Guidance properties of a cylindrical defocusing waveguide

Oldřich John, Charles A. Stuart

Abstract. We discuss the propagation of electromagnetic waves of a special form through an inhomogeneous isotropic medium which has a cylindrical symmetry and a nonlinear dielectric response. For the case where this response is of self-focusing type the problem is treated in [1]. Here we continue this study by dealing with a defocusing dielectric response. This tends to inhibit the guidance properties of the medium and so guidance can only be expected provided that the cylindrical stratification is such that guidance would occur for the linear response that is obtained in the limit of zero field strength. The guided modes that we seek correspond to solutions of the boundary value problem $-u'' + \frac{3}{4}\frac{u}{r^2} - q(r)u + p(r,u)u = \lambda u$ for $r > 0$ with $u \in H_0^1(0,\infty)$ and its linearisation is $-u'' + \frac{3}{4}\frac{u}{r^2} - q(r)u = \lambda u$ with $u \in H_0^1(0,\infty)$. This linear problem has the interval $[0,\infty)$ as its essential spectrum and the requirement that guidance should occur in the limit of zero field strength leads us to suppose that it has at least one negative eigenvalue. Solutions of the nonlinear problem are then obtained by bifurcation from such an eigenvalue. The main interest concerns the global behaviour of a branch of solutions since this determines the principal features of the waveguide. If the branch is bounded in $L^2(0,\infty)$ there is an upper limit to the intensity of the guided beams (high-power cut-off), whereas if the branch is unbounded in $L^2(0,\infty)$ then guidance is possible at arbitrarily high intensities. Our results show how these behaviours depend upon the properties of dielectric response.

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1. Introduction

In a non-conducting (dielectric), non-magnetic and charge free medium, Maxwell equations can be written as

$$rotE = -\frac{1}{c}\frac{\partial H}{\partial t}, \tag{1.1}$$

$$rotH = \frac{1}{c}\frac{\partial D}{\partial t}, \tag{1.2}$$

$$\text{div } H = 0 \quad \text{and} \quad \text{div } D = 0 \tag{1.3}$$

where E, H and D denote the electric, magnetic and displacement fields, respectively, and where c is the speed of light in a vacuum. Recalling that $rot\,rotA = -\triangle A + \bigtriangledown \text{div } A$, we see that (1.1) and (1.2) imply that

$$-\triangle E + \bigtriangledown \text{div } E = -\frac{1}{c^2}\frac{\partial^2 D}{\partial t^2}. \tag{1.4}$$

In the study of nonlinear optical waveguides it is common practice to consider monochromatic fields and to adopt a constitutive law having the form

$$D(x,t)=\varepsilon(x,<E^2>(x))E(x,t) \tag{1.5}$$

where ε is a scalar function and where $<E^2>$ denotes the time average of the intensity of the field E. That ε is a scalar means that at each point the medium is isotropic and its dependence on $x\in\mathbf{R^3}$ allows for inhomogeneity of the medium. Its dependence on the average value $<E^2>(x)$ of $\mid E(x,t)\mid^2$ at x, rather than on $\mid E(x,t)\mid^2$ itself, reflects the idea that, due to the high frequency of E, the dielectric constant at x can only adjust to this average effect.

Combining (1.4) and (1.5) we obtain an equation for E,

$$-\triangle E+\bigtriangledown \operatorname{div} E=-\frac{1}{c^2}\varepsilon(x,<E^2>(x))\frac{\partial^2 E}{\partial t^2} \tag{1.6}$$

which is sometimes called the vector electric field wave equation.

So far we have not placed any restriction on the way in which the properties of the medium vary from place to place. In what follows we suppose that this inhomogeneous composition enjoys a cylindrical symmetry. Using (r,φ,z) to denote the usual cylindrical polar coordinates, we suppose that the dielectric response depends only on r and $<E^2>$. Thus in (1.5), $\varepsilon=\varepsilon(r,<E^2>(x))$. In order to accomodate both smooth and abrupt changes in composition across cylindrical layers, we allow ε to be a piecewise continuous function of r. More precisely, we postulate a constitutive law of the form (1.5) where ε satisfies the following condition.

(**H1**) There are M intervals (r_i,r_{i+1}) with

$$0=r_1<r_2<\cdots<r_M<r_{M+1}=\infty$$

and M nonnegative functions

$$\chi_i\in \mathrm{C}^1([r_i,r_{i+1}]\times(0,\infty))\cap \mathrm{C}([r_i,r_{i+1}]\times[0,\infty)) \quad \text{for} \quad i=1,2,\dots,M$$

such that

$$\varepsilon(r,s)=1+4\pi\chi_i(r,s) \quad \text{on} \quad (r_i,r_{i+1})\times[0,\infty).$$

Furthemore,

$$\varepsilon(r,s)\le\varepsilon(r,0) \quad \text{for} \quad (r,s)\in[0,\infty)^2, \tag{1.7}$$

$$\lim_{r\to\infty}\varepsilon(r,0) \quad \text{exists and is finite} \tag{1.8}$$

$$\frac{\partial\varepsilon}{\partial r}(0,s)=0 \quad \text{for} \quad s\ge 0 \tag{1.9}$$

and there exists $\gamma>0$ such that

$$\lim_{s\to 0}\frac{\varepsilon(r,0)-\varepsilon(r,s)}{s^\gamma}=0 \quad \text{uniformly for} \quad r\ge 0. \tag{1.10}$$

Remark. 1. These conditions are similar to those laid down in (**A1**) of [1]. Note however that in [1], the condition (**A1**) requires $\varepsilon(r,s)$ to be a non-decreasing function of s and so it covers materials that have a self-focusing dielectric response. In (**H1**), the condition (1.7) constitutes a weakened form of the requirement that $\varepsilon(r,s)$ be a non-increasing function of s and so it covers materials having a defocusing response.

2. To simplify the subsequent analysis we introduce the following notations,

$$\varepsilon_L(r) = \varepsilon(r,0)\,, \quad \varepsilon_L(\infty) = \lim_{r\to\infty} \varepsilon_L(r), \tag{1.11}$$

$$q(r) = \frac{\omega^2}{c^2}[\varepsilon_L(r) - \varepsilon_L(\infty)], \tag{1.12}$$

$$p(r,s) = \frac{\omega^2}{c^2}[\varepsilon_L(r) - \varepsilon(r, \frac{1}{2}\frac{s^2}{r})], \tag{1.13}$$

and

$$\lambda = \frac{\omega^2}{c^2}\varepsilon_L(\infty) - k^2 \tag{1.14}$$

where ω is the frequency of the monochromatic fields.

This situation is similar to that set out in (3.1)–(3.5) of [1], but for convenience in the present case we have changed the sign of p. By (**H1**), we have

$$q \in C^1([r_i, r_{i+1}])\,, \quad i = 1, ..., M\,, \quad \text{with} \quad \lim_{r\to\infty} q(r) = 0 \tag{1.15 (i)}$$

and

$$p \in C([0,\infty) \setminus \{r_1, ..., r_M\} \times [0,\infty)) \text{ with } p(r,0) \equiv 0 \text{ for } r > 0. \tag{1.15 (ii)}$$

Furthermore,

$$0 \le p(r,s) \le \mathcal{A} \text{ for } r > 0 \text{ and } s \ge 0 \tag{1.15 (iii)}$$

where $\mathcal{A} = \frac{\omega^2}{c^2}\sup_{r\ge 0} \varepsilon_L(r) < \infty$, and there exists a constant $K > 0$ such that

$$0 \le p(r,s) \le K(\frac{1}{r}s^2)^\gamma \text{ for } r > 0 \text{ and } s \ge 0. \tag{1.16}$$

3. The usual Hilbert space norm on $L^2(0,\infty)$ is denoted by $\|\ \|$. For the Sobolev space

$$H_0^1((0,\infty)) = \{u \in L^2(0,\infty);\ u' \in L^2(0,\infty) \text{ and } u(0) = 0\}$$

we use the norm $\|\ \|_1$ defined by

$$\| u \|_1 = \{\| u \|^2 + \| u' \|^2\}^{\frac{1}{2}}.$$

We recall that for $u \in H_0^1(0,\infty)$

$$(1.17) \qquad \lim_{x\to\infty} u(x) = 0\,, \qquad \lim_{x\to 0} x^{-\frac{1}{2}} u(x) = 0\,,$$

$$(1.18) \qquad \max_{x\geq 0} \mid u(x) \mid = \parallel u \parallel_\infty \leq (\parallel u \parallel \parallel u' \parallel)^{\frac{1}{2}}\,,$$

$$(1.19) \qquad x^{-\frac{1}{2}} \mid u(x) \mid \leq \parallel u' \parallel \quad \text{for } x > 0$$

and

$$(1.20) \qquad \parallel \frac{u}{x} \parallel \leq 2 \parallel u' \parallel \quad \text{(Hardy's inequality)}$$

For $u \in H_0^1(0,\infty)$, it follows that

$$(1.21) \qquad 0 \leq p(r, u(r)) \leq K(r^{-1}u^2(r))^\gamma \leq K \parallel u' \parallel^{2\gamma}$$

and

$$(1.22) \qquad \lim_{r\to 0} p(r, u(r)) \quad = \quad \lim_{r\to\infty} p(r, u(r)) = 0.$$

In view of the cylindrical symmetry that underlies (**H1**) it is natural to seek a solution of Maxwell's equations that incorporates this symmetry. In particular we are interested in monochromatic fields propagating in the direction of the z-axis. This leads us to look for a solution of (1.6) in the form

$$(1.23) \qquad E = v(r)\cos(kz - \omega t) \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} \quad \text{for } r > 0,$$

where v is a scalar function. From (1.23) it follows that $< E^2 > (x) = \frac{1}{2}v^2(r)$ and div $E \equiv 0$.

As is discussed in more detail in [1], the field E given by (1.23) yields a complete solution of Maxwell's equations (1.1) to (1.3) satisfying (1.5) provided that

$$(1.24) \qquad v \in C^1((0,\infty)) \cap C^2((0,\infty) \setminus \{r_2, ..., r_M\}),$$

$$(1.25) \qquad \lim_{r\to 0} v(r) = 0 \quad \text{and} \quad \lim_{r\to 0} v'(r) \quad \text{exists and is finite,}$$

and

$$(1.26) \qquad \frac{1}{r}(rv'(r))' - \frac{1}{r^2}v(r) + \frac{\omega^2}{c^2}\varepsilon(r, \frac{1}{2}v^2(r))v(r) - k^2 v(r) = 0$$

for $r \in (0,\infty) \setminus \{r_2, ..., r_M\}$.

Furthermore, such a solution constitutes a guided wave provided that

$$\lim_{r\to\infty} v(r) = \lim_{r\to\infty} v'(r) = 0 \tag{1.27}$$

and

$$v \in H^1(\mathbf{R}^2) \quad \text{(as a function of } x \text{ and } y \text{ where } r = \sqrt{x^2+y^2}\ \text{)}. \tag{1.28}$$

These assertions are justified in §§1–3 of [1], where it is also shown that the problem of finding guided waves of the form (1.23) can be expressed in the following more compact way, by introducing the new variables λ (defined by (1.14)) and u defined by

$$u(r) = r^{\frac{1}{2}} v(r) \qquad \text{for } r > 0. \tag{1.29}$$

Problem G. *Given a dielectric response* ε *that satisfies* $(H1)$, *find a pair* $(\lambda, u) \in \mathbf{R} \times H_0^1((0,\infty))$ *such that* $-\infty < \lambda \leq \frac{\omega^2}{c^2}\varepsilon_L(\infty)$, $u \neq 0$ *and*

$$\int_0^\infty u'\varphi'\,dr = \int_0^\infty [q(r) - \frac{3}{4r^2} - p(r,u) + \lambda]\, u\varphi\,dr \quad \text{for all } \varphi \in H_0^1((0,\infty)). \tag{1.30}$$

In particular, a solution of **Problem G** has the property that $u \in C^2((0,\infty)\setminus\{r_2, ..., r_M\})$ and

$$-u'' + \frac{3}{4r^2}u - q(r)u + p(r,u)u = \lambda u \quad \text{for } r \neq r_2, ..., r_M. \tag{1.31}$$

(See **Theorem 3.1** of [1].)

2. The linear case

Prior to the analysis of the nonlinear eigenvalue **Problem G** we summarize the essential features of the special case where $p \equiv 0$. This means that the dielectric function ε is independent of the field strength and that **Problem G** is linear.

Associated with the differential expression $-u'' + \frac{3}{4r^2}u - q(r)u$ appearing on the left hand side of (1.31) there is a bilinear form defined by

$$a(u,v) = \int_0^\infty (u'v' + \frac{3}{4r^2}uv - q(r)uv)\,dr \quad \text{for } u, v \in H_0^1(0,\infty). \tag{2.1}$$

Clearly

$$a(u,u) \geq -\sup_{r\geq 0} q(r) \parallel u \parallel^2 \quad \text{for } u \in H_0^1(0,\infty) \tag{2.2}$$

where $q \in L^{\infty}(0,\infty)$ and recalling Hardy's inequality we also have that, for $u, v \in H_0^1(0,\infty)$

$$(2.3) \qquad |a(u,v)| \leq 4 \parallel u' \parallel \parallel v' \parallel + \sup_{r\geq 0} |q(r)| \parallel u \parallel \parallel v \parallel \leq C \parallel u \parallel_1 \parallel v \parallel_1 .$$

Hence a is a symmetric bilinear form that is continuous on $H_0^1(0,\infty)$ and bounded below. Furthermore, it is easily seen that a is closed in $L^2(0,\infty)$ and so, according to **Theorem 2.1**, Chapter VI, §2 in [4], there is a unique self-adjoint operator $S : D(S) \subset L^2(0,\infty) \longrightarrow L^2(0,\infty)$ associated with a through the relation

$$(2.4) \qquad a(u,v) = \int_0^{\infty} (Su)v\, dr \quad \text{for all } \ u \in D(S) \ \text{ and } \ v \in H_0^1(0,\infty).$$

As in Chapter VI, §4 of [4],

$$(2.5) \qquad D(S) = \{u \in H_0^1(0,\infty); -u'' + \frac{3}{4r^2}u \in L^2(0,\infty)\}$$

and

$$(2.6) \qquad Su = -u'' + \frac{3}{4r^2}u - q(r)u \quad \text{for } u \in D(S).$$

For a self-adjoint operator T, the spectrum and essential spectrum are denoted by $\sigma(T)$ and $\sigma_e(T)$. Let

$$(2.7) \qquad \Lambda = \inf\{a(u,u);\ u \in H_0^1(0,\infty) \text{ and } \parallel u \parallel = 1\}.$$

Then $\Lambda \geq -\sup_{r\geq 0} q(r) > -\infty$ by (2.2) and

$$(2.8) \qquad \Lambda = \inf \sigma(S).$$

To obtain some further information about the spectrum of S we regard S as a perturbation of the operator $S_0 : D(S_0) \subset L^2(0,\infty) \to L^2(0,\infty)$ defined by $D(S_0) = D(S)$ and $S_0 u = -u'' + \frac{3}{4r^2}u$ for $u \in D(S_0)$.

As for S_0 it follows from Chapter VI, §4.1 of [4] that S_0 is a positive self-adjoint operator and by the analysis following (4.5) there,

$$\int_0^{\infty} (u'v' + \frac{3}{4r^2}uv)\, dr = \int_0^{\infty} (S_0 u)v\, dr \quad \text{for } u \in D(S_0) \text{ and } v \in H_0^1(0,\infty).$$

In particular,

$$(2.9) \qquad \parallel u' \parallel^2 \leq \ \frac{1}{2}\{\parallel S_0 u \parallel^2 + \parallel u \parallel^2\} \qquad \forall\, u \in D(S_0).$$

Defining a linear operator Q by $Qu = qu$, it follows easily from the properties $q \in L^\infty(0,\infty)$ and $\lim_{r\to\infty} q(r) = 0$ that $Q : H_0^1(0,\infty) \to L^2(0,\infty)$ is compact.

From (2.9), we conclude that Q is S_0-compact in the terminology of [5]. Since $S = S_0 + Q$ we can now assert that $\sigma_e(S) = \sigma_e(S_0)$ and that the graph norms of S and S_0 are equivalent on $D(S) = D(S_0)$. In particular, $\exists C > 0$ such that

$$\| u \|_1 \leq C \| u \|_2 \quad \forall\, u \in D(S) \tag{2.10}$$

where $\| u \|_2 = \{\| Su \|^2 + \| u \|^2\}^{\frac{1}{2}}$ is the graph norm of S.

By explicit calculation, $\sigma_e(S_0) = [0,\infty)$ and hence $0 = \inf \sigma_e(S)$. We assume that q is such that the following condition is fulfilled:

(**H2**) $$\Lambda < 0.$$

It follows that

$$\Lambda \ \textit{ is a simple eigenvalue of } S \tag{2.11}$$

with an eigenfunction φ that can be normalized so that $\varphi > 0$ on $(0,\infty)$ and $\| \varphi \| = 1$.

It is not hard to give explicit properties of q that imply (**H2**).

3. A bifurcation result

With the notation and hypothesis of Sections 1 and 2, let H_2 denote the real Hilbert space that is obtained by considering $D(S)$ equipped with the graph norm $\| \, \|_2$ of S. By (2.10), H_2 is continuously embedded in $H_0^1(0,\infty)$. For $u \in H_0^1(0,\infty)$ we define a nonlinear operator M by

$$M(u)(r) = p(r,u(r))u(r) \qquad \text{for } r > 0. \tag{3.1}$$

As in [1] the continuity, boundedness and compactness of $M : H_0^1(0,\infty) \to L^2(0,\infty)$ are easy consequences of (1.21) and (1.22).

Furthermore, $\| M(u) \| \leq \sup_{r>0} p(r,u(r)) \| u \| \leq K \| u \| \| u' \|^{2\gamma} \leq K \| u \|_1^{1+2\gamma}$ for $u \in H_0^1(0,\infty)$.

The study of the **Problem G** can be replaced by the analysis of the set of solutions of the equation

$$Su + M(u) = \lambda u \qquad \text{in } \mathbf{R} \times H_2. \tag{3.2}$$

In fact by strengthening slightly the assumption (**H2**) it can be shown that **Problem G** has no solutions with $\lambda > 0$. For example if the response has the following properties in addition to (**H1**):

(i) $\varepsilon_L \in C^1((r_M,\infty))$ with $\lim_{r\to\infty} r\frac{\partial \varepsilon_L(r)}{\partial r}(r) = 0$

and

(ii) $\varepsilon(r,s) \to \varepsilon_L(r)$ and $r\frac{\partial \varepsilon}{\partial r}(r,s) \to r\frac{\partial \varepsilon_L}{\partial r}(r)$ as $s \to 0$ uniformly on (r_M,∞), then by a slight modification of the proof of **Theorem 3.1** in [10] it follows that $\lambda \leq 0$ for all solutions of **Problem G**.

Setting

$$E = \{(\lambda, u) \in \mathbf{R} \times H_2;\ \lambda < 0,\ u \neq 0 \text{ and } Su + M(u) = \lambda u\}$$

it is easy to see that (λ, u) is a solution of **Problem G** with $\lambda < 0$ if and only if $(\lambda, u) \in E$. Here we shall restrict our attention to one part of this set ; that which corresponds to the fundamental modes of the guidance problem. Let $\mathcal{C}$ denote the component of $E \cup \{(\Lambda, 0)\}$ that contains the point $(\Lambda, 0)$ and let $\bar{\mathcal{C}}$ be its closure in $\mathbf{R} \times H_2$. According to **Theorem 1.2** of [2] (or **Theorem 2.1** of [3]), $\mathcal{C}$ has at least one of the following properties:

(i) C is unbounded in $\mathbf{R} \times H_2$,
(ii) $\sup\{\lambda;\quad (\lambda, u) \in \mathcal{C}\} = 0$,
(iii) there exists $\mu \in \sigma(S) \cap]\Lambda, 0[$ ($(\Lambda, 0)$ is an open interval here) such that $(\mu, 0) \in \bar{\mathcal{C}}$.

However, using the additional properties of M that are available in the present setting this conclusion can immediately be sharpened. First of all we claim that for $(\lambda, u) \in \mathcal{C} \setminus \{(\Lambda, 0)\}$, $u^2(r) > 0$ for all $r > 0$. See [2], [3] for two different ways of obtaining this conclusion, and the corollary which states that $\mathcal{C}$ cannot have the property (iii).

Next, we observe that for $(\lambda, u) \in E$, $\lambda < 0$ and

$$\lambda \parallel u \parallel^2 = \int_0^\infty [Su + M(u)]u\, dr \geq \int_0^\infty (Su)u\, dr = a(u, u) \geq \Lambda \parallel u \parallel^2 . \tag{3.3}$$

Hence $\{\lambda;\ (\lambda, u) \in \mathcal{C}\} \subset [\Lambda, 0[$.

Furthermore for $(\lambda, u) \in E$, it follows from (1.15) that

$$\parallel Su \parallel \leq | \lambda | \parallel u \parallel + \parallel M(u) \parallel \leq (| \Lambda | + \mathcal{A}) \parallel u \parallel . \tag{3.4}$$

Thus we can assert that the following result has been established.

Theorem 3.1. *Under the hypotheses* (**H1**) *and* **H**(**2**) *the component* $\mathcal{C}$ *has exactly one of the following properties:*

(I) $a = \infty$ *and* $b < 0$,
(II) $a < \infty$ *and* $b = 0$,
(III) $a = \infty$ *and* $b = 0$

where $a = \sup\{\parallel u \parallel;\ (\lambda, u) \in \mathcal{C}\}$ *and* $b = sup\{\lambda;\ (\lambda, u) \in \mathcal{C}\}$.

Finally we observe that since $(\lambda, -u) \in \mathcal{C}$ whenever $(\lambda, u) \in \mathcal{C}$, the same result holds with $\mathcal{C}$ replaced by $\mathcal{C}^+ = \{(\lambda, u) \in \mathcal{C} :\ u(r) > 0 \text{ for all } r > 0\}$.

The main aim of our paper is to establish conditions on the function ε (equivalently, on the functions p and q) that enable us to predict which one of the possibilities (I), (II) or (III) occurs.

4. The case of small nonlinearity

The nearer to the zero (in L^∞-norm) the nonlinearity is, the closer the component $\mathcal{C}^+$ is to the ray of positive eigenfunctions of S.

Theorem 4.1. *Let (**H1**) and (**H2**) hold and suppose that* $0 \le p(r,s) \le d$ *for all* $r > 0$ *and* $s \ge 0$ *where* $0 < d < -\Lambda$.

Then for each $(\lambda, u) \in \mathcal{C}^+$ *we have* $\lambda \;\le\; \Lambda + d \;<\; 0$.

Remark. According to the assertion (I) of **Theorem 3.1** we have $\sup\{\| u \|;\ (\lambda,u) \in \mathcal{C}^+\} = \infty$ in this case.

Proof: If $(\lambda,u) \in \mathcal{C}^+$, then λ is the first eigenvalue of the self-adjoint operator T defined by:

$$Tw = -w'' + \frac{3w}{4r^2} - q(r)w + p(r,u(r))w, \qquad \text{for } w \in D(T) = D(S)\,,$$

with the corresponding eigenfunction u. According to the variational property we obtain that

$$\begin{aligned}\lambda = \inf\{\, a(w,w) + \int_0^\infty p(r,u(r))w^2(r)\,dr;\ w \in H_0^1(0,\infty) \ \text{ and } \ \| w \| = 1\} \\ \le\ \inf\{\, a(w,w);\ w \in H_0^1 \ \text{ and } \ \| w \| = 1\} + d = \Lambda + d\end{aligned}$$

which is what we had to prove. $\square$

5. Sufficient condition for $\sup\{\lambda;\ (\lambda,u) \in \mathcal{C}^+\} = 0$

We begin with a result concerning the uniform exponential decay of solutions.

Lemma 5.1. *Let (**H1**) and (**H2**) hold and let* γ *be given such that* $\Lambda < \gamma < 0$.

Then there exist $L > 0$ *and* $\beta > 0$ *such that for each* $(\lambda,u) \in \mathcal{C}^+$ *with* $\lambda \le \gamma$, *the function* $u(r)e^{\beta r}$ *is decreasing on the interval* $[L,\infty)$.

Proof: Let $(\lambda,u) \in \mathcal{C}^+$ with $\lambda \le \gamma$. As $u \in W_2^2(A,\infty)$ for all $A > 0$, we have that both u and u' are continuous on the interval $(0,\infty)$ and

$$u''(r) = (-\lambda - q(r) + \frac{3}{4r^2})u(r) + p(r,u(r))u(r)$$

almost everywhere. (In (r_M,∞) it holds pointwise.)

As $\lim_{r\to\infty} q(r) = 0$, there exists $L > 0$ such that $-q(r) + \frac{3}{4r^2} \ge \frac{\gamma}{2}$ for all $r \in (L,\infty)$, and so (taking account of $p(r,u) \ge 0$)

$$u''(r) \ge -\frac{\gamma}{2}u(r) > 0, \qquad r \in (L,\infty). \tag{5.1}$$

If now $u'(r_o) \ge 0$ in some $r_o > L$, we would have from (5.1) that u' is a strictly increasing positive function on (r_o,∞), which contradicts to the fact that $\lim_{r\to\infty} u'(r) = 0$. So $u' < 0$ on (L,∞).

Multiplying the relation (5.1) by the derivative $u'(r)$ on the interval (L,∞) and setting $-\frac{\gamma}{2}=\beta^2$ with $\beta>0$, we obtain the inequality

$$[(u')^2]' \leq \beta^2 (u^2)'.$$

Integrating it over any interval (r,∞) with $r>L$, we obtain

$$(u')^2(r) \geq \beta^2 u^2(r), \qquad r\in(L,\infty).$$

As $u>0$ and $u'<0$ on the interval (r,∞), we have

$$-u'(r) \geq \beta u(r), \qquad r\in(L,\infty). \tag{5.2}$$

From (5.2) easily follows that $\frac{\mathrm{d}}{\mathrm{dr}}(\log u(r)) \leq -\beta, \quad r\in(L,\infty)$. After the integration over the interval (r_1,r_2), $r_1>L$, we obtain finally

$$u(r_2)e^{\beta r_2} < u(r_1)e^{\beta r_1}, \qquad L<r_1<r_2. \tag{5.3}$$

Corollary. *Let the assumptions of the* **Lemma 5.1** *be satisfied. Then there are positive constants L, β and Q such that $(\lambda,u)\in\mathcal{C}^+$ and $\lambda\leq\gamma$ imply that*

$$u^2(r)+(u')^2(r) \leq Qe^{-2\beta r}, r\in(L,\infty). \tag{5.4}$$

(L, β and Q depend only on γ, Λ and on the bounds for q and p.)

Proof: The estimate for u^2 follows immediately from (5.3). Rewrite now the equation (1.31) as

$$-u''=\{-\frac{3}{4r^2}+q(r)+\lambda-p(r,u(r))\}u.$$

Taking account of the facts that $\lambda\geq\Lambda$, that $-\frac{3}{4r^2}+q(r)-p(r,u(r))$ is bounded from below on (L,∞) and that $u'(r)<0$ on (L,∞), we obtain

$$-[(u')^2]' \leq \mathcal{M}(u^2)' \quad \text{on} \quad (L,\infty)$$

with some negative constant $\mathcal{M}$. Integrating from $r>L$ to ∞, we obtain finally the relation

$$(u')^2(r) \leq -\mathcal{M}u^2(r), \qquad r\in(L,\infty),$$

which completes the proof of (5.4). □

Now we are able to prove that if the nonlinearity $p(r,s)$ is sufficiently big with respect to the expression $q(r)-\frac{3}{4r^2}$ for large values of s, then $\sup\{\lambda;\ (\lambda,u)\in\mathcal{C}^+\}=0$.

Theorem 5.1. *Let* (**H1**) *and* (**H2**) *hold and suppose that there exists* $s_o > 0$ *so that*

$$\forall_{s \geq s_o} \, \forall_{r>0} \quad p(r,s) \geq q(r) - \frac{3}{4r^2}. \tag{5.5}$$

Then

$$b \equiv \sup\{\, \lambda;\ (\lambda, u) \in \mathcal{C}^+\} = 0. \tag{5.6}$$

PROOF: Suppose that $b < 0$ and that $(\lambda, u) \in \mathcal{C}^+$. Let $r_o \in (0, \infty)$ be a point where $u(r_o) = \| u \|_\infty$. If $\| u \|_\infty \geq s_o$, we have, according to (5.5)

$$-u''(r_o) = [\lambda + q(r_o) - \frac{3}{4r_o^2} - p(r_o, u(r_o))]u(r_o) \leq bu(r_o) < 0,$$

which contradicts to the fact that r_o is a point of maxima of u on the interval $(0, \infty)$. (In case that $r_o = r_i$ for some $i = 2, \dots, M$, we can do similar reasoning using one-sided limits at the point r_o.) So we have proved that

$$(\lambda, u) \in \mathcal{C}^+ \Rightarrow \| u \|_\infty < s_o. \tag{5.7}$$

Taking γ for b in **Lemma 5.1**, we have the existence of positive numbers L and β such that $u(r)e^{\beta r}$ is decreasing on the interval (L, ∞) for all functions u for which $(\lambda, u) \in \mathcal{C}^+$.

This yields the estimate,

$$\| u \|^2 = \int_0^L u^2(r)\, dr + \int_L^\infty u^2(r)\, dr \leq s_o^2 L + s_o^2 \int_L^\infty e^{2\beta(L-r)}\, dr \leq s_o^2 (L + \frac{1}{2\beta})$$

and so

$$\sup\{\| u \|;\ (\lambda, u) \in \mathcal{C}^+\} \quad \text{is finite.} \tag{5.8}$$

According to **Theorem 3.1**, (5.8) is incompatible with $b < 0$. So we have proved that $b = 0$. □

6. The existence of a supersolution

In this section we prove the existence of a supersolution ψ of the problem under the condition that $p = p(r, s)$ is sufficiently big even for small values of s.

Lemma 6.1. *In addition to* (**H1**) *and* (**H2**), *suppose that the following conditions are satisfied,*

$$\overline{\lim_{r\to\infty}}[r^2q(r)-\frac{3}{4}]\leq 0, \tag{6.1}$$

$$\exists_{s_o>0}\ \forall_{r>0}\ \forall_{s\geq s_o}\quad p(r,s)\geq q(r)-\frac{3}{4r^2}, \tag{6.2}$$

$$\exists_{z>0}\exists_{\rho\in[0,2)}\exists_{K>0}\exists_{\sigma>0}\forall_{r>z}\forall_{s\in(0,s_o)}\quad p(r,s)\geq Ks^{\sigma}r^{-\rho}. \tag{6.3}$$

Then there exist positive constants A and d such that the function

$$\psi(r)=\frac{A}{(1+d^2r^2)^{\alpha}},\qquad \alpha=\frac{2-\rho}{2\sigma} \tag{6.4}$$

is a supersolution, i.e.,

$$-\psi''(r)+p(r,\psi(r))\psi(r)-q(r)\psi(r)+\frac{3}{4r^2}\psi(r)\geq 0\qquad \textit{a.e. in}\ \ (0,\infty). \tag{6.5}$$

Proof: Put $\psi(r)=\frac{A}{(1+d^2r^2)^{\alpha}}$ with α, A and d positive. After elementary calculations we obtain

$$\psi''(r)=d^2h(dr)\psi(r), \tag{6.6}$$

where

$$h(x)=\frac{2\alpha}{1+x^2}\{-1+\frac{2(\alpha+1)x^2}{1+x^2}\}. \tag{6.7}$$

So the inequality (6.5) can be rewriten as

$$p(r,\psi(r))-q(r)+\frac{3}{4r^2}\geq d^2h(dr),\qquad r\in(0,\infty). \tag{6.8}$$

Recall that the condition (**H1**) implies that p and q have the properties (1.15 (i)) to (1.15 (iii)).

I. STEP. The function $h(dr)$ is non-positive for $r\in(0,X(d))$ where

$$X(d)=\frac{1}{d\sqrt{2\alpha+1}}. \tag{6.9}$$

Put

$$\alpha=\frac{2-\rho}{2\sigma}. \tag{6.10}$$

(The purpose of this choice we shall see later.) Trying to find A in a way that $\psi(r) \ge s_o$ on $(0, X(d))$ (and $\psi(r) < s_o$ on $(X(d), \infty)$) we get

$$A = s_o \left(\frac{2\alpha+1}{2\alpha+2} \right)^{\alpha}, \text{ which does not depend on } d. \tag{6.11}$$

With this choice of A, $X(d)$ and α and using ((6.2)) we get

$$p(r, \psi(r)) - q(r) + \frac{3}{4r^2} \ge 0 \ge d^2 h(dr), \qquad r \in (0, X(d))$$

and so (6.8) is valid on $(0, X(d))$.

Observe that we have the possibility to change free parameter $d > 0$ (without any disturbing α and A) so that the point $X(d)$ shifts as much to the right as we need.

<u>II. STEP</u>. Taking sufficiently small $d > 0$ we have $X(d) > z$. So from (6.3) we get (as $\psi(r) < s_o$ for $r \in (X(d), \infty)$)

$$r^2 p(r, \psi(r)) \ge K \psi^{\sigma}(r) r^{-\rho+2}, \qquad r \in (X(d), \infty). \tag{6.12}$$

For $r > X(d)$ it follows from (6.9) that $1 < (2\alpha+1) r^2 d^2$ and thus we get

$$\psi(r) \ge \frac{A}{(2\alpha+2)^{\alpha} d^{2\alpha} r^{2\alpha}}, \qquad r \in (X(d), \infty).$$

Substituting it into (6.12) and using (6.10) (the choice of α !) we have

$$r^2 p(r, \psi(r)) \ge \frac{K A^{\sigma}}{(2\alpha+2)^{\alpha\sigma}} \frac{1}{d^{2\alpha\sigma}} r^{-\rho+2-2\alpha\sigma} = \frac{K^*}{d^{2\alpha\sigma}}, \qquad r \in (X(d), \infty), \tag{6.13}$$

where we put $K^* = \frac{K A^{\sigma}}{(2\alpha+2)^{\alpha\sigma}}$.

Supposing now $d > 0$ sufficiently small, we have from (6.1)

$$-r^2 q(r) + \frac{3}{4} \ge -\frac{1}{2} K^* > -\frac{1}{2} \frac{K^*}{d^{2\alpha\sigma}}, \qquad r \in (X(d), \infty). \tag{6.14}$$

Putting (6.13) and (6.14) together we can write

$$r^2 \{ p(r, \psi(r)) - q(r) + \frac{3}{4r^2} \} \ge \frac{1}{2} \frac{K^*}{d^{2\alpha\sigma}}, \qquad r \in (X(d), \infty). \tag{6.15}$$

Multiplied by r^2, the expression (6.7) can be estimated on $(X(d), \infty)$ as follows

$$r^2 d^2 h(dr) \le 2\alpha(2\alpha+1). \tag{6.16}$$

If d is sufficiently small, then obviously $2\alpha(2\alpha+1) \le \frac{1}{2} \frac{K^*}{d^{2\alpha\sigma}}$. It follows now from (6.15) and (6.16) that for such a choice of $d > 0$ we obtain (6.8) on $(X(d), \infty)$, which completes the proof of **Lemma 6.1**. $\square$

Remember that $\mathcal{C}^+$ is a connected subset of

$$\{ (\lambda, u);\ \lambda < 0, \quad (\lambda, u) \text{ solves } \textbf{Problem G}, \text{ and } u(r) > 0 \text{ on } (0, \infty) \}$$

and that $(\Lambda, 0) \in \bar{\mathcal{C}}^+$.

This implies the following result.

Lemma 6.2. *Let the assumptions of* **Lemma 6.1** *be satisfied. Then for each* $(\lambda, u) \in \mathcal{C}^+$ *we have* $u(r) \leq \psi(r)$ *for all* $r \in [0, \infty)$.

Proof: Denote

$$D = \{ (\lambda, u) \in \mathcal{C}^+;\ u(r) \leq \psi(r),\ r \in [0, \infty)\} \cup \{(\Lambda, 0)\}. \tag{6.17}$$

We shall prove that D is a nonempty subset of $\mathcal{C}^+ \cup \{(\Lambda, 0)\}$, which is simultaneously closed and open in $\mathcal{C}^+ \cup \{(\Lambda, 0)\}$ considered as a connected metric space equipped with the topology induced by $\mathbf{R} \times H_2$. It follows that $D = \mathcal{C}^+ \cup \{(\Lambda, 0)\}$.

I. STEP. As $(\Lambda, 0) \in D$, $D \neq \emptyset$.

II. STEP. (D is closed.) Let $(\lambda_n, u_n) \in D$ and $(\lambda_n, u_n) \to (\lambda_o, u_o) \in \mathcal{C}^+ \cup \{(\Lambda, 0)\}$ in $\mathbf{R} \times H_2$. Because of $u_n \to u_o$ in H_2 and H_2 is continuously imbedded into H_0^1, we have that $u_n \to u_o$ pointwisely in $[0, \infty)$ from which follows that $u_o(r) = \lim_{n\to\infty} u_n(r) \leq \psi(r)$.

III. STEP. (D is open.) Note that for $(\lambda_o, u_o) \in D$ we have that $u_o(r) < \psi(r)$ for all $r \in [0, \infty)$. If this were false, we would have the existence of $\bar{r} \in (0, \infty)$ such that $u_o(\bar{r}) = \psi(\bar{r})$. Then $\bar{r}$ is the point of minima of the (nonnegative) function $w = \psi - u_o$.

So we would have

$$\begin{aligned} 0 \leq w''(\bar{r}) = \psi''(\bar{r}) - u_o''(\bar{r}) \leq \frac{3}{4\bar{r}^2}[\psi(\bar{r}) - u_o(\bar{r})] \\ - q(\bar{r})[\psi(\bar{r}) - u_o(\bar{r})] + p(\bar{r}, \psi(\bar{r}))\psi(\bar{r}) - p(\bar{r}, u_o(\bar{r}))u_o(\bar{r}) + \lambda_o u_o(\bar{r}) \\ = \lambda_o u_o(\bar{r}) < 0, \end{aligned}$$

which is a contradiction. (Here we used (6.5), the fact that u_o solves the equation (1.31) and that $\psi(\bar{r}) = u_o(\bar{r})$. In the points of discontinuity of coefficients we can use the reasoning for one-sided limits.)

Let now $(\lambda_o, u_o) \in D$. Then $\lambda_o < 0$ and we put $I_{\lambda_o} = [\Lambda, 0) \cap (\frac{3}{2}\lambda_o, \frac{1}{2}\lambda_o)$. From **Lemma 5.1** we have that

$$\exists_{L>0}\exists_{\beta>0} \text{ such that } u(r)e^{\beta r} \text{ is non} - \text{increasing on } (L, \infty) \\ \text{for all } (\lambda, u) \in \mathcal{C}^+ \cup \{(\Lambda, 0)\} \text{ with } \lambda \in I_{\lambda_o}. \tag{6.18}$$

Because of $\lim_{r\to\infty} \psi(r)e^{\beta r} = \infty$, there exists $Y > L$ such that

$$u_o(r) < \frac{1}{2}\psi(r), \qquad r \in (Y, \infty). \tag{6.19}$$

Let now $(\lambda, u) \in \mathcal{C}^+ \cup \{(\Lambda, 0)\}$ with $\lambda \in I_{\lambda_o}$ and $\| u - u_o \|_2 < \frac{1}{2}\psi(Y)$. For $r \in (Y, \infty)$ we have

$$\begin{aligned} u^2(r) \leq u^2(Y)e^{2\beta(Y-r)} \leq \{u_o^2(Y) + u^2(Y) - u_o^2(Y)\}e^{2\beta(Y-r)} \\ \leq \{\frac{1}{4}\psi^2(Y) + 2 \| u - u_o \|_2^2\}e^{2\beta(Y-r)} \leq \frac{3}{4}\psi^2(Y)e^{2\beta(Y-r)} < \psi^2(r), \end{aligned}$$

and so

$$u(r) < \psi(r) \qquad \text{on} \qquad (Y,\infty). \tag{6.20}$$

(Here we used the fact that $\psi(r)e^{\beta r}$ is increasing on (Y,∞).)

From the observation at the beginning of this part of the proof,

$$\Delta = \min\{\psi(r) - u_o(r);\ r \in [0,Y]\} > 0. \tag{6.21}$$

Let now $(\lambda,u) \in \mathcal{C}^+ \cup \{(\Lambda,0)\}$ with $\lambda \in I_{\lambda_o}$ and $\| u - u_o \|_2 < \frac{\Delta}{\sqrt{Y}}$. For $r \in (0,Y)$ we have

$$u(r) = u_o(r) + (u(r) - u_o(r)) \le u_o(r) + \sqrt{Y} \| u - u_o \|_2 < (\psi(r) - \Delta) + \Delta = \psi(r),$$

and so

$$u(r) < \psi(r) \qquad \text{on} \qquad (0,Y)\,. \tag{6.22}$$

Hence we have proved that for each $(\lambda_o,u_o) \in D$ there exists a neighbourhood of (λ_o,u_o) in $\mathcal{C}^+ \cup \{(0,\Lambda)\}$ which lies in D.

7. Sufficient condition for $\sup\{\| u \|;\ (\lambda,u) \in \mathcal{C}^+\} < \infty$. Compactification

Here the supersolution obtained in §6 is used to control the behaviour of $\mathcal{C}^+$ near $\lambda = 0$.

Theorem 7.1. *Let the conditions* (**H1**), (**H2**), (6.1), (6.2) *and* (6.3) *be satisfied.*

(a) *If there exists* $\{(\lambda_n,u_n)\} \subset \mathcal{C}^+$ *with* $\lambda_n \to 0$ *and* $\sup\{\| u_n \|;\ n \in \mathbf{N}\} < \infty$, *then there exists* $\bar{u} \in H_2$ *such that* $0 < \bar{u} \le \psi$ *on* $(0,\infty)$ *and* $\bar{u}$ *satisfies* (1.31) *with* $\lambda = 0$. (*Here* ψ *is the supersolution defined by* (6.4).)

(b) *If the exponents* σ *and* ρ *in* (6.3) *satisfy*

$$\frac{\sigma}{2} + \rho < 2$$

then $\sup\{\| u \|;\ (\lambda,u) \in \mathcal{C}^+\} < \infty$, *and* $\mathcal{C}^+$ *is relatively compact in* $\mathbf{R} \times H_2$.

Proof: (a) By (3.4), $\sup \| u_n \|_2 < \infty$ and so we may suppose (by passing to a subsequence if necessary) that $u_n \rightharpoonup \bar{u}$ weakly in H_2. Hence $\bar{u} \in H_2$ and using (2.10) we may suppose that $u_n \rightharpoonup \bar{u}$ weakly in $H_0^1(0,\infty)$ and $u_n \to \bar{u}$ uniformly on bounded subsets of $[0,\infty)$. From **Lemma 6.2** we can conclude that $0 \le \bar{u} \le \psi$ on $[0,\infty)$.

Since $\lim_{r\to\infty} \psi(r) = 0$ we can now assert that $u_n \to \bar{u}$ uniformly on $[0,\infty)$. Since (λ_n,u_n) satisfies (1.30) it now follows that for all $\varphi \in H_0^1(0,\infty)$

$$\begin{aligned}\int_0^\infty (S\bar{u})\varphi\,dr &= \lim_{n\to\infty}\int_0^\infty (Su_n)\varphi\,dr = \lim_{n\to\infty}\int_0^\infty \{u_n'\varphi' + (\frac{3}{4r^2} - q)u_n\varphi\}\,dr \\ &= \lim_{n\to\infty}\int_0^\infty \{\lambda_n - p(r,u_n)\}u_n\varphi\,dr = -\int_0^\infty p(r,\bar{u})\bar{u}\varphi\,dr.\end{aligned}$$

Hence $\bar{u}$ satisfies (1.31) with $\lambda = 0$. It remains to prove that $\bar{u}(r) > 0$ on $(0, \infty)$.

Let $r_o > 0$ be such that $\bar{u}(r_o) = 0$. As $\bar{u}$ then attains its minimum at r_o, we have $\bar{u}'(r_o) = 0$. But $\bar{u}$ satisfies almost everywhere the linear equation

$$-w'' + \frac{3}{4r^2} w(r) - q(r)w(r) + p(r, \bar{u}(r))w(r) = 0$$

with the conditions $\bar{u}(r_o) = \bar{u}'(r_o) = 0$, and it follows that $\bar{u} \equiv 0$.

Hence we have either $\bar{u} \equiv 0$ or $\bar{u} > 0$ on $(0, \infty)$.

Suppose that $\bar{u} = 0$ and put $v_n = \frac{\bar{u}_n}{\|\bar{u}_n\|}$. Since $v_n > 0$ on $(0, \infty)$, λ_n is the lowest eigenvalue of

$$-w'' + \frac{3}{4r^2} w - qw + p(r, u_n(r))w = \lambda w, \quad w \in H_2.$$

Hence

$$\lambda_n = \inf\{\, a(v, v) + \int_0^\infty p(r, u_n(r))v^2(r)\, dr;\ v \in H_0^1(0, \infty) \text{ with } \| v \| = 1 \,\}.$$

By (3.3), $\Lambda \le \lambda_n$ and by (2.11),

$$\lambda_n \le a(\varphi, \varphi) + \int_0^\infty p(r, u_n(r))\varphi^2\, dr \le \Lambda + \int_0^\infty p(r, u_n(r))\varphi^2\, dr.$$

But $u_n \to \bar{u} \equiv 0$ uniformly on $[0, \infty)$ and so we have $\lambda_n \to \Lambda$. By (**H2**), $\Lambda < 0$ whereas by hypothesis $\lambda_n \to 0$. This means that $\bar{u} \ne 0$ and we must have $\bar{u} > 0$ on $(0, \infty)$.

(b) Under these conditions $\psi \in L^2(0, \infty)$ and by **Lemma 6.2**

$$\sup\{ \| u \|;\ (\lambda, u) \in \mathcal{C}^+ \} \le \| \psi \| < \infty. \tag{7.1}$$

Let $\{(\lambda_n, u_n)\} \subset \mathcal{C}^+$. By (3.4) and (7.1), $\sup\{\| u_n \|_2\} < \infty$. As in the proof of part (a), by passing to a subsequence we may suppose that $\lambda_n \to \lambda \le 0$ and that $u_n \rightharpoonup u$ weakly in H_2 where $0 \le u \le \psi$ on $(0, \infty)$ and (λ, u) satisfies (1.31).

We also have that $u_n \to u$ uniformly on $(0, \infty)$ and, since $\psi \in L^2(0, \infty)$, we conclude that $\| u_n - u \| \to 0$.

Furthermore $\| M(u_n) - M(u) \| \to 0$ by dominated convergence since $p(r, u_n(r))u_n(r) \to p(r, u(r))u(r)$ a.e. on $(0, \infty)$ and by (1.15 (iii)),

$$|p(r, u_n(r))u_n(r) - p(r, u(r))u(r)|^2 \le 2\mathcal{A}^2 \psi(r)^2$$

Hence $\| S(u_n - u) \| \to 0$ and so $\| u_n - u \|_2 \to 0$. □

8. Sufficient condition for $\sup\{\lambda;\ (\lambda, u) \in \mathcal{C}^+\} = 0$ and $\sup\{\|u\|;\ (\lambda, u) \in \mathcal{C}^+\} = \infty$.

We suppose throughout this section that (**H1**), (**H2**), (6.1), (6.2) and (6.3) are satisfied. By **Theorem 5.1** $\sup\{\lambda;\ (\lambda, u) \in \mathcal{C}^+\} = 0$ and our purpose here is to give conditions ensuring that $\sup\{\|u\|;\ (\lambda, u) \in \mathcal{C}^+\} = \infty$. According to **Theorem 3.1** it is sufficient to show that

$$\sup\{\|u\|; (\lambda, u) \in \mathcal{C}^+\} < \infty \quad \text{is impossible.} \tag{8.1}$$

In fact, if (8.1) holds, then **Theorem 7.1** (a) shows that there exists $\bar{u} \in C^2((r_M, \infty)) \cap H^1(r_M, \infty)$ such that $0 < \bar{u}(r) \le \psi(r)$ for $r > r_M$ and

$$-\bar{u}''(r) + \{\frac{3}{4r^2} - q(r) + p(r, \bar{u}(r))\}\bar{u}(r) = 0 \quad \text{for} \quad r > r_M. \tag{8.2}$$

In particular

$$\bar{u}(r) > 0 \text{ on } (r_M, \infty), \quad \bar{u} \in L^2(r_M, \infty) \quad \text{and} \quad \lim_{r\to\infty} \bar{u}(r) = 0. \tag{8.3}$$

We now give conditions which imply that (8.2) cannot have a solution satisfying (8.3), and hence that (8.1) cannot occur. These conditions are as follows,

$$rq(r) \in L^1(r_M, \infty), \tag{8.4}$$

$$\varliminf_{r\to\infty} r^2(\log r)q(r) \ge 0 \tag{8.5}$$

and

$$\exists_{\widetilde{z}>0}\exists_{\widetilde{s}_o>0}\exists_{\widetilde{\rho}\in[0,2)}\exists_{\widetilde{K}>0}\exists_{\widetilde{\sigma}>0}\forall_{r>\widetilde{z}}\forall_{s\in(0,\widetilde{s}_o)} \qquad p(r, s) \le \widetilde{K} s^{\widetilde{\sigma}} r^{-\widetilde{\rho}}. \tag{8.6}$$

Since (6.3) holds, we must have $\widetilde{\sigma} \le \sigma$ and $\widetilde{\rho} \le \rho$ in (8.6).

The main conclusion of this section can now be stated.

Theorem 8.1. *Let the conditions* (**H1**), (**H2**), (6.1), (6.2), (6.3), (8.4), (8.5) *and* (8.6) *hold. Suppose that either*

$$\frac{\widetilde{\sigma}}{2} + \widetilde{\rho} > 2 \tag{8.7}$$

or

$$\frac{\widetilde{\sigma}}{2} + \widetilde{\rho} = 2 \qquad \text{and} \qquad \widetilde{\sigma} \ge 2 \tag{8.8}$$

Then $\sup\{\|u\|;\ (\lambda, u) \in \mathcal{C}^+\} = \infty$.

The proof of this result is a consequence of the following lemmas concerning the asymptotic behaviour of solutions of (8.2).

Lemma 8.2. *Let the hypothesis of* **Theorem 8.1** *hold with the additional restriction that* $\widetilde{\sigma} > 2$ *when* (8.8) *occurs. Let* u *be a solution of* (8.2) *and* (8.3). *Then*

$$\text{(8.9)} \qquad \exists\, H > 0 \quad \text{such that} \qquad 0 < u(r) < (r \log r)^{-\frac{1}{2}} \qquad \text{for } r > H.$$

Proof: Set

$$V(r) = u(r) - (r \log r)^{-\frac{1}{2}}.$$

Calculating $V''(r)$ and taking account in the conditions we obtain that for sufficiently big $l > 0$

$$\text{(8.11)} \qquad V''(r) < r^{-\frac{5}{2}} \log^{-\frac{3}{2}} r \{ -\frac{1}{2} - \frac{1}{\log r} + \widetilde{K} (\log r)^{-\frac{\widetilde{\sigma}}{2}+1} r^{-\frac{\widetilde{\sigma}}{2}-\widetilde{\rho}+2} \} < 0$$

at each point $r \in (l, \infty)$ in which $V(r) < 0$.

Now either $V < 0$ on the whole interval (l, ∞) or $V(r_o) \geq 0$ at some point $r_o > l$. If the second possibility takes place we take any point y_o, $y_o > r_o$, in which $V(y_o) < 0$. (Such a point exists, otherwise $V \geq 0$ on (r_o, ∞) which contradicts the assumption that $u \in L^2(0, \infty)$.)

Let (C, D) be the maximal interval containing y_o on which $V < 0$. Suppose that $D < \infty$. Then $V(C) = V(D) = 0$ and V attains its minimum on $[C, D]$ in some $z_o \in (C, D)$. So $V(z_o) < 0$ and $V''(z_o) \geq 0$ which contradicts to (8.11). Thus $D = \infty$.

The validity of **Lemma 8.2** is now obvious. □

Lemma 8.3. *Under the hypothesis of* **Theorem 8.1**, *let* u *be a solution of* (8.2), (8.3). *Set*

$$\text{(8.12)} \qquad Q(r) = -q(r) + p(r, u(r)).$$

Then

$$\text{(8.13)} \qquad rQ(r) \in C((r_M, \infty)) \cap L^1(r_M, \infty).$$

Proof: The integrability of $rq(r)$ is given by (8.4). Furthermore, there exists $Z \geq r_M$ such that

$$\text{(8.14)} \qquad 0 \leq rp(r, u(r)) \leq \widetilde{K} r u^{\widetilde{\sigma}}(r) r^{-\widetilde{\rho}} \qquad \text{for } r > Z$$

by (8.6) since $\lim_{r \to \infty} u(r) = 0$.

If (8.8) holds with $\widetilde{\sigma} = 2$ we have $\widetilde{\rho} = 1$ and so (8.14) becomes $0 \leq rp(r, u(r)) \leq \widetilde{K} u^2(r)$. Since $u \in L^2(r_M, \infty)$ we see that (8.13) holds if (8.8) occurs with $\widetilde{\sigma} = 2$.

In all other cases, it follows from (8.14) and **Lemma 8.2** that $0 \leq rp(r, u(r)) \leq \widetilde{K} r^{1-\frac{\widetilde{\sigma}}{2}-\widetilde{\rho}} (\log r)^{-\frac{\widetilde{\sigma}}{2}}$.

The restrictions on the exponents guarantee the integrability of the right hand side of this inequality and so (8.13) holds. □

Lemma 8.4. *Consider the equation*

$$-v''(r) + \frac{3}{4r^2}v(r) + Q(r)v(r) = 0 \quad \text{on} \quad (a,\infty) \tag{8.15}$$

where $Q \in C((a,\infty))$ *and* $rQ(r) \in L^1(a,\infty)$. *It has two solutions* ψ_1, ψ_2 *such that*

$$\lim_{r\to\infty} \frac{\psi_1(r)}{r^{\frac{3}{2}}} = \lim_{r\to\infty} \frac{\psi_2(r)}{r^{-\frac{1}{2}}} = 1. \tag{8.16}$$

Proof: The functions $r^{\frac{3}{2}}$ and $r^{-\frac{1}{2}}$ are linearly independent solutions of the equation $-v''(r) + \frac{3}{4r^2}v(r) = 0$ on $(0,\infty)$. By a slight modification of the proof of the **Theorem 3.6.1** in [6] we obtain the deserved result. □

Proof of Theorem 8.1: If (8.1) occurs, there is a function $\bar{u}$ satisfying (8.2) and (8.3). Setting

$$Q(r) = -q(r) + p(r,\bar{u}(r)) \qquad \text{for} \qquad r > r_M$$

we see that $\bar{u}$ satisfies (8.15).

By **Lemma 8.3**, Q satisfies (8.13) and hence **Lemma 8.4** implies that there exist constants A and B such that

$$\bar{u}(r) = A\psi_1(r) + B\psi_2(r) \qquad \text{for} \qquad r > a$$

where ψ_1, ψ_2 satisfy (8.16).

But $\lim_{r\to\infty} u(r) = 0$ by (8.3) and so $A = 0$. Then $\bar{u} = B\psi_2 \in L^2(a,\infty)$ by (8.3) and this means that $B = 0$, contradicting the fact that $\bar{u}(r) > 0$ on (r_M,∞). Hence (8.1) cannot occur and the result follows from **Theorem 3.1**. □

Remark. The above proof actually establishes a stronger conclusion, namely

$$\varliminf_{\mu\to 0} \{ \| u \|; \ (\lambda,u) \in \mathcal{C}^+ \quad \text{with} \quad \mu \le \lambda < 0\} = \infty.$$

9. Behaviour of the waveguide

We summarize our conclusions, giving the hypothesis in terms of the dielectric response function ε which satisfies the basic assumptions (**H1**), (**H2**). For convenience we set

$$\varepsilon_{NL}(r,s) = \varepsilon(r,s) - \varepsilon_L(r).$$

By (**H1**), $\varepsilon_{NL} \le 0$ and, setting $\mathcal{M} = \sup_{r,s\ge 0} |\varepsilon_{NL}(r,s)|$ we also have that $\mathcal{M} < \infty$.

If

$$\mathcal{M} < \frac{c^2}{\omega^2}|\Lambda| \tag{9.1}$$

it follows from **Theorem 4.1** that the component $\mathcal{C}$ of fundamental modes has the property (I) of **Theorem 3.1** so guidance is possible at all powers.

Instead of (9.1) let us now suppose that

$$\exists\,\delta > 2 \quad \text{such that} \qquad \lim_{r\to\infty} r^{\delta}\{\varepsilon_L(r) - \varepsilon_L(\infty)\} = 0. \tag{9.2}$$

Clearly this implies (6.1), (8.4) and (8.5).

We suppose also that

(9.3)

$$\exists\, s_o > 0 \quad \text{such that} \quad r^2\{\varepsilon(r, \frac{1}{2r}s^2) - \varepsilon_L(\infty)\} \le \frac{3c^2}{4\omega^2} \quad \text{for } r > 0 \text{ and } s \ge s_o,$$

whereas there exist $z > 0$, $K > 0$, $\nu \in (0, 2)$ and $\kappa \in [0, 2-\nu)$ such that

$$|\varepsilon_{NL}(r,s)| \ge K s^{\nu} r^{-\kappa} \qquad \forall\, r > z \quad \text{and} \quad s \le s_o. \tag{9.4}$$

By **Theorem 7.1** we see that the conditions (9.2), (9.3) and (9.4) ensure that $\mathcal{C}$ has the property (II) of **Theorem 3.1** provided that $2\nu + \kappa < 2$. This means that there are no guided modes of this kind with power above a certain level.

If (9.2), (9.3) and (9.4) are satisfied and also there exist $s_1 > 0$, $K_1 > 0$, $\nu_1 \in (0, 2)$ and $\kappa_1 \in [0, 2 - \nu_1)$ such that

$$|\varepsilon_{NL}(r,s)| \le K_1 s^{\nu_1} r^{-\kappa_1} \quad \forall\, r > z \text{ and } s \le s_1 \tag{9.5}$$

then **Theorem 8.1** shows that $\mathcal{C}$ has property (III) of **Theorem 3.1** provided that either $2\nu_1 + \kappa_1 > 2$ or $\nu_1 \ge 1$. This means that guidance is possible at all powers.

Finally as an example we deal with a special case in which the waveguide has a homogeneous exterior region (cladding) with a nonlinear response.

Example. In addition to (**H1**) and (**H2**), suppose that $\varepsilon(r, s) = \varepsilon^c(s)$ for all $r > r_M$ and $s \ge 0$, where ε^c is non-increasing and

$$\exists\, K > 0, \nu > 0 \quad \text{such that} \quad \lim_{s\to 0} \frac{\varepsilon^c(s) - \varepsilon^c(0)}{s^{\nu}} = -K. \tag{9.6}$$

Suppose also that there exists $s_1 > 0$ such that

$$\varepsilon(r,s) < \varepsilon^c(0), \qquad \forall\, r \in [0, r_M] \text{ and } s \ge s_1. \tag{9.7}$$

Clearly $\varepsilon_L(r) = \varepsilon^c(0)$ for $r > r_M$ and so (9.2) is satisfied. Setting $s_o = \sqrt{2r_M s_1}$ we find that (9.3) and (9.4) are satisfied with $\kappa = 0$ and $z = r_M$. Furthermore, (9.5) is also satisfied with $\kappa_1 = 0$, $\nu_1 = \nu$ and $z = r_M$.

Hence the component $\mathcal{C}$ has the property (II) if $0 < \nu < 1$ and it has the property (III) if $2 > \nu \ge 1$, where ν is given by (9.6). The case $\nu = 1$ corresponds to a cladding composed of a defocusing material whose response is approximated by the usual Kerr non-linearity [7]. But other values of ν (particularly $0 < \nu < 1$) do occur for some other types of material [8], [9].

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Faculty of Mathematics and Physics, Charles University, Sokolovská 83,
186 00 Praha 8, Czech Republic

Department of Mathematics, EPFL, CH-1015 Lausanne, Switzerland

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