On the range of a normal Jordan *-derivation

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Abstract. In this note, by means of the spectrum of the generating operator, we characterize the self-adjointness and closedness of the range of a normal and a self-adjoint Jordan *-derivation, respectively.

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In the last thirty years many beautiful results were obtained concerning the range of a derivation defined on the operator algebra $\mathcal{B}(H)$ on a complex infinite dimensional separable Hilbert space H (e.g. [1], [2], [3], [8], [9], mentioning only such papers which bear a closer relation to our present investigations).

In our recent paper [4], we started similar studies for the range of a Jordan *-derivation which is an additive function $J : \mathcal{R} \to \mathcal{R}$ on the *-ring $\mathcal{R}$ with the property that

$$J(a^2) = aJ(a) + J(a)a^* \qquad (a \in \mathcal{R}).$$

As for their significance we note that the excellent results of P. Šemrl show that the structure of these mappings plays essential role in the problem of representability of quasi-quadratic functionals by sesquilinear ones (cf. [5], [6]).

Concerning operator algebras, a rather straightforward computation shows that for every Jordan *-derivation $J : \mathcal{B}(H) \to \mathcal{B}(H)$ there exists an element $A \in \mathcal{B}(H)$ such that

$$J(T) = J_A(T) = TA - AT^* \qquad (T \in \mathcal{B}(H))$$

(see [7] for a much more general result). In our mentioned paper it turned out that the fact that a Jordan *-derivation is merely a real-linear operator causes some difficulties, but we were able to prove several theorems which are very common in feature with the results on derivations. To continue this work in this note we are going to deal with normal Jordan *-derivations that is the case when the generating operators are normal and, in contrast with the results of [4], here we gain statements that are rather different from their counterparts in the theory of derivations.

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We begin with the notation and some trivial observations. For every $A \in \mathcal{B}(H)$ let $\mathcal{R}_A$ denote the range of the Jordan *-derivation J_A, that is, let

$$\mathcal{R}_A = \{TA - AT^* \colon T \in \mathcal{B}(H)\}.$$

It is easy to see that $(\mathcal{R}_A)^* = \mathcal{R}_{A^*}$ and

$$i\mathcal{R}_A = \{TA + AT^* \colon T \in \mathcal{B}(H)\}.$$

The use of this set will make our proofs a little bit more elegant. In the sequel the sign of closure refers to the closure in the operator norm topology and for every $A \in \mathcal{B}(H)$, its spectrum will be denoted by $\sigma(A)$, as usual.

Our first result characterizes the self-adjointness of $\mathcal{R}_A$ (cf. [2, Corollary 2.2], [9, Theorem 3]).

Theorem 1. *If $A \in \mathcal{B}(H)$ is a normal operator, then the following statements are equivalent:*

(i) *$\mathcal{R}_A$ is self-adjoint,*
(ii) *$\overline{\mathcal{R}_A}$ is self-adjoint,*
(iii) *$\sigma(A) \subset \mathbb{R} \cup i\mathbb{R}$.*

Proof: Suppose that $A \neq 0$ and $\overline{\mathcal{R}_A}$ is self-adjoint. By the above observation we infer

$$A \in \overline{\{TA^* + A^*T^* \colon T \in \mathcal{B}(H)\}}.$$

Let $\lambda \in \sigma(A)$. Then the spectral theorem of normal operators implies that, for a fixed $0 < \epsilon < \|A\|$, there exist a unit vector $f \in H$ and an operator $0 \neq T \in \mathcal{B}(H)$ such that

$$\|Af - \lambda f\| < (\epsilon/4)\min\{1, 1/\|T\|\} \quad \text{and} \quad \|A - (TA^* + A^*T^*)\| < \epsilon/4.$$

As a consequence we have the following inequalities

$$|\langle Af, f\rangle - \lambda| < \epsilon/4$$
$$|\langle Af, f\rangle - (\langle A^*f, T^*f\rangle + \langle T^*f, Af\rangle)| < \epsilon/4$$
$$|\langle A^*f, T^*f\rangle - \langle \overline{\lambda}f, T^*f\rangle| \leq \|Af - \lambda f\|\|T\| < \epsilon/4$$
$$|\langle T^*f, Af\rangle - \langle T^*f, \lambda f\rangle| \leq \|Af - \lambda f\|\|T\| < \epsilon/4$$

which imply

$$|\lambda - \overline{\lambda}(\langle Tf, f\rangle + \langle f, Tf\rangle)| < \epsilon.$$

Since $\langle Tf, f\rangle + \langle f, Tf\rangle$ is a real number and ϵ is arbitrary, it follows that $\lambda \in \mathbb{R} \cup i\mathbb{R}$.

To the implication (iii)$\Longrightarrow$(i), let S be the operator obtained by integrating the function which is identically -1 on $i\mathbb{R}$ and 1 on $\mathbb{R} \setminus \{0\}$ with respect to the spectral measure corresponding to A. Then

$$TA^* + A^*T^* = (TS)A + A(TS)^* \qquad (T \in \mathcal{B}(H))$$

and it results in (i). □

Remark 1. It seems natural to ask whether our theorem holds true also in the case when we substitute the closure with respect to the strong (or weak) operator topology for the closure with respect to the operator norm in the statement (ii). The answer to this question is negative as the following counterexample shows (cf. [4, Proposition]). Let $(e_n)_{n\in\mathbb{Z}}$ be a complete orthonormal basis in H and let U denote the corresponding bilateral shift. If $i_0, j_0 \in \mathbb{Z}$ and $\alpha \in \mathbb{C}$, then for the operator

$$T = \sum_{k=0}^{\infty} \alpha e_{i_0+2k} \otimes e_{j_0+2k+1} + \sum_{k=0}^{\infty} \overline{\alpha} e_{j_0+2k+2} \otimes e_{i_0+2k+1}$$

we have

$$TU - UT^* = \alpha e_{i_0} \otimes e_{j_0}.$$

If Q_k denotes the orthogonal projection onto the subspace generated by the vectors $e_{-k}, \dots, e_0, \dots, e_k$ $(k = 0, 1, \dots)$, then it follows that for every operator A the relation $Q_n A Q_m \in \mathcal{R}_U$ holds true for any n, m. Consequently, the range of J_U is dense in $B(H)$ with respect to the strong (or weak) operator topology but the spectrum of U is the unit circle.

In our second theorem we consider the closedness of $\mathcal{R}_A$ (cf. [1, Theorem 3.3], [3, Corollary (4.5)], [8, Theorem 1]).

Theorem 2. *Let $A \in \mathcal{B}(H)$ be self-adjoint. Then the range $\mathcal{R}_A$ is closed in $\mathcal{B}(H)$ with respect to the operator norm topology if and only if 0 is not a limit point of $\sigma(A)$.*

Proof: First suppose that 0 is a limit point of $\sigma(A)$. In this case there is a sequence (λ_n) of pairwise different nonzero real numbers in $\sigma(A)$ converging to 0. With no loss of generality we may and do assume that the elements of this sequence are positive and less than 1. Let $g : \mathbb{R} \to \mathbb{R}$ be defined as

$$g(\lambda) = \begin{cases} 0 & \text{if} \quad \lambda \le 0 \\ \sqrt{\lambda} & \text{if} \quad \lambda > 0. \end{cases}$$

For every $n \in N$ let $g_n : \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$\lambda g_n(\lambda) \overset{n\to\infty}{\longrightarrow} g(\lambda)$$

uniformly on $\sigma(A)$. If B and T_n denote the spectral integral of $2g$ and g_n with respect to the spectral measure E of A, respectively, then we infer

$$T_n A + A T_n^* \xrightarrow{n\to\infty} B.$$

But if $\mathcal{R}_A$ is closed, it implies that there exists a $T \in \mathcal{B}(H)$ such that

$$B = TA + AT^*.$$

For every $n \in \mathbb{N}$, let I_n be the open interval with center λ_n and diameter $2(\lambda_n - (\sqrt{\lambda_n} - \lambda_n)^2)$ and define

$$P_n = E(I_n \cap \sigma(A)).$$

Then $P_n \neq 0$ and taking a unit vector f_n from its range, we have

$$\|Bf_n - \sqrt{\lambda_n} f_n\| \leq \lambda_n \quad \text{and} \quad \|Af_n - \lambda_n f_n\| \leq \lambda_n.$$

It follows that

$$|\langle Bf_n, f_n\rangle - \lambda_n(\langle Tf_n, f_n\rangle + \langle f_n, Tf_n\rangle)| \leq$$
$$|\langle TAf_n, f_n\rangle - \lambda_n\langle Tf_n, f_n\rangle| + |\langle T^* f_n, Af_n\rangle - \langle T^* f_n, \lambda_n f_n\rangle| \leq 2\|T\|\lambda_n$$

and we conclude

$$|\sqrt{\lambda_n} - \lambda_n(\langle Tf_n, f_n\rangle + \langle f_n, Tf_n\rangle)| \leq \lambda_n(1 + 2\|T\|)$$

for every n, which contradicts the boundedness of T. Consequently, $\mathcal{R}_A$ cannot be closed.

Let us now suppose that 0 is not a limit point of $\sigma(A)$. Then it is well-known that the range of A is closed and consequently, on $H = \operatorname{rng} A \oplus \ker A$, the operators

$$A \qquad \text{and} \qquad \begin{pmatrix} A_{|\operatorname{rng} A} & 0 \\ 0 & 0 \end{pmatrix}$$

can be identified, where the matrix entry $A_{|\operatorname{rng} A}$ is invertible. With this identification one can prove that

$$i\mathcal{R}_A = \left\{ \begin{pmatrix} S & B^* \\ B & 0 \end{pmatrix} : S \in \mathcal{B}(\operatorname{rng} A),\ \ S^* = S \ \text{ and } \ B \in \mathcal{B}(\operatorname{rng} A, \ker A) \right\}.$$

The closedness of $\mathcal{R}_A$ follows immediately. □

Remark 2. We note that the statement of the previous theorem is no longer valid, if A is supposed to be only normal. In fact, consider a bilateral shift U just as in Remark 1. Then 0 is not a limit point of its spectrum but the observations made there imply that the ideal of finite rank operators is contained in the closure of $\mathcal{R}_U$ and consequently, by [4, Corollary 2], the range of J_U cannot be closed in the operator norm topology.

We also remark that at the present we are not able to give such a simple characterization of normal operators that induce Jordan *-derivations with closed ranges, so we leave this problem as an open question.

Finally, as for a Caradus type statement (cf. [8, Theorem 2]) concerning Jordan *-derivations, we state our last result whose proof, that we omit, can be based on an argument similar to that was used in the proof of Theorem 2. In what follows, $\mathcal{C}(H)$ stands for the ideal of compact operators on H.

Theorem 3. *If $A \in \mathcal{B}(H)$ is self-adjoint, then $\mathcal{R}_A \cap \mathcal{C}(H) = J_A(\mathcal{C}(H))$ if and only if 0 is not a limit point of $\sigma(A)$.*

Remark 3. It is not too hard to see that normality is not sufficient to assure the validity of the previous statement. In fact, just as above, a bilateral shift gives a desired counterexample.

References

[1] Anderson J.H., Foias C., *Properties which normal operators share with normal derivations and related operators*, Pacific J. Math. **61** (1975), 313–325.

[2] Anderson J.H., Bunce J.W., Deddens J.A., Williams J.P., *C*-algebras and derivation ranges*, Acta Sci. Math. **40** (1978), 211–227.

[3] Johnson B.E., Williams J.P., *The range of a normal derivation*, Pacific J. Math. **58** (1975), 105–122.

[4] Molnár L., *The range of Jordan *-derivation*, submitted.

[5] Šemrl P., *Quadratic functionals and Jordan *-derivations*, Studia Math. **97** (1991), 157–165.

[6] ———, *Quadratic and quasi-quadratic functionals*, Proc. Amer. Math. Soc., to appear.

[7] ———, *Jordan *-derivations of standard operator algebras*, Proc. Amer. Math. Soc. **120** (1994), 515–519.

[8] Stampfli J.G., *On the range of a hyponormal derivation*, Proc. Amer. Math. Soc. **52** (1975), 117–120.

[9] ———, *On self-adjoint derivation ranges*, Pacific J. Math. **82** (1979), 257–277.

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