On the approximation of entire functions over Carathéodory domains

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Abstract. Let D be a Carathéodory domain. For $1 \le p \le \infty$, let $L^p(D)$ be the class of all functions f holomorphic in D such that $\|f\|_{D,p} = [\frac{1}{A}\int\int_D |f(z)|^p\, dx\, dy]^{1/p} < \infty$, where A is the area of D. For $f \in L^p(D)$, set

$$E_n^p(f) = \inf_{t\in\pi_n} \|f-t\|_{D,p}\,;$$

π_n consists of all polynomials of degree at most n. In this paper we study the growth of an entire function in terms of approximation error in L^p-norm on D.

Keywords: approximation error, generalized parameters, L^p norm and Fourier coefficients

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1. Introduction

Let B denote a Carathéodory domain, that is, a bounded simply connected domain such that the boundary of B coincides with the boundary of the domain lying in the complement of the closure of B and containing the point ∞. In particular, a domain bounded by a Jordan Curve is a Carathéodory domain. Let $L^p(B)$, $1 \le p \le \infty$, be the class of all functions f holomorphic on B and satisfying

$$\|f\|_{B,p} = \left[\int\int_B |f(z)|^p\, dx\, dy\right]^{1/p} < \infty\,,$$

where the last inequality is understood to be $\sup_{z\in B}|f(z)| < \infty$ for $p=\infty$. Then $\|\cdot\|_{B,p}$ is called the L^p-norm on $L^p(B)$. For $f \in L^p(B)$, let us define b_n's called the Fourier coefficients of f as follows:

$$b_n = \int\int_B f(z)\overline{p_n(z)}\, dx\, dy, \qquad \int\int_B p_n(z)\overline{p_m(z)}\, dx\, dy = \delta_m^n\,, \tag{1.1}$$

$\delta_m^n = 1$ for $m=n$ and $\delta_m^n = 0$, otherwise and $\{p_n\}_{n=0}^\infty$ is a sequence of polynomials, p_n being of degree n. It is known [10, p. 273] that $f \in L^p(B)$ is entire, if and only if,

$$\lim_{n\to\infty} |b_n|^{1/n} = 0\,. \tag{1.2}$$

Moreover, $f(z)=\sum_{n=0}^{\infty} b_n p_n(z)$ holds in the whole complex plane. For $f \in L^p(B)$, we define $E_n^p(f)$, the error in approximating the function f by polynomials of degree at most n in L^p-norm, as

$$E_n^p(f)=E_n^p(f,B)=\inf_{t\in\pi_n}\|f-t\|_{B,p}, \quad n=0,1,2,\dots, \tag{1.3}$$

where π_n consists of all polynomials of degree at most n.

Let L° denote the class of functions $h(x)$ satisfying conditions (H,i) and (H,ii):

(H,i) $h(x)$ is defined on $[a,\infty)$; is positive, strictly increasing, and differentiable; and tends to ∞ as $x\to\infty$.

(H,ii)
$$\lim_{x\to\infty}\frac{h[x(1+\phi(x))]}{h(x)}=1$$
for every function $\phi(x)$ such that $\phi(x)\to 0$ as $x\to\infty$.

Let Λ denote the class of functions $h(x)$ satisfying conditions (H,i) and (H,iii):

(H,iii)
$$\lim_{x\to\infty}\frac{h(cx)}{h(x)}=1 \quad \text{for every} \quad 0<c<\infty\,.$$

Seremeta [8], Shah [9] defined generalized growth parameters $\varrho(\alpha,\beta,f)$ and $\lambda(\alpha,\beta,f)$ of an entire function $f(z)$ as

$$\begin{matrix}\varrho(\alpha,\beta,f)\\ \lambda(\alpha,\beta,f)\end{matrix}=\lim_{r\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\alpha(\log M(r,f))}{\beta(\log r)}, \tag{1.4}$$

where $\alpha(x)\in\Lambda$ and $\beta(x)\in L^\circ$ and generalized various results, cf. [2], [6], [7], [11].

The generalized orders of an entire function in terms of the coefficients in its Taylor series have been characterized by Shah [9] and Nautiyal et al. [5]. Surprisingly, they have obtained these results under the condition:

$$\frac{d[\beta^{-1}(\alpha(x))]}{d(\log x)}=0(1) \quad \text{as} \quad x\to\infty\,. \tag{1.5}$$

Clearly, the corresponding results of Shah [9] and Nautiyal et al. [5] fail to exist for the functions, $\alpha(x)=\beta(x)$. To include this class of functions, Kapoor and Nautiyal [3] defined generalized growth parameters in a new setting as follows:

Let Ω be the class of functions $h(x)$ satisfying (H,i) and (H,iv):

(H,iv) There exists a $\delta(x)\in\Lambda$ and x_0, K_1 and K_2 such that

$$0<K_1\le\frac{d(h(x))}{d(\delta(\log x))}\le K_2<\infty \quad \text{for all} \quad x>x_0\,.$$

Let $\overline{\Omega}$ be the class of functions $h(x)$ satisfying (H,i) and (H,v):

(H,v)

$$\lim_{x\to\infty}\frac{d(h(x))}{d(\log x)} = K, \quad 0 < K < \infty\,.$$

Let $f(z) = \sum_{n=1}^{\infty} a_n z^{\lambda_n}$ be a nonconstant entire function. Here $\lambda_0 = 0$ and $\{\lambda_n\}_{n=1}^{\infty}$ is a strictly increasing sequence of positive integers such that no element of the sequence $\{a_n\}_{n=1}^{\infty}$ is zero.

The generalized growth parameters of an entire function $f(z)$ are defined as

$$\begin{matrix}\varrho(\alpha,\alpha,f)\\ \lambda(\alpha,\alpha,f)\end{matrix} = \lim_{r\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\alpha(\log M(r,f))}{\alpha(\log r)}\,, \tag{1.6}$$

where $\alpha(x)$ either belongs to Ω or $\overline{\Omega}$ and

$$M(r,f) = \max_{|z|=r}|f(z)|, \quad \mu(r,f) = \max_{n\geq 0}\left[|a_n| r^{\lambda_n}\right]\,.$$

Kapoor and Nautiyal [3] have characterized generalized growth parameters for entire functions of slow growth in terms of the sequence $\{E_n(f)\}$. It seems that, even for the unit disc and $[-1,1]$, the interrelation between the growth of an entire function and the approximation error in L^p-norm has not been studied so extensively as in the case of approximation error in the uniform norm. Further, the study of the growth of an entire function in terms of approximation error in L^p-norm on more generalized domains then the unit disc and $[-1,1]$ has been completely neglected.

In this paper we study the approximations of entire functions in L^p-norm on Carathéodory domains. The generalized growth parameters of an entire function have been characterized in terms of the error $E_n^p(f)$, defined by (1.3).

The text has been divided into three parts. Section 1 consists of an introductory exposition of the topic and in Section 2 we prove three lemmas, one of them connecting the generalized growth parameters of an entire function f to the maximum modulus and other connecting the growth of an entire function with its Fourier coefficients and in the last lemma, the growth parameters of an entire function f have been characterized in terms of the error $E_n^p(f)$. Finally we prove some theorems and a necessary condition on $E_n^p(f)$ for an entire function f to be of generalized regular growth.

Some of our results extend and improve the results contained in [3] and [5].

We shall use the following notation throughout the paper.

Notation:

$$\begin{aligned} P_\psi &= \max\{1,\nu\} && \text{if } \alpha(x)\in\Omega\,,\\ &= \psi+\nu && \text{if } \alpha(x)\in\overline{\Omega}\,. \end{aligned}$$

We shall write $P(\nu)$ for $P_1(\nu)$.

2. Preliminary lemmas

Let B^* be the component of the complement of the closure of the Carathéodory domain B that contains the point ∞. Set $B_r = \{z : |\overline{\phi}(z)| = r\}$, $r > 1$, where the function $w = \overline{\phi}(z)$ maps B^* conformally on to $|w| > 1$ such that $\overline{\phi}(\infty) = \infty$ and $\overline{\phi}'(\infty) > 0$.

Lemma 1. *Let f be an entire function having generalized growth parameters $\varrho(\alpha, \alpha, f)$ and $\lambda(\alpha, \alpha, f)$. Then*

$$\begin{matrix} \varrho(\alpha,\alpha,f) \\ \lambda(\alpha,\alpha,f) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(\log \overline{M}(r))}{\alpha(\log r)},$$

where

$$\overline{M}(r) \equiv \overline{M}(r,f) = \max_{z\in B_r} |f(z)|.$$

Proof: Let z_0 be a fixed point of the set B and $r > 1$. Then from [12],

$$r - 2|B| - |z_0| \leq |z| \leq r + |B| + |z_0|, \quad z \in B_r.$$

For $\xi < 1$ and $\eta > 1$, using $\log Kx \simeq \log x$ as $x \to \infty$, $0 < K < \infty$, we get

$$\log M(\xi r) \leq \log \overline{M}(r) \leq \log M(\eta r).$$

Now, Lemma 1 is immediate in view of (1.6). □

Lemma 2. *Let $f \in L^p(B)$, $1 \leq p \leq \infty$, be the restriction to B of an entire function having generalized growth parameters $\varrho(\alpha,\alpha,f)$ and $\lambda(\alpha,\alpha,f)$. Then $g(z) = \sum_{n=0}^{\infty} |b_n| z^n$, b_n's are given by (1.1), is an entire function. Further*

$$\varrho(\alpha,\alpha,f) = \varrho(\alpha,\alpha,g) \quad \text{and} \quad \lambda(\alpha,\alpha,f) = \lambda(\alpha,\alpha,g),$$

also hold.

Proof: Firstly, g is entire as follows from (1.2).

From [10, p. 272] we have

$$\max_{z\in B_{r'}} |p_n(z)| \leq C\, r'^{\,n}, \quad n = 1, 2, \dots,$$

where C is a constant independent of n, r' (> 1) is a fixed number. Thus, applying Bernstein's inequality (e.g. [1, p. 21], [4, p. 112]) for each term of the series $\sum_{n=0}^{\infty} b_n p_n(z)$, we get

$$|f(z)| \leq |b_0| + C \sum_{n=1}^{\infty} |b_n| (rr')^n, \quad z \in B_r. \tag{2.1}$$

$$\overline{M}(r,f) \leq |b_0| + CM(rr', g), \quad r > 1. \tag{2.2}$$

Thus using Lemma 1 and the fact that either $\alpha \in \Omega$ or $\overline{\Omega}$. (2.2) gives

$$\varrho(\alpha,\alpha,f) \le \varrho(\alpha,\alpha,g) \quad \text{and} \quad \lambda(\alpha,\alpha,f) \le \lambda(\alpha,\alpha,g). \tag{2.3}$$

Now, let $r^* > 1$ be a fixed constant. Since f is entire, it follows that ([4, p. 114]) there exists a sequence of polynomials $\{Q_n\}$, Q_n being of degree at most n, such that

$$|f(z) - Q_n(z)| < \frac{2}{3}\overline{M}(r)\frac{(r^*/r)^{n+1}}{1-(r^*/r)}, \quad z \in \overline{B}, \tag{2.4}$$

for all sufficiently large n and all $r > r^*$.

Now,

$$b_n = \int\int_B f(z)\overline{p_n(z)}\,dx\,dy = \int\int_B (f(z) - Q_{n-1}(z))\overline{p_n}(z)\,dx\,dy.$$

Since p_n is orthogonal to any polynomial of degree less than n, using Schwarz inequality, we get

$$|b_n| \le \|f - Q_n\|_{B,p} \le A^{1/p}\max_{z\in\overline{B}}|f(z) - Q_n(z)|, \quad 1 \le p < \infty\,,$$

where A is the area of B. Using (2.4) in above, we get

$$|b_n| \le \gamma\overline{M}(r)\left(\frac{r^*}{r}\right)^n \tag{2.5}$$

for all sufficiently large n and $r > 2r^*$, γ is a constant independent of n and r. Moreover, (2.5) gives

$$\mu(r/r^*; g) \le \gamma\overline{M}(r,f) \tag{2.6}$$

for all sufficiently large values of r. Thus using Theorem 3 of [3], Lemma 1 and the fact that either $\alpha \in \Omega$ or $\overline{\Omega}$, we obtain

$$\varrho(\alpha,\alpha,g) \le \varrho(\alpha,\alpha,f) \quad \text{and} \quad \lambda(\alpha,\alpha,g) \le \lambda(\alpha,\alpha,f). \tag{2.7}$$

Combining (2.3) and (2.7) we get the required result for $1 \le p < \infty$. For $p = \infty$, the lemma can easily be proved following Winiarski [12]. □

Lemma 3. *Let $f \in L^p(B)$, $1 \le p \le \infty$, be the restriction to B of an entire function having generalized growth parameters $\varrho(\alpha,\alpha,f)$ and $\lambda(\alpha,\alpha,f)$. Then $\tilde{g}(z) = \sum_{n=0}^{\infty} E_n^p(f)z^n$, $E_n^p(f)$ as given in (1.3), is also an entire function. Further, we have*

$$\varrho(\alpha,\alpha,f) = \varrho(\alpha,\alpha,\tilde{g}) \quad \textit{and} \quad \lambda(\alpha,\alpha,f) = \lambda(\alpha,\alpha,\tilde{g}). \tag{2.8}$$

PROOF: From the definition of $E_n^p(f)$, since $Q_n \in \pi_n$, we have

$$E_n^p(f) \leq \|f - Q_n\|_{B,p} \leq A^{1/p} \max_{z\in\overline{B}} |f(z) - Q_n(z)|, \tag{2.9}$$

where A is the area of B. Now using (2.4) and (2.9), we get

$$E_n^p(f) \leq \gamma \overline{M}(r)(r^*/r)^n\,. \tag{2.10}$$

If f is entire, then $\lim_{n\to\infty}(E_n^p(f))^{1/n} = 0$, for $r > 2r^*$ and $r \to \infty$. So $\tilde{g}(z)$ is an entire function. Further (2.10) gives

$$\begin{aligned} M(r/r^*, \tilde{g}) &\leq P(r) + \gamma\overline{M}(r+1, f)\sum_{n=0}^{\infty}[(r/r+1)]^n \\ &= P(r) + \gamma(r+1)\overline{M}(r+1, f), \end{aligned}$$

where $P(r)$ is a polynomial. Thus, using Lemma 1 and $\alpha \in \Omega$ or $\overline{\Omega}$, we get

$$\varrho(\alpha, \alpha, \tilde{g}) \leq \varrho(\alpha, \alpha, f) \ \text{ and } \ \lambda(\alpha, \alpha, \tilde{g}) \leq \lambda(\alpha, \alpha, f). \tag{2.11}$$

On the other hand, for any $w \in \pi_{n-1}$, $n \geq 1$, we get

$$|b_n| = \left|\int\int_B (f(z) - w(z))\overline{p_n(z)}\,dx\,dy\right| \leq Cr'^{\,n}\|f - \omega\|_{B,1}\,. \tag{2.12}$$

On applying Hölder's inequality, (2.12) gives

$$|b_n|/r'^{\,n} \leq CA^q\|f - w\|_{B,q}\,, \quad 1 \leq p < \infty,$$

where A is defined as earlier and $q = 1 - 1/p$. Since the above relation holds for any $w \in \pi_{n-1}$, we have

$$|b_n|/r'^{\,n} \leq CA^q E_{n-1}^p(f), \quad 1 \leq p < \infty. \tag{2.13}$$

Now using (2.12) and (2.13), we obtain

$$\begin{aligned} &\overline{M}(r, f) \leq |b_0| + C^2A^q\sum_{n=1}^{\infty}E_{n-1}^p(f)(rr'^{\,2})^n,\ 1 \leq p < \infty. \\ &\overline{M}(r, f) \leq |b_0| + C^2A^q rr'^{\,2}M[rr'^{\,2}, \tilde{g}]. \end{aligned} \tag{2.14}$$

In view of Lemma 1, from (2.14) and $\alpha \in \Omega$ or $\overline{\Omega}$, we have

$$\varrho(\alpha, \alpha, f) \leq \varrho(\alpha, \alpha, \tilde{g}) \ \text{ and } \ \lambda(\alpha, \alpha, f) \leq \lambda(\alpha, \alpha, \tilde{g}). \tag{2.15}$$

On combining (2.11) and (2.15), the lemma is proved for $1 \leq p < \infty$. For $p = \infty$, the lemma can be proved following Winiarski [12]. $\square$

3. Main results

Now we prove the following theorems:

Theorem 1. *Let $f \in L^p(B)$, $1 \le p \le \infty$, be the restriction to B of an entire function having generalized growth parameters $\varrho(\alpha,\alpha,f)$ and $\lambda(\alpha,\alpha,f)$. Then*

(i) $\varrho(\alpha,\alpha,f) = P(L)$,

(ii) $\varrho(\alpha,\alpha,f) \le P(L^*)$, *where*

$$L = \lim_{n\to\infty} \sup \frac{\alpha(n)}{\alpha\{\frac{1}{n}\log E_n^p(f)^{-1}\}},$$

and

$$L^* = \lim_{n\to\infty} \sup \frac{\alpha(n)}{\alpha\{\log(E_{n-1}^p(f)/E_n^p(f))\}}.$$

(iii) $\lambda(\alpha,\alpha,f) \ge P(\tilde{\ell})$, *where*

$$\tilde{\ell} = \lim_{n\to\infty} \inf \frac{\alpha(n)}{\alpha\{\frac{1}{n}\log E_n^p(f)^{-1}\}}.$$

(iv) *If we take $\alpha(x) = \alpha(a)$ on $(-\infty, a)$, then*
$\lambda(\alpha,\alpha,f) \ge P(\ell^*)$, *where*

$$\ell^* = \lim_{n\to\infty} \inf \frac{\alpha(n)}{\alpha\{\log(E_{n-1}^p(f)/E_n^p(f))\}}.$$

Theorem 2. *Let $f \in L^p(B)$, $1 \le p \le \infty$, be the restriction to B of an entire function having generalized growth parameters $\varrho(\alpha,\alpha,f)$, $\lambda(\alpha,\alpha,f)$ and if $(E_n^p(f)/E_{n+1}^p(f))$ is nondecreasing, then*

$$\varrho(\alpha,\alpha,f) = P(L) = P(L^*)$$

and

$$\lambda(\alpha,\alpha,f) = P(\tilde{\ell}) = P(\ell^*).$$

Theorem 3. *Let $f \in L^p(B)$, $1 \le p \le \infty$, be the restriction to B of an entire function having generalized lower order $\lambda(\alpha,\alpha,f)$. Then:*

(i) *If $\alpha(x) \in \Omega$, we have*

$$\lambda(\alpha,\alpha,f) = \max_{\{n_k\}}[P_X\{\ell'\}] \tag{3.1}$$

and if we further take $\alpha(x) = \alpha(a)$ on $(-\infty, a)$, then

$$\lambda(\alpha,\alpha,f) = \max_{\{n_k\}}[P_X\{\ell'^*\}], \tag{3.2}$$

where

$$X \equiv X(\{n_k\}) = \lim_{k\to\infty} \inf \frac{\alpha(n_{k-1})}{\alpha(n_k)}$$

and

$$\ell' \equiv \ell'(\{n_k\}) = \lim_{k\to\infty} \inf \frac{\alpha(n_{k-1})}{\alpha\{\frac{1}{n_k} \log E^p_{n_k}(f)^{-1}\}}$$

and

$$\ell'^* = \ell'^*(\{n_k\}) = \lim_{k\to\infty} \inf \frac{\alpha(n_{k-1})}{\alpha\{\frac{1}{(n_k-n_{k-1})} \log(E^p_{n_{k-1}}(f)/E^p_{n_k}(f))\}}\,.$$

The maximum in (3.1) and (3.2) is taken over all increasing sequences $\{n_k\}$ of positive integers.

Further if $\{n_m\}$ is the sequence of the principal indices of the entire function $\tilde{g}(z) = \sum_{n=0}^{\infty} E^p_n(f) z^n$ and $\alpha(n_m) \sim \alpha(n_{m+1})$ as $m \to \infty$, then (3.1) and (3.2) also hold for $\alpha(x) \in \overline{\Omega}$.

Proof of Theorems 1,2,3: Theorems 1,2, and 3 follow easily from [3, Theorems 4–6, Lemma 1] and Lemma 3.

For $f \in L^p(B)$, $1 \le p \le \infty$, let $\{n_i\}_{i=0}^{\infty}$ with $n_0 = 0$, be the sequence of positive integers defined as follows:

$$\text{(3.3)}\quad E^p_{n_{i-1}}(f) > E^p_{n_i}(f) \quad\text{and}\quad E^p_n(f) = E^p_{n_{i-1}}(f) \quad\text{for}\quad n_{i-1} \le n < n_i, \quad i = 1, 2, 3, \ldots$$

□

We now obtain a relation that shows how this sequence influences the growth of an entire function. Thus we have

Theorem 4. *Let $f \in L^p(B)$, $1 \le p \le \infty$, be the restriction to B of an entire function having generalized growth parameters $\varrho(\alpha,\alpha,f)$ and $\lambda(\alpha,\alpha,f)$. Then*

$$\lambda(\alpha,\alpha,f) \le \varrho(\alpha,\alpha,f) \lim_{i\to\infty} \inf \frac{\alpha(n_i)}{\alpha(n_{i+1})}\,,$$

where n_i is defined by (3.3).

Proof: Let us define a function $\theta(z)$ as

$$\theta(z) = \sum_{n=1}^{\infty} (E^p_{n-1}(f) - E^p_n(f)) z^n = \sum_{i=1}^{\infty} \pi_i z^{n_i}\,,$$

where

$$\pi_i \equiv \pi_i(f) = E^p_{n_{i-1}}(f) - E^p_{n_i}(f).$$

Clearly $\theta(z)$ has the generalized order $\varrho(\alpha,\alpha,f)$, the generalized lower order $\lambda(\alpha,\alpha,f)$, and so applying Lemma 1 and Theorem 4 of [3] to $\theta(z)$ we get

$$\begin{aligned}\lambda(\alpha,\alpha,f) &= \sup_{\{i_k\}}\left[\lim_{k\to\infty}\inf\frac{\alpha(n_{i_{k-1}})}{\alpha(\frac{1}{n_{i_k}})\log(\pi_{i_k}^{-1})}\right]\\ &\le \sup_{\{i_k\}}\left[\lim_{k\to\infty}\sup\frac{\alpha(n_{i_k})}{\alpha(\frac{1}{n_{i_k}})\log(\pi_{i_k}^{-1})}\right]\sup_{\{i_k\}}\left[\lim_{k\to\infty}\inf\frac{\alpha(n_{i_{k-1}})}{\alpha(n_{i_k})}\right]\\ &\le \varrho(\alpha,\alpha,f)\lim_{i\to\infty}\inf\frac{\alpha(n_{i-1})}{\alpha(n_i)}\,.\end{aligned}$$

This proves the theorem. □

Corollary. *Suppose $f\in L^p(B)$, $1\le p\le\infty$, be the restriction to B of an entire function having generalized regular growth. Further, let $\alpha\in\Omega$ or $\overline{\Omega}$. Then*

$$\alpha(n_i)\sim\alpha(n_{i+1})\quad\text{as}\quad i\to\infty,$$

where $\{n_i\}$ is defined by (3.3).

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