, the* RK*-predecessors of ${}^{\lambda}p$ are rapid.*

The previous results and Theorem 3.6 in [GT 1] imply, in particular, that $C_\pi(\mathbb{R})$ does not have a degree of p-sequentiality $< \omega_1$ if p is selective. We will obtain a stronger result in 1.20: if p is selective, then $C_\pi(\mathbb{R})$ is not p-sequential.

The following theorem can be proved in a similar way as Theorem 3.6 in [GT 1].

1.18 Theorem. *Let $M \subset \omega^*$ such that every* RK*-predecessor of any element of M is rapid. If $X \subset \mathbb{R}$ satisfies $W\gamma_M$, then X has measure zero.*

Because of 1.11, 1.17 and 1.18 we obtain

1.19 Theorem. *Let $p \in \omega^*$ be selective and $X \subset \mathbb{R}$. If X has δ_p, then X has measure zero.*

1.20 Corollary. *Let $p \in \omega^*$ be selective and let $X \subset \mathbb{R}$. If $C_\pi(X)$ is p-sequential, then X has measure zero. In particular, $C_\pi(\mathbb{R})$ is not p-sequential.*

2. Strictly Fréchet-Urysohn (M)-function spaces and semiselective ultrafilters

2.1 Definitions. Let $\emptyset \neq M \subset \omega^*$.

(a) A space X has *strictly $W\gamma_M$* if for each sequence $(\mathcal{G}_n)_{n<\omega}$ of open ω-covers of X, there are $p \in M$ and, for each $n < \omega$, $G_n \in \mathcal{G}_n$ such that $X = \lim_p G_n$.

(b) A space X has *strictly $S\gamma_M$* if for each sequence $(\mathcal{G}_n)_{n<\omega}$ of open ω-covers of X, there is $G_n \in \mathcal{G}_n$ for each $n < \omega$ such that $X = \lim_p G_n$ for every $p \in M$.

In particular, we say that a space X has *strictly* γ_p if X has strictly $W\gamma_{\{p\}}$ (or, equivalently, if X has strictly $S\gamma_{\{p\}}$).

With a slight modification in the proof of the equivalence (a) $\Leftrightarrow$ (b) of Theorem 1.6, we can show the following result.

2.2 Theorem ([GT 1]). *Let X be a space and $p \in \omega^*$. $C_\pi(X)$ is a strictly* FU(p)*-space if and only if X has strictly γ_p.*

By applying the same techniques used so far, it is not difficult to prove the following generalization of Theorem 2.2.

2.3 Theorem. *Let X be a space, and let $\emptyset \neq M \subset \omega^*$. Then*

(a) *$C_\pi(X)$ is a strictly* WFU(M)*-space if and only if X has strictly $W\gamma_M$.*

(b) *$C_\pi(X)$ is a strictly* SFU(M)*-space if and only if X has strictly $S\gamma_M$.*

We recall that a space X has property C'' provided that for each sequence $(\mathcal{G}_n)_{n<\omega}$ of open covers of X, there is a sequence $(G_n)_{n<\omega}$, with $G_n \in \mathcal{G}_n$ and $X = \bigcup_{n<\omega} G_n$. A space Y has *Id-fan tightness* if for every $y \in Y$ and every sequence $(A_n)_{n<\omega}$ of subsets of Y such that $y \in \bigcap_{n<\omega} \operatorname{cl} A_n$, there is $F_n \in [A_n]^{\leq n}$ for each $n < \omega$, such that $y \in \operatorname{cl} \bigcup_{n<\omega} F_n$. This concept was introduced in [GT 3] and the authors pointed out that the following result holds (the equivalence between (a) and (c) was shown by Sakai in [S]).

2.4 Theorem. *For a space X, the following are equivalent.*

(a) *$C_\pi(X)$ has countable strong fan tightness.*

(b) *$C_\pi(X)$ has Id-fan tightness.*

(c) *Each finite product of X has property C''.*

Since every strictly WFU(M)-space, for $\emptyset \neq M \subset \omega^*$, has countable strong fan tightness, then we obtain the following corollary of Theorem 2.4.

2.5 Corollary. *Let $\emptyset \neq M \subset \omega^*$ and let X be a space. If X satisfies strictly $W\gamma_M$, then X^n has C'' for every $n < \omega$. In particular, if X is a subset of $\mathbb{R}$ having strictly $W\gamma_M$, then X has strong measure zero.*

We may also obtain the conclusions in Corollary 2.5 by developing a proof similar to that given in [D, p. 100] of the fact that γ implies Rothberger property C'' (see [GN 2]).

Corollary 2.5 implies that γ_p and strictly $W\gamma_{T(p)}$ are not necessarily equivalent because $\mathbb{R}$ satisfies γ_p for some $p \in \omega^*$ (Theorem 2.15, [GT 1]), but $\mathbb{R}$ does not satisfy strictly $W\gamma_{T(p)}$. Nevertheless, the two concepts coincide when p is selective:

2.6 Theorem. *If $p \in \omega^*$ is selective, then a space X has γ_p if and only if X has strictly $W\gamma_{T(p)}$.*

Proof: It is not difficult to verify that strictly $W\gamma_{T(p)}$ implies property γ_p for every $p \in \omega^*$. Let $p \in \omega^*$ be selective and assume that X has γ_p. Let $(\mathcal{G}_m)_{m<\omega}$

be a sequence of open ω-covers of X. If X is finite, there is nothing to prove, so we suppose that we can take $\{x_n : n < \omega\} \subset X$ such that $x_i \neq x_j$ if $i \neq j$. For each $m < \omega$, we put $\mathcal{F}_m = \{G - \{x_m\} : G \in \mathcal{G}_m\}$. Then $\mathcal{F} = \bigcup_{m<\omega} \mathcal{F}_m$ is an open ω-cover of X. Since X satisfies γ_p, there is a sequence $(F_n)_{n<\omega}$ in $\mathcal{F}$ such that $X = \lim_p F_n$. The sets $A_k = \{n < \omega : F_n \in \mathcal{F}_k \setminus \bigcup_{i<k} \mathcal{F}_i\}$, for $k < \omega$, constitute a partition of ω. Besides, $A_k \notin q$ for every $k < \omega$ because, on the contrary, if $A_k \in q$ for some $k < \omega$, then $X = \bigcup_{n \in A_k} F_n$ which is in contradiction with the construction of $\mathcal{F}_k$. Hence, since q is selective, there is $A \in q$ such that $|A \cap A_k| \leq 1$ for every $k < \omega$. By adding points if it is necessary, we may assume that $A \cap A_k = \{n_k\}$ for each $k < \omega$ and hence $A = \{n_k : k < \omega\}$. Thus, the function $f : \omega \to \omega$ defined by $f(i) = n_i$ is injective and $f[\omega] = A \in p$. Let $q \in \omega^*$ satisfying $\overline{f}(q) = p$. Then, by 0.12, $X = \lim_{k\to\infty} {}_q F_{n_k}$. Since f is one-to-one, $q \simeq_{\mathrm{RK}} p$ (see [CN] or [C 1]). Now, for each $k < \omega$ we choose $G_k \in \mathcal{G}_k$ so that $F_{n_k} \subset G_k$. Then $X = \lim_q G_k$ and $q \in T(p)$. This completes our proof. $\square$

2.7 Corollary. *Let $p \in \omega^*$ be selective. Then $C_\pi(X)$ is an* FU(p)*-space if and only if $C_\pi(X)$ is a strictly* WFU$(T(p))$*-space.*

2.8 Problem. Are γ_p and strictly γ_p the same property whenever p is selective?

2.9. In the following diagrams we summarize some of the results that we have already proved in this article and in [GT 1] concerning generalizations of the FU-property in $C_\pi(X)$ and cover-properties in X. The expression $p \leq^*_{\mathrm{RK}} q$, for $p, q \in \omega^*$, will mean that ${}^\lambda p \leq_{\mathrm{RK}} q$ (equivalently, $p^\lambda \leq_{\mathrm{RK}} q$) for every $\lambda < \omega_1$.

$$
\begin{array}{ccc}
C_\pi(X) \text{ is strictly SFU}(T(p))\text{-space} & \Longleftrightarrow & X \text{ has strictly } S\gamma_{T(p)} \\
\Downarrow & & \Downarrow \\
\text{strictly FU}(p)\text{-space} & \Longleftrightarrow & \text{strictly } \gamma_p \\
\Downarrow & & \Downarrow \\
\text{strictly WFU}(T(p))\text{-space} & \Longleftrightarrow & \text{strictly } W\gamma_{T(p)} \\
\Downarrow & & \Downarrow \\
\text{FU}(p)\text{-space} & \Longleftrightarrow & \gamma_p \\
\Downarrow & & \Downarrow \\
p\text{-sequential space} & \Longrightarrow & \delta_p \\
\Downarrow & & \Downarrow \\
\text{WFU}(L(p))\text{-space} & \Longleftrightarrow & W\gamma_{L(p)} \\
\Downarrow & & \Downarrow \\
\text{FU}(q)\text{-space } \forall\, p \leq^*_{\mathrm{RK}} q & \Longleftrightarrow & \gamma_q \forall\, p \leq^*_{\mathrm{RK}} q
\end{array}
$$

Moreover, if $p \in M \subset \omega^*$, then

$$
\begin{array}{ccccccc}
X \text{ has} & & & C_\pi(X) \text{ is} & & & X \text{ has} \\
\text{strictly } S\gamma_M & \Longleftrightarrow & & \text{strictly SFU}(M)\text{-space} & & \Longleftrightarrow & \text{strictly } S\gamma_M \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
S\gamma_M & \Longleftrightarrow & \text{SFU}(M)\text{-space} & & \text{strictly FU}(p) & \Longleftrightarrow & \text{strictly } \gamma_p \\
\Downarrow & & \Downarrow & \Downarrow & \Downarrow & & \Downarrow \\
\gamma_p & \Longleftrightarrow & \text{FU}(p)\text{-space} & & \text{strictly WFU}(M) & \Longleftrightarrow & \text{strictly } W\gamma_M \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
W\gamma_M & \Longleftrightarrow & & \text{WFU}(M)\text{-space} & & \Longleftrightarrow & W\gamma_M \\
\Downarrow & & & & & & \Downarrow \\
\delta_M & & & & & & \delta_M
\end{array}
$$

2.10 Problems. (1) Under what conditions on $p \in \omega^*$, and on $M \subset \omega^*$ (and on $N \subset \omega^*$), does $W\gamma_M$ imply γ_p (resp., $W\gamma_M$ implies $W\gamma_N$)?

(2) Is p-sequentiality a consequence of δ_p if p is selective?

Now we are going to see that semiselective ultrafilters can be characterized in terms of strictly WFU(M)-properties in $\xi(p)$-type spaces, where $p \in \omega^*$ and $\xi(p)$ is the subspace $\omega \cup \{p\}$ of $\beta\omega$.

2.11 Lemma. (a) *Let $\emptyset \neq M \subset \omega^*$ and $p \in \omega^*$. If $\xi(p)$ is a strictly* WFU(M)-*space, then p is semiselective.*

(b) *Let $p, q \in \omega^*$. If $\xi(p)$ is a strictly* FU(q)-*space, then p is semiselective and $p <_{\mathrm{RK}} q$.*

(c) *For every $p \in \omega^*$, $\xi(p)$ is not a strictly* FU(q)-*space for every $q \in T(p)$.*

Proof: (a) Let $\{A_n\}_{n<\omega} \subset p$. For each $n < \omega$ there are $x_n \in A_n$ and $q \in M$ such that $p = q\text{-}\lim x_n$, because $p \in \mathrm{cl}_{\xi(p)} A_n$ for all $n < \omega$. Take $A = \{x_n : n < \omega\}$. If $A \notin p$, then $\omega \setminus A \in p$, and so $\emptyset = \{n < \omega : x_n \in \omega \setminus A\} \in q$, a contradiction. Therefore, $A \in p$.

(b) By (a), p is semiselective. Let $A_n \in p$ such that $n \notin A_n$ for $n < \omega$. Then, there is $x_n \in A_n$, for $n < \omega$, such that $p = q\text{-}\lim x_n$. If $f(n) = x_n$ for each $n < \omega$, then $\overline{f}(q) = p$. Suppose that f is one-to-one. Let B_n be such that $f(n) \notin B_n$ for each $n < \omega$. By assumption, there is $g : \omega \to \omega$ such that $g(n) \in B_n$, for all $n < \omega$, and $\overline{g}(q) = \overline{f}(q)$. By [G-F 1, Lemma 2.1], $\{n < \omega : g(n) = f(n)\} \in p$ which is a contradiction. Thus, $p <_{\mathrm{RK}} q$.

(c) is a consequence of (b). $\square$

2.12 Theorem. *Let $p \in \omega^*$. p is semiselective if and only if $\xi(p)$ is a strictly* WFU$(T(p))$-*space.*

Proof: According to Lemma 2.11 (a), we only have to prove the necessity. Assume that $p \in \omega^*$ is semiselective and let $p \in \bigcap_{n<\omega} \mathrm{cl}_{\xi(p)} A_n$, where $A_n \subset \xi(p)$ for every $n < \omega$. Let $B = \{n < \omega : p \in A_n\}$. We have two cases:

First case: $B \in p$.
In this case, for each $n \in B$ we take $x_n = p$, and for every $n \notin B$ we choose some $x_n \in A_n$. If $W \in p$, $\{n < \omega : x_n \in \widehat{W}\} \supset B \in p$; so $p = p\text{-}\lim x_n$.

Second case: $B \notin p$.
Let $\omega \setminus B = \{n_k : k < \omega\}$ such that $n_i < n_j$ if $i < j$. By assumption, for every $k < \omega$, there is $x_{n_k} \in A_{n_k}$ such that $A = \{x_{n_k} : k < \omega\} \in p$. We take $y_0 = x_{n_0}$ and

$$y_{k+1} = \begin{cases} x_{n_{k+1}} & \text{if } x_{n_{k+1}} \notin \{x_{n_s} : 0 \le s \le k\} \\ x \in A_{n_{k+1}} \setminus \{x_{n_s} : 0 \le s \le k\} & \text{otherwise.} \end{cases}$$

We have that $A \subset \{y_k : k < \omega\} = A'$ and so $A' \in p$. Let $f : \omega \to A'$ defined by $f(n) = y_k$ if $n = n_k$ and $f(n) = y_0$ if $n \in B$. Let $q \in \omega^*$ such that $\overline{f}(q) = p$. It follows that $p = q\text{-}\lim y_k$. On the other hand, since $f|_{\omega\setminus B}$ is a one-to-one function and $\omega \setminus B \in p$, $p \simeq_{\mathrm{RK}} q$ (see [C 1, Lemma 3.2 (c)]). So $\xi(p)$ is a strictly WFU$(T(p))$-space. □

Note that, according to 2.11 (b) and 2.12, for every semiselective ultrafilter p, $\xi(p)$ is a strictly WFU$(T(p))$-space which is not a strictly FU(p)-space.

2.13 Problem. Let $p \in \omega^*$ be selective. Is there any $q \in \omega^*$ such that $p <_{\mathrm{RK}} q$ and $\xi(p)$ is a strictly FU(q)-space?

3. Products, subspaces and sums

For a nonempty subset M of ω^*, S_M will denote one of the p-sequential like properties that we have considered in this work so far: weakly and strongly M-sequentiality, WFU(M) and SFU(M)-property and strictly WFU(M) and strictly SFU(M)-property. Let $\mathfrak{I}$ be the set of all S_M with $\emptyset \neq M \subset \omega^*$ plus sequentiality and Fréchet-Urysohn property. In a similar way, C_M is one of the cover properties $W\gamma_M$, $S\gamma_M$, δ_M, strictly $W\gamma_M$ or strictly $S\gamma_M$, and $\mathfrak{C} = \{C_M : \emptyset \neq M \subset \omega^*\} \cup \{\varepsilon, \gamma\}$.

Properties in $\mathfrak{I}$ are not invariants under continuous functions and products, even the square of a space X having $S \in \mathfrak{I}$ does not necessarily satisfy S. In fact, there exists a Fréchet-Urysohn space X whose square $X \times X$ has uncountable tightness ([AR 1], [AR 2]; see also Example 1 in [GN 1]).

On the other hand, the properties in $\mathfrak{C}$ satisfy the following result, proof of which is similar to that given for Theorem 2.1 in [GT 1].

3.1 Theorem. *Let $C \in \mathfrak{C}$.*

(a) *C is preserved under continuous functions.*
(b) *If F is an F_σ-subset of a space X having C, then F has C.*
(c) *If X has C, then X_n satisfies C for every $n < \omega$.*

3.2 Remarks. (1) Every countable space has C for every $C \in \mathfrak{C}$.

(2) Let X, Y be two spaces and $C \in \mathfrak{C}$. $X \times Y$ has C if and only if the disjoint sum $X \coprod Y$ has C. In fact, $X \coprod Y$ is homeomorphic to a closed subset of $X \times Y$, and $X \times Y$ is homeomorphic to a closed subset of $(X \coprod Y) \times (X \coprod Y)$.

(3) Malykhin and Shakhmatov have shown in [MS] that if we add a single Cohen real to a countable model of MA $+\neg$CH, then there exist two spaces X and Y in the generic extension satisfying γ such that $X \times Y$ does not even have ε. On the other hand, as was remarked by van Douwen (see p. 1222 in [C 2]), there are (without assuming any additional set theoretic axioms) two spaces X and Y of weight not bigger that 2^ω satisfying ε such that $X \times Y$ is not Lindelöf. Thus, because of Theorem 2.3 in [GT 1], X and Y have γ_p for some $p \in \omega^*$ but $X \times Y$ does not satisfy any of the properties in $\mathfrak{C}$.

3.3 Theorem. *If X satisfies $C \in \mathfrak{C}$ and N is a countable space, then $X \times N$ has C.*

Proof: Without loss of generality we may suppose that X and N are infinite. The proof for the case $C = \gamma_p$ suffices here. Let $\{a_n : n < \omega\}$ be a faithful indexation of N, let $\{x_n : n < \omega\} \subset X$ such that $x_i \neq x_j$ if $i \neq j$ and let $\mathcal{G}$ be an open ω-cover of $X \times N$. For each $n < \omega$ set $\mathcal{H}_n = \{V \subset X : V$ is open and there is $G \in \mathcal{G}$ such that $V \times \{a_j : j < n\} \subset G\}$ and $\mathcal{H}'_n = \{V \setminus \{x_n\} : V \in \mathcal{H}_n\}$. We claim that $\mathcal{H} = \bigcup_{n<\omega} \mathcal{H}'_n$ is an open ω-cover of X. In fact, if $\{y_0, \dots, y_r\} \subset X$ there are $s < \omega$ and $G \in \mathcal{G}$ such that $x_s \notin \{y_0, \dots, y_r\}$ and $\{y_0, \dots, y_r\} \times \{a_0, \dots, a_s\} \subset G$. Then, there is an open subset V of X containing $\{y_0, \dots, y_r\}$ such that $V \times \{a_0, \dots, a_s\} \subset G$. So, $V \in \mathcal{H}_s$ and $\{y_0, \dots, y_r\} \subset V \setminus \{x_s\} \in \mathcal{H}'_s$. Thus, $\mathcal{H}$ is an open ω-cover of X. By hypothesis, there is a sequence $(H'_k)_{k<\omega}$ in $\mathcal{H}$ such that $X = \lim_p H'_k$, where $H'_k = V_k \setminus \{x_{n_k}\}$ for some open subset V_k of X and some natural number n_k. For each k there is $G_k \in \mathcal{G}$ such that $V_k \times \{a_j : j < n_k\} \subset G_k$. Now, we are going to prove that $X \times N = \lim_p G_n$. In order to achieve this goal, we take a point $(x, a_n) \in X \times N$. If $D = \{k < \omega : n_k \leq n\} \in p$, then there is $m \leq n$ such that $E = \{k < \omega : n_k = m\} \in p$; but this implies that $X = \bigcup_{k \in E} V_k \setminus \{x_m\}$, which is a contradiction. So, $\{k < \omega : x \in V_k \setminus \{x_{n_k}\},\ n < n_k\} \in p$. It follows that $\{k < \omega : (x, a_n) \in V_k \times \{a_j : j < n_k\} \subset G_k$ and $n < n_k\} \in p$. $\square$

3.4 Corollary. *Let $S \in \mathfrak{I} \setminus \{$weakly M-sequentiality, strongly M-sequentiality : $\emptyset \neq M \subset \omega^*\}$.*

(a) *If $C_\pi(X)$ satisfies S, then $C_\pi(X^n)$ satisfies S.*
(b) *If $C_\pi(X)$ satisfies S, then $C_\pi(X)^\omega$ has S.*
(c) *$C_\pi(X) \times C_\pi(Y)$ has S if and only if $C_\pi(X \times Y)$ has S.*
(d) *If Y is a quotient space of a space X and Y satisfies $C \in \mathfrak{C}$, then X has C.*

Proof: We obtain (a) as a consequence of Theorem 3.1; (b) follows from 3.1, 3.3 and the fact that $C(X)^\omega \cong C(X \times \omega)$ (see Corollary 2.4.7 in [MN]); 3.2 (2) and

$C_\pi(X) \times C_\pi(Y) \cong C_\pi(X \coprod Y)$ implies (c); and (d) results from Theorems 2.2.8 and 2.2.10 in [MN]. $\square$

It is pointed out in [G, p. 258] that if a space X has γ, then X must be zero-dimensional. For $\emptyset \neq M \subset \omega^*$, the following holds.

3.5 Corollary. *If all* RK*-predecessors of each $p \in M \subset \omega^*$ are rapid, then every space X with $W\gamma_M$ is zero-dimensional.*

Proof: Assume that X has property $W\gamma_M$. Let $x \in X$ and U an open neighborhood of x. Choose a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ and $f(y) = 1$ for all $y \in X \setminus U$. From 3.1 (a) and 1.18 it follows that $f[X]$ has measure zero. Hence, there is $r \in [0,1]$ such that $r \notin f[X]$. Then $x \in f^{-1}([0,r)) \subset f^{-1}([0,r]) \subset U$ and $f^{-1}([0,r])$ is a clopen subset of X. $\square$

We know ([GT 1, Theorem 2.3]) that there is $p \in \omega^*$ such that $\beta\omega$ and ω^* have property γ_p. However, if $p \in \omega^*$ is selective, then $\beta\omega$ and ω^* cannot have property γ_p since none of them have property C''. In fact, by induction we may define a partition $\{A_{t(0),\dots,t(n-1)} : t : n \to \{0,1\}$ is a function$\}$ of ω in infinite subsets such that $\{A_{t(0),\dots,t(n-1),0}, A_{t(0),\dots(n-1),1}\}$ is a partition of $A_{t(0),\dots,t(n-1)}$ in infinite sets, for each $n < \omega$ and each function $t : n \to \{0,1\}$. Now define, for each $n < \omega$, $\mathcal{G}_n = \{\widehat{A}_{t(0),\dots,t(n-1)} : t : n \to \{0,1\}$ is a function$\}$. We have that $\mathcal{G}_n$ is a cover of $\beta\omega$. Assume that $\beta\omega$ has property C''. Then, for every $n < \omega$ there is $t_n : n \to \{0,1\}$ such that $\beta\omega = \bigcup_{n<\omega} \widehat{A}_{t_n(0),\dots,t_n(n-1)}$. We may find $\sigma : \omega \to \{0,1\}$ so that $\sigma(n-1) \neq t_n(n-1)$ for all $1 < n < \omega$. We may choose $q \in [\bigcap_{n<\omega} \widehat{A}_{\sigma(0),\dots,\sigma(n-1)}] \cap \omega^*$. It is then evident that $q \notin \widehat{A}_{t_n(0),\dots,t_n(n-1)}$ for all $1 < n < \omega$, a contradiction. $\square$

Using analogous proofs of those given for Theorems 3.18 and 3.20 in [GT 1], we have

3.6 Theorem. *Let $C \in \mathfrak{C}$.*

(1) *the following are equivalent.*

(a) *$\mathbb{R}$ satisfies C.*

(b) *The Cantor space 2^ω satisfies C.*

(c) *Every metrizable separable locally compact space has C.*

(2) *the following are equivalent.*

(a) *$\mathbb{R}^\omega$ satisfies C.*

(b) *The set of irrational numbers ω^ω satisfies C.*

(c) *Every completely metrizable space has C.*

If $|X| > 1$, then the Cantor set 2^ω is homeomorphic to a closed subspace of X^ω. On the other hand, 2^ω does not have γ ([GN 2, Theorem 6]). Thus, we obtain

3.7. *If X has more than one point, then X^ω does not satisfy γ.*

Note also that if X^ω has $C \in \mathfrak{C}$, then X has C, because X is homeomorphic to some closed subset of X^ω. But the converse is not necessarily true, for example the discrete space $\{0,1\}$ satisfies C for every $C \in \mathfrak{C}$ but 2^ω does not have γ_p if p is semiselective ([GT 1, Theorems 3.9 and 3.18]). We obtain something different when we take X into some convenient class of spaces:

3.8 Theorem. *Let X be a compact second countable and non-countable space and let $C \in \mathfrak{C} \setminus \{\gamma\}$, then X has C if and only if X^ω has C.*

Proof: Since X is second countable and non-countable, there is $\emptyset \neq F \subset X$ which is perfect (see [E, p. 59, 1.7.11]). Furthermore, F is completely metrizable because X is, then F contains a copy of the Cantor set ([E, p. 290, 4.5.5]). Thus, 2^ω satisfies C. By Theorem 3.6, every compact metric space has C; in particular X^ω has C. □

3.9 Corollary. *Let X be a compact second countable non-countable space, and let $S \in \mathfrak{I} \setminus \{$weakly M-sequentiality, strongly M-sequentiality $: \emptyset \neq M \subset \omega^*\}$. Then $C_\pi(X)$ has S if and only if $C_\pi(X^\omega)$ satisfies S.*

3.10 Problems. (1) Let $C \in \mathfrak{C}$. Does $\mathbb{R}^\omega$ satisfy C if $\mathbb{R}$ does?

(2) Let $p \in \omega^*$ be a P-point and let X be a space having γ. Does X^ω satisfy γ_p?

3.11 Examples. Let $C \in \mathfrak{C}$.

(1) If X contains a closed discrete set of cardinality $> \aleph_0$, then X does not satisfy C. So, the Moore Plane and $L \times L$, where L is the Sorgenfrey Line do not satisfy C, and by Theorem 3.1, L does not satisfy C either. Also for every cardinal $\alpha \geq \omega$, $\alpha^{(\alpha^+)}$ does not have C. In particular, if X contains a closed copy of ω, X^α does not satisfy C if $\alpha > \omega$.

(2) Let X be a space satisfying ε and such that X contains a point x_0 with the property that for every neighborhood V of x_0, $|X \setminus V| \leq \aleph_0$. Then X satisfies C. Thus, the one point compactification of a discrete space and $[o, \omega_1]$ (with the order topology) satisfy C.

Because of Theorem 2.12 in [GT 1] we know that 2^{2^ω} has γ_p for some $p \in \omega^*$. So it is natural to ask:

3.12 Problem. Does $2^{(2^\omega)^+}$ satisfy γ_p for any $p \in \omega^*$?

For a space X, $n(X)$ denotes the Novak number of X (that is, $n(X)$ is the smaller power of a family of nowhere dense sets covering X). Our last result is a consequence of Theorem 1 in [Ma].

3.13 Theorem ($n(\omega^*) > c$)**.** *Let X be a space. The following statements are equivalent.*

(a) *X satisfies γ.*
(b) *X satisfies γ_p for every $p \in \omega^*$.*
(c) *X satisfies strictly γ_p for every $p \in \omega^*$.*
(d) *$C_\pi(X)$ is p-sequential for every $p \in \omega^*$.*

3.14 Problems. (1) Is it consistent with ZFC that every space satisfying δ_p for every $p \in \omega^*$ has γ_p for every $p \in \omega^*$?

(2) Is it consistent with ZFC that there exists a space X satisfying γ_p (resp., δ_p, strictly γ_p) for every $p \in \omega^*$ and $\neg\gamma$?

(3) Is it consistent with ZFC that $\mathbb{R}$ satisfies γ_p for every $p \in \omega^*$?

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