Comment.Math.Univ.Carolin. 35,4 (1994)779–788

On open light mappings

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Abstract. Whyburn has proved that each open mapping defined on arc (a simple closed curve) is light. Charatonik and Omiljanowski have proved that each open mapping defined on a local dendrite is light. Theorem 3.8 is an extension of these results.

Keywords: continuum, locally connected, open, light, fan, order, component number
Classification: 54C10, 54F50

1. Introduction

In [2], J.J. Charatonik and K. Omiljanowski proved the following results.

Theorem 1.1 ([2, Theorem 5, p. 214]). *Let a metric space X be locally dendritic, and let a space Y have no isolated points. Then each continuous open surjection $f : X \to Y$ is light.*

Thus each nonconstant open mapping defined on a local dendrite is light ([2, Corollary 6, p. 216]).

Theorem 1.2 ([2, Theorem 8, p. 216]). *Let a space X satisfy the condition: each nondegenerate closed connected subset of X has the nonempty interior. If a space Y has no isolated points, then each open mapping from X onto Y is light.*

Hence if each nondegenerate subcontinuum of a metric continuum X has the nonempty interior, then each nonconstant open mapping on X is light ([2, Corollary 9, p. 217]).

J.J. Charatonik and K. Omiljanowski in [2] also asked:

Problem 1.3 ([2, Problem 3, p. 214]). What topological spaces X and Y have the property that each open mapping from X onto Y is light?

Problem 1.4 ([2, Problem 11, p. 217]). Characterize all metric continua X such that each open mapping on X is light.

The above problems were the inspiration to the present paper. However, being not completely solved, they remain still open.

All mappings considered in this paper are continuous and all spaces are assumed to be metric. A *continuum* means a compact connected space. A locally connected continuum containing no simple closed curve is called a *dendrite*. If each point of a space has a neighborhood being a dendrite, then the space is said to be *locally dendritic*. If, moreover, the space is a continuum, then it is called a *local dendrite*.

Let $\mathbb{N}$ stand for the set of all positive integers. Given two point u and w in a Euclidean space, we denote by uw the straight line segment with the end points u and w.

If the Euclidean plane $\mathbb{R}^2$ is equipped with the Cartesian coordinate system, then put $v = (0,0)$, $e_0 = (0,1)$ and $e_n = ((1/n),1)$ for $n \in \mathbb{N}$. The set $F_H = \bigcup\{ve_n : n \in \mathbb{N} \cup \{0\}\}$ is called the *harmonic fan*. Let $\{\rho_n : n \in \mathbb{N}\}$ be the set of all rationals in the open unit interval $[0,1] \setminus \{0,1\}$. For each $n \in \mathbb{N}$ take $x_n \in ve_n$ such that $d(v,x_n) = \rho_n \cdot d(v,e_n)$, where d denotes the Euclidean metric in the plane; put $x_0 = e_0$ and define $F_{HS} = \bigcup\{vx_n : n \in \{0\} \cup \mathbb{N}\} \subseteq F_H$. We call F_{HS} the *harmonic shredded fan* (see [1, p. 31]). Finally, we define the locally connected fan F_ω putting $F_\omega = \bigcup\{vy_n : n \in \mathbb{N}\} \subseteq F_H$, where $y_n \in ve_n$ and $d(v,y_n) = (1/n) \cdot d(v,e_n)$, for $n \in \mathbb{N}$.

Recall that a surjective mapping $f : X \to Y$ is:

- *open* if f maps every set open in X onto a set open in Y;
- *confluent* if for every subcontinuum Q of Y each component of the inverse image $f^{-1}(Q)$ is mapped by f onto Q;
- *light* if for every point $y \in Y$ each component of the inverse image $f^{-1}(y)$ is a singleton (equivalently: $f^{-1}(y)$ is zero-dimensional, if X is compact).

It is known that each open mapping of a compact space is confluent.

2. The class L. Definition and properties

Let L denote the class of all such continua X that each nonconstant open mapping on X is light. It is known that an arc and a simple closed curve belongs to L ([6, 1.2, 1.3, p. 184]). Every local dendrite is in the class L ([2, Corollary 6, p. 216]). Also some nonlocally connected continua are in the class L, e.g. the harmonic shredded fan ([1, Proposition 7.6, p. 32]). Continua X such that each nondegenerate subcontinuum of X has the nonempty interior belong to L ([2, Corollary 9, Remark 10, p. 217]).

Example 2.1. The continuum X which is the union of the locally connected fan F_ω and the harmonic fan F_H as in Figure 1 is in the class L.

Proof: Suppose on the contrary that X does not belong to L, i.e. there are a space Y, a point $y \in Y$, and an open mapping $f : X \to Y$ such that the inverse image $f^{-1}(y)$ contains a nondegenerate component C. Since each subcontinuum of X with the empty interior is contained in the segment rp, then $C \subseteq rp$. We prove that $f^{-1}(y) \cap rp_i \neq \emptyset$ for infinitely many indices i. Suppose on the contrary that there is a natural number M such that for $i \geq M$ the set $f^{-1}(y) \cap (rp_i \setminus \{r\})$ is empty. Take an open set U in X such that $U \cap rp \subseteq C$ and that $U = (ab \setminus \{a,b\}) \cup \bigcup\{A_i : i \geq M\}$, where $ab \subseteq rp$, $A_i = a_ib_i \setminus \{a_i,b_i\}$, $a_ib_i \subseteq rp_i \setminus \{r\}$. Then $f(U) = \bigcup\{f(A_i) : i \geq M\} \cup \{y\}$. The sets $f(A_i)$ are open and the family $\{f(A_i) : i \geq M\}$ forms a null sequence converging to $\{y\}$. Hence $f(U)$ is not open.

Take $x_i \in f^{-1}(y) \cap (rp_i \setminus \{r\})$. The continuum X is locally connected at the point x_i, and $y = f(x_i)$, thus Y is locally connected at y ([1, Proposition 3.11, p. 14]) and $\operatorname{ord}_y Y \leq 2$ ([6, 7.31, p. 147]). Consider the family S of all such segments rp_i that contain the points x_i with $f(x_i) = y$. Since rp_i is a free arc in X, then $f(rp_i)$ is a free arc in Y. Let rp_i and rp_j belong to S. The common part $f(rp_i) \cap f(rp_j)$ contains the points y and $f(r)$. It is easy to see that $f(rp_i) = f(rp_j)$. This equality holds for every two elements of the family S. By continuity of f, we have

(1) $\qquad f(rp_i) = f(rp)$ with $f(p_i) = f(p)$ for sufficiently large indices i.

Since p_i is an end point, $f(p_i)$ is an end point too; hence

$$f(p) \text{ is an end point.} \tag{2}$$

Observe that $\operatorname{ord}_q X = \omega$. The components of $X \setminus \{q\}$ are open and form a null sequence, and the images of these components under f form a null sequence too, whence it follows that $\operatorname{ord}_{f(q)} Y = \omega$. Each point of the set $X \setminus (rp \cup \{q\})$ is of order 1 or 2 in X. The points of $f(rp)$ are of a finite order, by (1) and ([6, 7.31, p. 147]). Therefore we have

$$f^{-1}(f(q)) = \{q\}. \tag{3}$$

Take such a point $z \in rp \cup \bigcup\{rp_i : i \in \mathbb{N}\}$ that $f(z) \neq f(p)$, and take in Y the arc L from $f(q)$ to $f(z)$. Thus L does not contain the point $f(p)$, by (2); hence

$$p \notin f^{-1}(L). \tag{4}$$

Let K be the component of $f^{-1}(L)$ that contains the point z. By confluence of f, we have $f(K) = L$, which contradicts (3) and (4). The proof is complete. □

Now we prove that the class L is not closed under any of the following operations: the union of two elements (even if their intersection is a singleton), the product, the inverse limit, and taking a closed subspace.

Example 2.2. There exists a continuum Y which does not belong to L, and which is the one-point union of two continua belonging to L.

Proof: The continuum Y is the one-point union of the locally connected fan F_ω and the continuum X defined in Example 2.1 (see Figure 2). Define the mapping $f : Y \to f(Y)$ by $f \mid (Y \setminus F_H) = \mathrm{id}$ and $f \mid F_H$ is as in Example 7.1 of [1, p. 29]. Let $s : f(Y) \to f(Y)$ be the symmetry with respect to the line ℓ (see Figure 2). Let $Q = pq \cup \bigcup\{qq_i : i \in \mathbb{N}\}$. Define $g : f(Y) \to f(Y)$ by $g \mid Q = s$ and $g \mid (f(Y) \setminus Q) = \mathrm{id}$. The space $g(f(Y))$ is homeomorphic to F_ω. The mapping $h = gf$, being the composition of f and g, is open and nonlight. □

Observation 2.3. *If nondegenerate continua X and Y belong to L, then their product $X \times Y$ does not belong to L.*

In fact, the projection from $X \times Y$ onto X (or Y) is open and nonlight.

Recall that an *n-od* is the union of n arcs, any two of which intersect at their common end point only.

Example 2.4. There exists an inverse sequence (X_n, f_n) such that each X_n is an n-od (and thus belongs to L), each f_n is open and light and $\lim\operatorname{inv}(X_n, f_n)$ is not in L.

PROOF: For $n \geq 3$, let X_n be contained in F_H and $X_n = \bigcup\{ve_i : i \in \{1, 2, \dots, n\}\}$. Thus $X_n \subseteq X_{n+1}$. Define a retraction $f_n : X_{n+1} \to X_n$ as $f_n \mid X_n = \mathrm{id}$ and $f_n \mid ve_{n+1}$ as a linear homeomorphism from ve_{n+1} onto ve_n. Then $\lim\operatorname{inv}(X_n, f_n)$ $= F_H$ is not in L because there exists a nonlight open mapping from F_H onto an arc ([1, Example 7.1, p. 29]). □

Example 2.5. There exists a hereditarily locally connected continuum X such that X belongs to L and X contains a subcontinuum Z which does not belong to L.

PROOF: In the rectangular coordinates in the plane put for each $n \in \{0\} \cup \mathbb{N}$ and for $k \in \{0, 1, 2, \dots, 2^n\}$ $p_k^n = \big((2k+1)/2^{n+1}, 1/2^{n+1}\big)$, where $k < 2^n$ and $q_k^n = (k/2^n, 0)$.

Define

$$Z = q_0^0 q_1^0 \cup \bigcup\Big\{\bigcup\{p_k^n q_k^n \cup p_k^n q_{k+1}^n : k \in \{0, 1, \dots, 2^n - 1\}\} : n \in \{0\} \cup \mathbb{N}\Big\}.$$

The projection $(x, y) \mapsto y$ of Z onto the closed interval $[0, 1/2]$ is an open nonlight mapping ([2, Example 7, p. 216]). Let $r_k^n = (k/2^n, -1/2^n)$, where $k \in \{1, 2, \dots, 2^n - 1\}$, and define

$$X = Z \cup \bigcup\Big\{\bigcup\{q_k^n r_k^n : k \in \{1, 2, \dots, 2^n - 1\}\} : n \in \mathbb{N}\Big\} \quad \text{(see Figure 3).}$$

The continuum X belongs to L because it satisfies the assumptions of Theorem 3.8 below. □

Question 2.6. Does there exist a non-arcwise connected continuum which is in the class L?

Let M be an arbitrary class of mappings that contains the class of homeomorphisms. The following inclusions among the classes of mappings on continua are known (see e.g. [4, Table II, p. 28]).

$$\begin{tikzcd} \text{homeomorphisms} \arrow[r] \arrow[d] & \text{local homeomorphisms} \arrow[rr] & & \text{open mappings} \arrow[d] \\ \text{atomic mappings} \arrow[r] & \text{hereditarily monotone mappings} \arrow[r] & \text{monotone mappings} \arrow[r] & \text{MO-mappings} \end{tikzcd}$$

We study the classes M for which the implication holds:

$$(*) \qquad \text{if } f \in \mathsf{M} \text{ and } X \in \mathsf{L}, \text{ then } f(X) \in \mathsf{L}.$$

Proposition 2.7. *The implication* $(*)$ *holds if* M *is the class of open mappings.*

PROOF: Suppose on the contrary that $f(X) \notin \mathsf{L}$. Then there are a continuum K, a nonlight open mapping $g : f(X) \to K$ and a point $x \in K$ such that $g^{-1}(x)$ contains a nondegenerate component C. Thus C is a subcontinuum of $f(X)$. By confluence of f, each component of the set $f^{-1}(C)$ is mapped under f onto C. Hence the components of $f^{-1}(C)$ are not one-point sets.

On the other hand, the mapping $h = gf$, being the composition of f and g, is open. Since X belongs to L, we infer that h is light. But $h^{-1}(x) = f^{-1}(g^{-1}(x))$ contains a nondegenerate component, a contradiction. $\square$

Proposition 2.8. *If* M *is the class of hereditarily monotone mappings, then implication* $(*)$ *is not true.*

Given continua X and Z as in Example 2.5. Define mapping $f : X \to Z$ by $f \mid Z = \mathrm{id}$ and $f(r_k^n q_k^n) = \{q_k^n\}$. We see that the mapping f is hereditarily monotone and $f(X) = Z$ does not belong to L.

Proposition 2.9. *The implication* $(*)$ *holds if* M *is the class of atomic mappings and the continuum* X *is arcwise connected* ([4, 6.3, p. 51]).

If Question 2.6 has a positive answer, we have the next one.

Question 2.10. Is the implication $(*)$ true if the continuum X is not arcwise connected and M is the class of atomic maps?

3. Local connectedness and the class L

We are going to use the following theorem.

Theorem 3.1 (Whyburn, [6, 7.1, p. 148])**.** *Let* X *and* Y *be compact and let* $f : X \to Y$ *be open. If* X *is locally connected,* B *is any closed set in* Y*, and* R *is any component of* $Y \setminus B$*, then* $f^{-1}(R)$ *has just a finite number of components and each one of these components maps onto all of* R *under* f*.*

Lemma 3.2. *Let X be a locally connected continuum and let $f : X \to Y$ be a nonconstant open surjection. If A is a subcontinuum of X satisfying the following conditions:*

(i) *$f(A) = \{p\}$ (the image of A is one-point set), and*

(ii) *$X \setminus A$ is not connected and has infinitely many components,*

then $Y \setminus \{p\}$ is not connected and has infinitely many components.

Proof: Observe that each component of $X \setminus A$ is an open set in X ([6, 14.1, p. 20]), thus no one of them maps under f onto the point p. Hence the number of components of $X \setminus f^{-1}(p)$ is not lower than the number of components of $X \setminus A$.

Suppose on the contrary that $Y \setminus \{p\}$ is connected. Putting $B = \{p\}$ in Theorem 3.1 we see that the set $f^{-1}(Y \setminus \{p\})$ has just a finite number of components, but $f^{-1}(Y \setminus \{p\}) = X \setminus f^{-1}(p)$ has infinitely many components. Therefore $Y \setminus \{p\}$ is not connected.

Let R be a component of $Y \setminus \{p\}$. We have $f^{-1}(R) \subseteq X \setminus f^{-1}(p)$ and each component of $f^{-1}(R)$ is contained in some component of $X \setminus f^{-1}(p)$. The set $f^{-1}(R)$ has only a finite number of components and the number of components of $X \setminus f^{-1}(p)$ is infinite. Therefore the number of components of $Y \setminus \{p\}$ is infinite. $\square$

Recall that the number $\alpha(p)$ (finite or infinite) of components of the set $M \setminus \{p\}$ for any point $p \in M$ in a connected set M is called the *component number* of p in M.

Corollary 3.3. *Let X be a locally connected continuum, let $p \in X$ and the component number $\alpha(p)$ be infinite. If $f : X \to Y$ is a nonconstant open surjection, then $\alpha(f(p))$ is infinite.*

Let X be a local dendrite and $p \in X$. Then the equality $\operatorname{ord}_p X = \omega$ means that $\alpha(p)$ is infinite ([3, § 51, VII, Theorem 4, p. 303]). Thus we have

Corollary 3.4. *Let X be a local dendrite, $p \in X$, and $\operatorname{ord}_p X = \omega$. If $f : X \to Y$ is nonconstant open surjection, then $\operatorname{ord}_{f(p)} Y = \omega$.*

Remarks 3.5. (1) The assumption of Corollary 3.3 that the continuum X is locally connected, is necessary. There is a (nonlight) open mapping g from the harmonic fan F_H onto a simple triod. For the vertex v of F_H, we have $\alpha(v)$ is infinite and $\alpha(g(v)) = 3$ (see [1, Example 7.2, p. 30]).

(2) The assumption of Corollary 3.4 that X is a local dendrite is essential. K. Menger ([5, p. 179]) gave an example of a curve that is the union of two arcs with a common end point p being of order ω in the curve. Let K be the union of this curve and of four edges of the square – see Figure 4. Let f be the orthogonal projection from K onto the diagonal pq of the square. The map f is open and $\operatorname{ord}_p pq = 1$.

Theorem 3.6. *Let X be a locally connected continuum satisfying the condition:*

for every nondegenerate subcontinuum A with the empty interior in X, the set $X \setminus A$ has infinitely many components.

Let Y be a space such that for each $y \in Y$ the component number $\alpha(y)$ is finite. If $f : X \to Y$ is nonconstant open surjection, then f is light.

Proof: Suppose on the contrary that f is nonlight. Then there is a point $y \in Y$ such that the inverse image $f^{-1}(y)$ contains a nondegenerate component A. Since f is open, A has the empty interior. By the assumption, $X \setminus A$ has infinitely many components. Further $f(A) = \{y\}$, and $\alpha(y)$ is infinite by Lemma 3.2. This is a contradiction. $\square$

Example 3.7. There exists a nonlight open mapping defined on a locally connected continuum W such that for every nondegenerate subcontinuum $A \subseteq W$ with the empty interior the set $W \setminus A$ has infinitely many components.

Proof: The continuum W is contained in the product of the locally connected fan F_ω and the straight line segment I. Namely, W is the union of countably many copies of the continuum Z defined in Example 2.5. In the first leaf of the book $F_\omega \times I$, there is the continuum Z situated in such a way, that the segment $q_0^0 q_1^0$ coincides with the segment $\{v\} \times I$, where v denotes the top of the fan F_ω. In the n-th leaf, there are 2^{n-1} copies of the continuum Z located so that the union of the segments with the empty interior covers $\{v\} \times I$. In Figure 5, we have the first leaf of the book and the second one. If A is the subcontinuum of W and A has the empty interior, then $A \subseteq \{v\} \times I$, and $W \setminus A$ has infinitely many components.

The projection $f : W \to F_\omega$ is open and nonlight, because $f(\{v\} \times I) = \{v\}$. But the component number $\alpha(v)$ is infinite. $\square$

We use the symbol $\operatorname{bd} A$ to denote the boundary of a set A. Recall that a set S is said to be a *boundary set* if its complement is dense.

Theorem 3.8. *Let X be a locally connected continuum satisfying the condition:*

> *if A is a subcontinuum of X with the empty interior, then $X \setminus A$ has infinitely many components, and the boundary of each of these components, except for a finite number of them, is a boundary set in A.*

Then X belong to the class L.

Proof: Suppose that there are a space Y and a nonconstant nonlight open mapping $f : X \to Y$. Hence there are a point $p \in Y$ and a subcontinuum $A \subseteq X$ such that $f(A) = \{p\}$. Since f is open, A has the empty interior. By Lemma 3.2, the set $Y \setminus \{p\}$ has infinitely many components. Let B be such a component of $Y \setminus \{p\}$ that the boundary of each component of $f^{-1}(B)$ is a boundary set in A. By Theorem 3.1, the number of these components is finite. Denote them by $C_1, \dots, C_k$. The union of the boundaries of these components, i.e. $\bigcup\{\operatorname{bd} C_i : i \in \{1, \dots, k\}\}$, is a closed, boundary set in A. Therefore there is an open set U in X such that $U \cap A \neq \emptyset$ and $U \cap \bigcup\{C_i : i \in \{1, \dots, k\}\} = \emptyset$. Take a point $x \in U \cap A$. Then U is a neighborhood of x. Next, $f(x) = p$, but $f(U)$ is not a neighborhood of p in Y because $f(U) \cap B = \emptyset$. This contradicts the openness of f. $\square$

Let us observe that for the continua, Theorems 1.1 and 1.2 are consequences of Theorem 3.8. Indeed, if a continuum X is locally dendritic or satisfies the condition of Theorem 1.2, then the assumptions of Theorem 3.8 are satisfied.

The author cannot give any necessary condition (and if a continuum X is not locally connected – neither necessary nor a sufficient one) that a continuum X belongs to the class L.

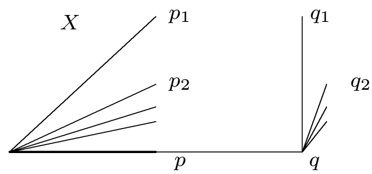

Figure 1

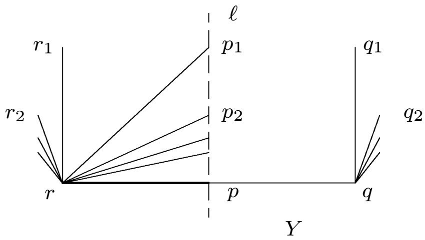

Figure 2

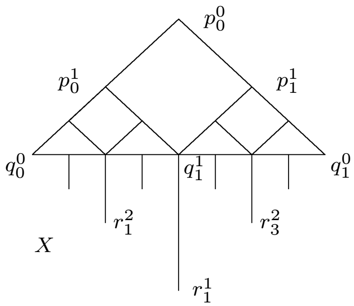

Figure 3

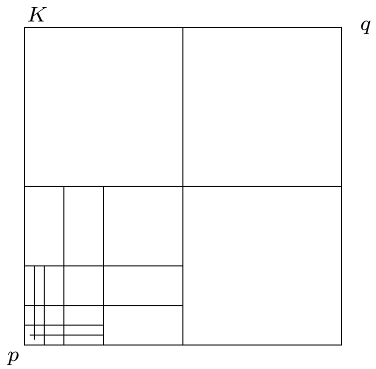

Figure 4

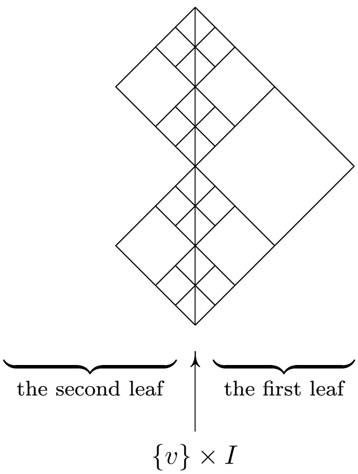

Figure 5

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(*Received* December 22, 1993)