# Galerkin approximations for nonlinear evolution inclusions

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*Abstract.* In this paper we study the convergence properties of the Galerkin approximations to a nonlinear, nonautonomous evolution inclusion and use them to determine the structural properties of the solution set and establish the existence of periodic solutions. An example of a multivalued parabolic p.d.e. is also worked out in detail.

*Keywords:* Galerkin approximations, evolution triple, monotone operator, hemicontinuous operator, compact embedding, periodic trajectory, tangent cone, connected set, acyclic set

*Classification:* 34G99, 35B10, 35K22, 35R70

## 1. Introduction

In this paper we study the properties of the solution set of a class of nonlinear, nonautonomous evolution inclusions and also establish the existence of periodic trajectories. This is done by developing a general abstract approximation framework and a convergence theory for the Galerkin approximations of the system under consideration. We employ standard Galerkin techniques (see for example the book of Strang-Fix [11]) to obtain a sequence of approximating multivalued systems. Under readily verifiable hypotheses on the data, we demonstrate that the solutions of the finite dimensional approximations converge to those of the original infinite dimensional evolution inclusion. This approximation procedure allows us to establish certain useful properties of the solution set and also prove the existence of periodic trajectories. More precisely, we show that the solution set is compact and connected in the Lebesgue-Bochner space $L^p(T, H)$. This is done for both systems with and without state constraints. For the first as expected, we employ a tangential condition. Also using a well-known fixed point theorem for pseudo-acyclic multifunctions on the Galerkin approximations and then passing to the limit, we prove the existence of periodic solutions under a weak tangential condition. An example of a nonlinear multivalued distributed parameter system is also worked out in detail. We note that evolution inclusions are the right device to model distributed parameter control systems with a priori feedbacks, as well as other infinite dimensional systems with multivalued terms. Furthermore, our analytical framework based on Galerkin approximations can be useful in computational considerations.

*This work was done while the first author was on sabbatical leave at Florida Tech.

## 2. Preliminaries

Let $T = [0, b]$, $H$ a separable Hilbert space and $X$ a subspace of $H$ carrying the structure of a reflexive separable Banach space, which embeds into $H$ continuously and densely. Identifying $H$ with its dual (pivot space), we have $X \to H \to X^*$ with all embeddings being continuous and dense. A triple $(X, H, X^*)$ is known as an "evolution triple" (cf. Zeidler [12]). We will also assume that the embeddings are compact. By $\|\cdot\|$ (resp. $|\cdot|$, $\|\cdot\|_*$) we will denote the norm of $X$ (resp. of $H$, $X^*$). Also by $(\cdot,\cdot)$ we will denote the inner product in $H$ and by $\langle\cdot,\cdot\rangle$ the duality brackets for the pair $(X, X^*)$. The two are compatible in the sense that $(\cdot,\cdot) = \langle\cdot,\cdot\rangle|_{X\times H}$. Let $1 < p,\ q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. We define the space $W_{pq}(T) = \{x \in L^p(T, X) : \dot{x} \in L^q(T, X^*)\}$. In this definition the derivative of $x(\cdot)$ is understood in the sense of vector valued distributions. Furnished with the norm $\|x\|_{W_{pq}(T)} = [\|x\|^2_{L^p(T,X)} + \|\dot{x}\|^2_{L^q(T,X^*)}]^{1/2}$, $W_{pq}(T)$ becomes a reflexive separable Banach space. In fact, if $p = q = 2$, then $W_{22}(T) = W(T)$ is a separable Hilbert space with inner product $(x, y)_{W_{pq}(T)} = (x, y)_{L^2(T,X)} + (\dot{x}, \dot{y})_{L^2(T,X^*)}$. Recall (cf. Zeidler [12]) that $W_{pq}(T)$ embeds continuously in $C(T, H)$ and compactly in $L^p(T, H)$.

In what follows, by $P_{fc}(H)$ we will denote the family of nonempty, closed and convex subsets of $H$. A multifunction (set-valued function) $F : T \to P_{fc}(H)$ is said to be measurable, if for all $y \in H$, $t \to d(y, F(t)) = \inf\{\|y - x\| : x \in F(t)\}$ is measurable. Given a multifunction $G : H \to P_{fc}(H)$, its graph is the set $GrG = \{(x, y) \in H \times H : y \in G(x)\}$. We will say that $G(\cdot)$ is upper semicontinuous (u.s.c.), if for every $U \subseteq H$ open, the set $G^+(U) = \{x \in H : G(x) \subseteq U\}$ is open. Recall (cf. DeBlasi-Myjak [2]) that if $G(\cdot)$ is u.s.c., then $GrG$ is a closed subset of $H \times H$.

Let $Y$ be any Banach space and $P_f(Y) = \{C \subseteq Y : \text{nonempty and closed}\}$. Let $B_1 = \{y \in Y : \|y\|_Y \le 1\}$. For $C, D \in P_f(Y)$, we define

$$
\begin{aligned}
h^*(C, D) &= \inf\{\varepsilon > 0 : C \subseteq D + \varepsilon B_1\} = \sup[d(c, D) : c \in C] \\
\text{and}\ \ h^*(D, C) &= \inf\{\varepsilon > 0 : D \subseteq C + \varepsilon B_1\} = \sup[d(b, C) : b \in D].
\end{aligned}
$$

Then we set $h(C, D) = \max[h^*(C, D), h^*(D, C)]$. It is well known that $h(\cdot,\cdot)$ is a generalized metric on $P_f(Y)$ (a metric on the bounded sets in $P_f(Y)$), known as the Hausdorff metric and $(P_f(Y), h)$ is a complete metric space, with $P_{fc}(Y)$ a closed subset of it. Also if $\{C_n\}_{n\ge1} \subseteq 2^Y \setminus \{\emptyset\}$ we define

$$
\begin{aligned}
\underline{\lim}\, C_n &= \{y \in Y : \lim d(y, C_n) = 0\} \\
&= \{y \in Y : y = \lim y_n,\ y_n \in C_n,\ n \ge 1\} \\
\text{and}\ \ \overline{\lim}\, C_n &= \{y \in Y : \underline{\lim}\, d(y, C_n) = 0\} \\
&= \{y \in Y : y = \lim y_{n_k},\ y_{n_k} \in C_{n_k},\ n_1 < n_2 < \dots < n_k < \dots\}.
\end{aligned}
$$

Clearly we always have that $\underline{\lim}\, C_n \subseteq \overline{\lim}\, C_n$ and both sets are closed, maybe empty. We say that the $C_n$'s convergence to $C$ in the Kuratowski sense, denoted by $C_n \xrightarrow{K} C$, if $\underline{\lim}\, C_n = \overline{\lim}\, C_n = C$. Finally, convergence in the Hausdorff metric will be denoted by $\xrightarrow{h}$; i.e. $C_n \xrightarrow{h} C$ if and only if $h(C_n, C) \to 0$ as $n \to \infty$.

In the rest of this section we will prove some auxiliary results that we will need in the sequel.

**Proposition 1.** *If* $\{C_n\}_{n\geq 1} \subseteq P_f(Y)$, $C_n \xrightarrow{K} C$ *and there exists a nonempty compact set* $V$ *such that* $C_n \subseteq V$ *for all* $n \geq 1$, *then* $C_n \xrightarrow{h} C$ *as* $n \to \infty$.

Proof: Note that since $V$ is compact, $h^*(C_n, C) = d(c_n, C)$ with $c_n \in C_n \subseteq V$. So by passing to a subsequence if necessary, we may assume that $c_n \to c$, with $c \in C$ since by hypothesis $C_n \xrightarrow{K} C$. Hence $d(c_n, C) \to 0 \Rightarrow h^*(C_n, C) \to 0$. Also $h^*(C, C_n) = d(\widehat{c}_n, C_n)$ with $\widehat{c}_n \in C$. Again we may assume that $\widehat{c}_n \to \widehat{c}_{\in} C$. Then note that $d(\widehat{c}_n, C_n) \leq \|\widehat{c}_n - \widehat{c}\|_Y + d(\widehat{c}, C_n) \to 0$, since $C_n \xrightarrow{K} C$. So $d(\widehat{c}_n, C_n) = h^*(C, C_n) \to 0 \Rightarrow h(C_n, C) \to 0$. □

The next auxiliary result shows that connectedness is preserved by Hausdorff convergence.

**Proposition 2.** *If* $\{C_n\}_{n\geq 1} \subseteq P_k(Y)$, *for every* $n \geq 1$, $C_n$ *is connected and* $C_n \xrightarrow{h} C$ *as* $n \to \infty$, *then* $C \in P_f(Y)$ *is connected, too.*

Proof: Suppose not. Then there exists $U_1, U_2 \subseteq Y$ open such that $U_1 \cap U_2 = \emptyset$, $C \subseteq U_1 \cup U_2$ and $C \cap U_1 \neq \emptyset$, $C \cap U_2 \neq \emptyset$. Let $\varepsilon > 0$ be such that $C \subseteq \overset{\circ}{C}_\varepsilon \subseteq U_1 \cup U_2$, where $\overset{\circ}{C}_\varepsilon = \{y \in Y : d(y, C) < \varepsilon\}$. Then since $C_n \xrightarrow{h} C$, we can find $N(\varepsilon) \geq 1$ such that if $n \geq N(\varepsilon)$, $C_n \subseteq \overset{\circ}{C}_\varepsilon \subseteq U_1 \cup U_2$ and $C_n \cap U_1 \neq \emptyset$, $C_n \cap U_2 \neq \emptyset$, $\Rightarrow C_n$ is disconnected for $n \geq N(\varepsilon)$, a contradiction. □

Recall that if $C \in P_{fc}(H)$, then the metric projection map $\operatorname{proj}[\cdot; C] : H \to C$ defined by $\operatorname{proj}[x; C] = \{c \in C : |x - c| = d(x, C)\}$ is a single-valued, nonexpansive map. We have the following result:

**Proposition 3.** *If* $F : T \to P_{fc}(H)$ *is a measurable function, then* $(t, x) \to \operatorname{proj}[x; F(t)]$ *is measurable in* $t$ *and continuous and* $x$ (*i.e. a Carathéodory function*).

Proof: We only need to show the measurability in $t$. Note that $Gr \operatorname{proj}[x; F(\cdot)] = \{(t, v) \in GrF : d(x, F(t)) = |x - v|\}$. But since $F(\cdot)$ is measurable, $GrF \in B(T) \times B(H)$, with $B(T)$ (resp. $B(H)$) being the Borel $\sigma$-field of $T$ (resp. of $H$). Also for the same reason, $t \to d(x, F(t))$ is measurable. Therefore $Gr \operatorname{proj}[x; F(\cdot)] \in B(T) \times B(H)$ and so $t \to \operatorname{proj}[x; F(t)]$ is Lebesgue measurable. □

**Remark.** This proposition implies that $(t, x) \to \operatorname{proj}[x; F(t)]$ is superpositionally measurable; i.e. if $T \to H$ is measurable, then $t \to \operatorname{proj}[x(t); F(t)]$ is measurable.

Now let us introduce our Galerkin approximation scheme. For each $n \geq 1$, let $H_n$ be a finite dimensional subspace of $H$ which is also contained in $X$. Let

$p_n : H \to H_n$ be the orthogonal projection of $H$ onto $H_n$ with respect to the inner product $(\cdot,\cdot)$. We assume that the approximating spaces $H_n$ and the projections $p_n(\cdot)$ satisfy

$(*)$ "for each $x \in X$, we have $\lim \|p_n x - x\| = 0$."

Note that $(*)$ and the Uniform Boundedness Principle imply that there exists $\gamma > 0$ such that $\|p_n x - x\| \le \gamma\|x\|$ for all $n \ge 1$ and all $x \in X$. Furthermore, since $X$ embeds continuously and densely into $H$, we also have $\lim |p_n x - x| = 0$ for all $x \in H$. In what follows, by $X_n$ we will denote the linear space $H_n$ equipped with the $X$-norm (i.e. $X_n$ is considered as a subspace of $X$ rather than of $H$). Since $\dim H_n < \infty$, we see that $X_n^*$ is the space $H_n$ equipped with the $X^*$-norm.

We will be studying the following evolution inclusion defined on $T = [0, b]$ and the evolution triple $(X, H, X^*)$:

$$\left\{ \begin{array}{c} \dot{x}(t) + A(t, x(t)) \in F(t, x(t)) \text{ a.e.} \\ x(0) = x_0. \end{array} \right\} \tag{1}$$

Here $A : T \times X \to X^*$ and $F : T \times H \to P_{fc}(H)$. By a solution of (1) we mean a function $x(\cdot) \in W_{pq}(T)$ such that $\dot{x}(t) + A(t, x(t)) = f(t)$ a.e. in $X^*$, with $f \in L^q(T, H)$, $f(t) \in F(t, x(t))$ a.a. and $x(0) = x_0$. We will denote the solution set of (1) by $S \subseteq W_{pq}(T) \subseteq L^p(T, H)$.

For each $n \ge 1$, define $A_n : T \times X_n \to X_n^*$ to be the restriction of the operator $A(t,\cdot)$ to $X_n$, by $A_n(t, x) = y$ for $x \in X_n$, where $y$ satisfies

$$\langle A(t, x), v\rangle = \langle y, v\rangle \quad \text{for all} \quad v \in X_n.$$

Then from the Riesz Representation theorem, this is a well defined operator which furthermore is measurable in $t$, if $t \to A(t, x)$ is.

We consider the following sequence of Galerkin approximations to (1):

$$\left\{ \begin{array}{c} \dot{x}_n(t) + A_n(t, x_n(t)) \in p_n F(t, x_n(t)) \text{ a.e.} \\ x_n(0) = p_n x_0 = x_0^n. \end{array} \right\} \tag*{$(1)_n$}$$

We denote the solution set of $(1)_n$ by $S_n \subseteq W_{pq}(T) \subseteq L^p(T, H)$.

## 3. Convergence results

In this section, we examine how the solution sets $S_n$ approximate $S$.

We will need the following hypotheses on the data:

<u>$H(A)$</u>: $A : T \times X \to X^*$ is an operator such that

(1) $t \to A(t, x)$ is measurable,
(2) $x \to A(t, x)$ is hemicontinuous, monotone (i.e. for every $x, y, z \in X$, $\lambda \to \langle A(t, x + \lambda y), z\rangle$ is continuous from $[0, 1]$ into $\mathbb{R}$ (hemicontinuity) and $\langle A(t, x) - A(t, y), x - y\rangle \ge 0$ for all $x, y \in X$ (monotonicity)),
(3) $\langle A(t, x), x\rangle \ge c\|x\|^p$ for all $x \in X$ and almost all $t \in T$, with $c > 0$,
(4) $\|A(t, x)\|_* \le a_1(t) + c_1\|x\|^{p-1}$ for all $x \in X$ and almost all $t \in T$, with $a_1(\cdot) \in L^q(T)$, $c_1 > 0$ and $2 \le p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$.

$\underline{H(F)}$: $\quad F : T \times H \to P_{fc}(H)$ is a multifunction such that

(1) $t \to F(t, x)$ is measurable,
(2) $x \to F(t, x)$ has a graph which is sequentially closed in $H \times H_w$ with $H_w$ being the Hilbert space $H$ equipped with the weak topology,
(3) $|F(t, x)| = \sup\{|v| : v \in F(t, x)\} \le a_2(t) + c_2|x|^{2/q}$ a.e. in $t$ and for all $x \in H$, with $a_2(\cdot) \in L^q(T)$, $c_2 > 0$.

**Theorem 4.** *Iff hypotheses $H(A)$, $H(F)$ hold and $x_0 \in H$, then $\overline{\lim}\, S_n \subseteq S$ and $h^*(S_n, S) \to 0$ in $L^p(T, H)$.*

Proof: Using hypothesis $H(F)$ (1), we see that the multifunction $t \to p_n F(t, x) = G_n(t, x)$ is measurable. Also from hypothesis $H(F)$ (2) we see that the multifunction $x \to p_n F(t, x) = G_n(t, x)$ has a closed graph. Combining this with hypothesis $H(F)$ (3) we get that $G_n(t, \cdot)|_K$ is u.s.c. for every $K \subseteq X_n$ compact. Invoking Lemma 8 of Papageorgiou [8], we get that $G_n(t, \cdot)$ is u.s.c. Furthermore $(t, x) \to A_n(t, x)$ is Carathéodory, monotone in $x$ and

$$|p_n F(t, x)| \le \|p_n\|_\ell |F(t, x)| \le a_2(t) + c_2|x|^{2/q} \text{ a.e.}$$

while $\|A_n(t, x)\|_* \le a_1(t) + c_1\|x\|^{p-1}$ a.e.

So from Theorem 3.1 of Papageorgiou [10], we see that $S_n \subseteq W_{pq}(T) \subseteq L^p(T, H)$ is nonempty and closed.

Now we will establish some a priori bounds for the sets $S_n$ which are uniform in $n \ge 1$. So let $x_n(\cdot) \in S_n$, $n \ge 1$. We have:

$$\langle \dot{x}_n(t), x_n(t)\rangle + \langle A_n(t, x_n(t)), x_n(t)\rangle = (p_n f_n(t), x_n(t)) \text{ a.e.}$$
$$\Rightarrow \langle \dot{x}_n(t), x_n(t)\rangle + \langle A(t, x_n(t)), x_n(t)\rangle = (p_n f_n(t), x_n(t)) \text{ a.e.}$$

with $f_n \in L^q(T, H)$, $f_n(t) \in F(t, x_n(t))$ a.e. So we get

$$\frac{d}{dt}|x_n(t)|^2 + 2c\|x_n(t)\|^p \le 2|f_n(t)| \cdot |x_n(t)| \le 2|f_n(t)|\beta\|x_n(t)\| \text{ a.e.}$$

with $\beta > 0$ such that $|\cdot| \le \beta\|\cdot\|$. Such a $\beta > 0$ exists since $X$ embeds into $H$ continuously. Applying on the right hand side of the last inequality, Young's inequality with $\varepsilon > 0$, we get

$$\frac{d}{dt}|x_n(t)|^2 + 2c\|x_n(t)\|^p \le 2\beta\left(\frac{\varepsilon^q}{q}|f_n(t)|^q + \frac{1}{\varepsilon^p p}\|x_n(t)\|^p\right) \text{ a.e.}$$

We choose $\varepsilon > 0$ so that $\frac{2\beta}{\varepsilon^p p} = 2c \Rightarrow \varepsilon = \left(\frac{\beta}{cp}\right)^{1/p}$. Hence we get

$$\frac{d}{dt}|x_n(t)|^2 \le \widehat{c}|f_n(t)|^q \quad \text{a.e. with } \widehat{c} = \frac{2\beta}{q}\left(\frac{\beta}{cp}\right)^{p-1}$$
$$\Rightarrow \frac{d}{dt}|x_n(t)|^2 \le \widehat{c}(a_2(t) + c_2|x_n(t)|^{2/q})^q$$
$$\le 2^{q-1}\widehat{c}a_2(t) + 2^{q-1}c_2|x_n(t)|^2 \text{ a.e.}$$
$$\Rightarrow |x_n(t)|^2 \le |x_0|^2 + 2^{q-1}\widehat{c}\|a\|_q^q + 2^{q-1}c_2\int_0^t |x_n(s)|^2\, ds \quad (\text{recall } |x_0^n| \le |x_0|).$$

Apply Gronwall's lemma to get $M_1 > 0$ such that

$$|x_n(t)| \leq M_1 \quad \text{for all} \quad n \geq 1 \quad \text{and all} \quad t \in T.$$

Then we have

$$\begin{aligned} &\frac{d}{dt}|x_n(t)|^2 + 2c\|x_n(t)\|^p \leq 2M_1|f_n(t)| \quad \text{a.e.} \\ \Rightarrow\ &2c\int_0^b \|x_n(t)\|^p\,dt \leq |x_0|^2 + 2M_1\int_0^b |f_n(t)|\,dt \\ &\leq |x_0|^2 + 2M_1\int_0^b (a(t) + c_1 M_1^{2/q})\,dt. \end{aligned}$$

Hence there exists $M_2 > 0$ such that

$$\|x_n\|_{L^p(T,X)} \leq M_2 \quad \text{for all} \quad n \geq 1.$$

Finally, since $\dot{x}_n(t) = -A_n(t, x_n(t)) + f_n(t)$ a.e., and by using hypothesis $H(A)\,(4)$ and the definition of $A_n(t)$ from the above estimates, we get $M_3 > 0$ such that

$$\|\dot{x}_n\|_{L^p(T,X^*)} \leq M_3 \quad \text{for all} \quad n \geq 1.$$

Let $V = \{x \in W_{pq}(T) : \|x\|_{L^p(T,X)} \leq M_2,\ \|\dot{x}\|_{L^q(T,X^*)} \leq M_3\}$. This is a bounded closed convex subset of $W_{pq}(T)$. By the Eberlein-Smulian theorem (see for example Lakshmikantham-Leela [6, Theorem 1.1.12, p. 7]), we have that $V$ is sequentially weakly compact in $W_{pq}(T)$. Also since $W_{pq}(T)$ embeds compactly into $L^p(T,H)$ (see Zeidler [12, p. 450]), we get that $V$ is a compact subset of $L^p(T,H)$ and furthermore for all $n \geq 1$, $S_n \subseteq V$.

Now let $x_n \in S_n$, $n \geq 1$, and assume that $x_n \to x \in L^p(T,H)$. Since $\{x_n\}_{n\geq 1} \subseteq V$ by passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{w} x$ in $W_{pq}(T)$. By definition we have

$$\left\{ \begin{array}{c} \dot{x}_n(t) + A_n(t, x_n(t)) = p_n f_n(t) \text{ a.e.} \\ x_n(0) = x_0^n \end{array} \right\}$$

with $f_n(t) \in F(t, x_n(t))$ a.e., $f_n(\cdot) \in L^q(T,H)$. Since $|f_n(t)| \leq a_2(t) + c_2|x_n(t)|^{2/q} \leq a_2(t) + c_2 M_1^{2/q} = \widehat{a}_2(t)$ a.e. with $\widehat{a}_2(\cdot) \in L^q(T)$, we may assume that $f_n \xrightarrow{w} f$ in $L^q(T,H)$ (recall that $L^q(T,H)$ is a separable, reflexive Banach space). Invoking Theorem 3.1 of Papageorgiou [7], we get

$$f(t) \in \overline{\text{conv}}\, w\text{-}\overline{\lim}\{f_n(t)\}_{n\geq 1} \subseteq \overline{\text{conv}}\, w\text{-}\overline{\lim}\, F(t, x_n(t)) \subseteq F(t, x(t)) \quad \text{a.e.},$$

the last inclusion being a consequence of hypothesis $H(F)\,(2)$. In what follows, let $\widehat{A}_n(\cdot)$ be the Nemitsky (superposition) operator corresponding to $A_n(t,x)$; i.e. $\widehat{A}_n : L^p(T, X_n) \to L^q(T, X_n^*)$ is defined by $\widehat{A}_n(x)(\cdot) = A_n(\cdot, x(\cdot))$. Then if by $((\cdot,\cdot))_0$ we denote the duality brackets for the pair $(L^p(T,X),L^q(T,X^*))$ (i.e. $((f,g))_0=\int_0^b\langle f(t),g(t)\rangle\,dt$, $f\in L^p(T,X)$, $g\in L^q(T,X^*)$), we have

$$((\dot{x}_n,x_n-x))_0+((\widehat{A}(x_n),x_n-x))_0=((p_nf_n,x_n-x))_0\,.$$

From the integration by parts formula for functions in $W_{pq}(T)$ (see Zeidler [12, Proposition 23.23, p. 422]), we get that

$$\begin{aligned}((\dot{x}_n-\dot{x},x_n-x))_0&=\frac{1}{2}|x_n(b)-x(b)|^2-\frac{1}{2}|p_nx_0-x_0|^2\Rightarrow((\dot{x}_n,x_n-x))_0\\&=\frac{1}{2}|x_n(b)-x(b)|^2-\frac{1}{2}|p_nx_0-x_0|^2+((\dot{x},x_n-x))_0\to 0\ \text{ as }\ n\to\infty.\end{aligned}$$

Also we have

$$\begin{aligned}((p_nf_n,x_n-x))_0&=\int_0^b\langle p_nf_n(t),x_n(t)-x(t)\rangle\,dt=\int_0^b(p_nf_n(t),x_n(t)-x(t))\,dt\\&=\int_0^b(f_n(t),x_n(t)-p_nx(t))\,dt.\end{aligned}$$

Recall that $|p_nx(t)-x(t)|\to 0$ as $n\to\infty$. So we get

$$((p_nf_n,x_n-x))_0=\int_0^b(f_n(t),x_n(t)-p_nx(t))\,dt\to 0\ \text{ as }\ n\to\infty.$$

Thus finally we have

$$\lim((\widehat{A}_n(x_n),x_n-x))_0=0.$$

Then we write

$$\begin{aligned}&((\widehat{A}(x_n),x_n-x))_0=((\widehat{A}(x_n)-\widehat{A}_n(x_n),x_n-x))_0+((\widehat{A}_n(x_n),x_n-x))_0\\&=((\widehat{A}(x_n)-\widehat{A}_n(x_n),x_n-p_nx))_0\\&+((\widehat{A}(x_n)-\widehat{A}_n(x_n),p_nx-x))_0+((\widehat{A}_n(x_n),x_n-x))_0\\&=((\widehat{A}(x_n)-\widehat{A}_n(x_n),p_nx-x))_0+((\widehat{A}_n(x_n),x_n-x))_0\end{aligned}$$

(recall the definition of $A_n(t,x)$). Since $\|\widehat{A}(x_n)\|_{L^q(T,X^*)}$, $\|\widehat{A}_n(x_n)\|_{L^q(T,X^*)}\le M_4$ for all $n\ge 1$ and some $M_4>0$ (cf. hypothesis $H(A)\,(4)$) and $p_nx(\cdot)\to x(\cdot)$ in $L^p(T,X)$, we get

$$\lim((\widehat{A}(x_n),x_n-x))_0=0.$$

But $\widehat{A}(\cdot)$ is hemicontinuous monotone, since $A(t,\cdot)$ is. Hence it has property $(M)$ (cf. Zeidler [12, pp. 583–588]). Therefore, $\widehat{A}(x_n) \xrightarrow{w} \widehat{A}(x)$ in $L^q(T,X^*) \Rightarrow \widehat{A}_n(x_n) \xrightarrow{w} \widehat{A}(x)$ in $L^q(T,X^*)$. Therefore for every $u \in L^p(T,X)$, we have

$$\begin{aligned}
&((\dot{x}_n,u))_0 + ((\widehat{A}_n(x_n),u))_0 = ((p_n f_n,u))_0 \\
&\to ((\dot{x},u))_0 + ((\widehat{A}(x),u))_0 = ((f,u))_0 \\
&\Rightarrow \dot{x}(t) + A(t,x(t)) = f(t) \ \text{ a.e.}, x(0) = x_0 \\
&\text{with } \ f \in L^q(T,H), f(t) \in F(t,x(t)) \ \text{ a.e.}
\end{aligned}$$

Thus $x \in S$ and so we have proved that

$$\overline{\lim}\, S_n \subseteq S.$$

Recalling that $S_n \subseteq V =$ compact subset of $L^p(T,H)$, from the proof of Proposition 1, we also conclude that

$$h^*(S_n,S) \to 0 \ \text{ as } \ n \to \infty.$$

$\square$

If we strengthen our hypotheses, we can improve the conclusion of Theorem 4 above.

<u>$H(A)_1$</u>: $A : T \times X \to X^*$ is an operator such that

(1) $t \to A(t,x)$ is measurable,
(2) $x \to A(t,x)$ is continuous, monotone,
(3) $\langle A(t,x),x\rangle \geq c\|x\|^p$ for all $x \in X$ and almost all $t \in T$, with $c > 0$,
(4) $\|A(t,x)\|_* \leq a_1(t) + c_1\|x\|^{p-1}$ for all $x \in X$ and almost all $t \in T$, with $a_2(\cdot) \in L^q(T)$, $c_1 > 0$ and $2 \leq p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$.

<u>$H(F)_1$</u>: $F : T \times H \to P_{fc}(H)$ is a multifunction such that

(1) $t \to F(t,x)$ is measurable,
(2) $h(F(t,x)F(t,y)) \leq k(t)|x-y|$ a.e. with $k(\cdot) \in L^1(T)$,
(3) $|F(t,x)| \leq a_2(t) + c_2|x|^{2/q}$ a.e. with $a_2(\cdot) \in L^q(T)$, $c_2 > 0$.

**Theorem 5.** <u>*If*</u> *hypotheses $H(A)_1$, $H(F)_1$ hold and $x_0 \in H$,* <u>*then*</u> $S_n \xrightarrow[h]{K} S$ *in $L^p(T,H)$.*

Proof: From Theorem 4, we know that

$$\overline{\lim}\, S_n \subseteq S \ \text{ in } \ L^p(T,H). \tag{2}$$

In what follows we will show that we also have $S \subseteq \underline{\lim}\, S_n$ in $L^p(T,H)$.

To this end, let $x \in S$. Then by definition we have

$$\left\{\begin{array}{c} \dot{x}(t) + A(t, x(t)) = f(t) \text{ a.e.} \\ x(0) = x_0 \end{array}\right\}$$

with $f \in L^q(T, H) \in F(t, x(T))$ a.e. Define $g_n(t) = \operatorname{proj}[f(t); p_n F(t, x(t))]$ and $v_n(t, x) = \operatorname{proj}[g_n(t); p_n F(t, x)]$. From Proposition 3, we have $g_n(\cdot) \in L^q(T, H)$ and $t \to v_n(t, x)$ is measurable. Also from Theorem 3.33, p. 322 of Attouch [1], we have that $x \to v_n(t, x)$ is continuous. Then consider the following problem:

$$\left\{\begin{array}{c} \dot{x}_n(t) + A_n(t, x_n(t)) = v_n(t, x_n(t)) \text{ a.e.} \\ x_n(0) = p_n x_0 = x_0^n. \end{array}\right\} \tag{3}$$

From Papageorgiou [10], we know that problem (3) above has at least one solution $x_n(\cdot) \in W_{pq}(T)$. Then we have

$$\begin{aligned} &\langle \dot{x}(t) - \dot{x}_n(t), x(t) - x_n(t)\rangle + \langle A(t, x(t)) - A_n(t, x_n(t)), x(t) - x_n(t)\rangle \\ &= (f(t) - v_n(t, x_n(t)), x(t) - x_n(t)) \quad \text{a.e.} \\ &\Rightarrow \frac{1}{2}\frac{d}{dt}|x(t) - x_n(t)|^2 = (f(t) - v_n(t, x_n(t)), x(t) - x_n(t)) \\ &+ \langle A(t, x(t)) - A_n(t, x_n(t)), x_n(t) - x(t)\rangle \quad \text{a.e.} \\ &\Rightarrow \frac{1}{2}\frac{d}{dt}|x(t) - x_n(t)|^2 \leq \frac{1}{2}|x_0 - x_0^n|^2 \\ &+ \int_0^t (f(s) - v_n(s, x_n(s)), x_n(s) - x(s))\, ds \\ &+ \int_0^t \langle A(s, x(s)) - A_n(s, x_n(s)), x_n(s) - x(s)\rangle\, ds. \end{aligned} \tag{4}$$

We investigate the third summand in the right-hand side of the above inequality. We have:

$$\begin{aligned} &\int_0^t \langle A(s, x(s)) - A_n(s, x_n(s)), x_n(s) - x(s)\rangle\, ds \\ &= \int_0^t \langle A(s, x(s)) - A(s, p_n x(s)) + A(s, p_n x(s)) - A_n(s, x_n(s)), x_n(s) - x(s)\rangle\, ds \\ &= \int_0^t \langle A(s, x(s)) - A(s, p_n x(s)), x_n(s) - x(s)\rangle\, ds \\ &+ \int_0^t \langle A(s, p_n x(s)) - A_n(s, x_n(s)), x_n(s) - x(s)\rangle\, ds. \end{aligned}$$

Note that since $p_n x(s) \to x(s)$ in $X$, we have $A(s, p_n x(s)) \to A(s, x(s))$ in $X^*$ (cf. hypothesis $H(A)_1$ (2)). So since $\|x_n\|_{L^p(T,X)} \leq M_2$, $n \geq 1$ (check the proof of Theorem 4), we have

$$\int_0^t \langle A(s, x(s)) - A(s, p_n x(s)), x_n(s) - x(s)\rangle\, ds \to 0 \quad \text{as} \quad n \to \infty.$$

Also we have

$$\begin{aligned}
&\int_0^t \langle A(s,p_n x(s)) - A_n(s,x_n(s)), x_n(s) - x(s)\rangle\, ds \\
&= \int_0^t \langle A(s,p_n x(s)) - A_n(s,x_n(s)), x_n(s) - p_n x(s)\rangle\, ds \\
&+ \int_0^t \langle A(s,p_n x(s)) - A_n(s,x_n(s)), p_n x(s) - x(s)\rangle\, ds \\
&= \int_0^t \langle A(s,p_n x(s)) - A(s,x_n(s)), x_n(s) - p_n x(s)\rangle\, ds \\
&+ \int_0^t \langle A(s,p_n x(s)) - A_n(s,x_n(s)), p_n x(s) - x(s)\rangle\, ds \\
&\le \int_0^t \langle A(s,p_n x(s)) - A_n(s,x_n(s)), p_n x(s) - x(s)\rangle\, ds \to 0 \quad \text{as} \quad n\to\infty,
\end{aligned}$$

since $p_n x(s) \to x(s)$ in $X$ and $\{A(\cdot,p_n x(\cdot))\}_{n\ge1}$, $\{A_n(\cdot,x_n(\cdot))\}_{n\ge1}$ are both bounded in $L^q(T,X^*)$. Then going back to (4), we have

$$\begin{aligned}
&\frac{1}{2}|x(t)-x_n(t)|^2 \le \frac{1}{2}|x_0 - x_0^n|^2 + \int_0^t (f(s) - v_n(s,x_n(s)), x(s) - x_n(s))\, ds \\
&+ \int_0^t \langle A(s,x(s)) - A(s,p_n x(s)), x_n(s) - x(s)\rangle\, ds \\
&+ \int_0^t \langle A(s,p_n x(s)) - A_n(s,x_n(s)), x_n(s) - x(s)\rangle\, ds.
\end{aligned}$$

From the above convergence observations, we see that given $\varepsilon > 0$, we can find $n_0(\varepsilon) \ge 1$ such that for $n \ge n_0(\varepsilon)$ we have

$$\begin{aligned}
&|x_n(t) - x(t)|^2 \le \varepsilon + 2\int_0^t (f(s) - v_n(s,x_n(s)), x(s) - x_n(s))\, ds \\
&\le \varepsilon + 2\int_0^t |f(s) - v_n(s,x_n(s))|\cdot|x(s) - x_n(s)|\, ds \\
&\le \varepsilon + 2\int_0^t (|f(s) - v_n(s,x(s))| + |v_n(s,x(s)) - v_n(s,x_n(s))|)|x_n(s) - x(s)|\, ds \\
&\le \varepsilon + 2\int_0^t [d(f(s), p_n F(s,x(s))) + d(g_n(s), p_n F(s,x_n(s)))]|x_n(s) - x(s)|\, ds \\
&\le \varepsilon + 2\int_0^t [d(f(s), p_n F(s,x(s))) + h(p_n F(s,x(s)), p_n F(s,x_n(s)))]|x_n(s) - x(s)|\, ds \\
&\le \varepsilon + 2\int_0^t [|f(s) - p_n f(s)| + k(s)|x_n(s) - x(s)|]|x_n(s) - x(s)|\, ds.
\end{aligned}$$

Note that $|f(s) - p_n f(s)| \to 0$ as $n \to \infty$. So we can find $n_1(\varepsilon) \geq n_0(\varepsilon)$ such that for $n \geq n_1(\varepsilon)$, we have

$$|x(t) - x_n(t)|^2 \leq 2\varepsilon + 2\int_0^t k(s)|x(s) - x_n(s)|^2\,ds$$
$$\Rightarrow |x(t) - x_n(t)|^2 \leq 2\varepsilon \exp 2\|k\|_1 \quad \text{for all} \quad t \in T \quad \text{and all} \quad n \geq n_1(\varepsilon).$$

Thus $x_n \to x$ in $C(T,H)$, hence in $L^p(T,H)$. Since $x_n \in S_n$, we have

$$S \subseteq \underline{\lim}\, S_n. \tag{5}$$

From (2) and (5), we deduce that $S_n \xrightarrow{K} S$ in $L^p(T,H)$. Since $S_n \subseteq V$ with $V$ compact in $L^p(T,H)$, from Proposition 1, we also have that $S_n \xrightarrow{h} S$. $\square$

## 4. The structure of the solution set and periodic solutions

In this section we use Theorems 4 and 5 to examine the structural properties of $S$ even when state constraints are present, and to establish the existence of periodic solutions.

**Theorem 6.** *If hypotheses $H(A)_1$, $H(F)_1$ hold and $x_0 \in H$, then $S \subseteq L^p(T,H)$ is compact and connected.*

PROOF: From DeBlasi-Myjak [3], we know that for every $n \geq 1$, $S_n \subseteq C(T,H)$ is compact and connected in $L^p(T,H)$. So from Proposition 2 and Theorem 5, we conclude that $S$ is compact and connected in $L^p(T,H)$. $\square$

**Remark.** In fact, a careful reading of the proof of DeBlasi-Myjak [3] reveals that for each $n \geq 1$, $S_n$ is the Hausdorff limit of a sequence $\{S_{nm}\}_{m\geq 1} \subseteq L^p(T,H)$ of contractible sets and for all $n,m \geq 1$, $S_{nm} \subseteq K$, with $K$ being compact in $L^p(T,H)$. Hence from Theorem 5 and Corollary 1.18, p. 37 of Attouch [1], we deduce that there exists a sequence $m \to n(m)$ with $n(m) \to \infty$ as $m \to \infty$, such that $S_{n(m)m} \xrightarrow{h} S$.

We can have a similar structural result for the solution set when state constraints are present. So we consider the following problem:

$$\left\{\begin{array}{c} \dot{x}(t) + A(t,x(t)) \in F(t,x(t)) \text{ a.e.} \\ x(0) = x_0 \in K \\ x(t) \in K \quad \text{for all} \quad t \in [0,b]. \end{array}\right\} \tag{6}$$

Here $K \subseteq H$ is a nonempty, bounded, closed and convex subset of $H$ such that $K_n = p_n(K) = K \cap H_n$, $n \geq 1$. In what follows by $T'_K(x)$ we will denote the Bouligand tangent cone to $K$ at $x \in K$ in $X^*$; i.e.
$T'_K(x) = \{h \in X^* : \underline{\lim}_{\lambda\downarrow 0} \frac{d_*(x+\lambda h,K)}{\lambda} = 0\}$ with $d_*(x+\lambda h, K) = \inf\{\|x+\lambda h - k\|_* : k \in K\}$. It is well-known that this is a closed and convex cone in $X^*$.

We will need the following strong tangential condition:

$\underline{H_{\mathcal{T}}}$: for all $(t,x) \in T \times (K \cap X)$, $[F(t,x) - A(t,x)] \subseteq T'_K(x)$.

Also denote by $S(K)$ the solution set of (6). We have the following structural result:

**Theorem 7.** *If hypotheses $H(A)_1$, $H(F)_1$ and $H_{\mathcal{T}}$ hold, then $S(K)$ is compact and connected in $L^p(T,H)$.*

Proof: We claim that for every $n \geq 1$ and every $t \in T$, $x \in K \cap X_n = K_n$, we have

$$p_n F(t,x) - A_n(t,x) \subseteq T_{K_n}(x).$$

To this end, fix $(t,x) \in T \times K_n$ and let $v \in F(t,x) - A(t,x)$. Then $v = g - A(t,x)$ with $g \in F(t,x)$. We will show that $p_n g - A_n(t,x) \in T_{K_n}(x)$. So let $w \in N_{K_n}(x) = T_{K_n}(x)^- = \{h \in X_n : \langle h,u\rangle \leq 0 \text{ for all } u \in T_{K_n}(x)\} = \{h \in X_n : \langle h,x\rangle = \sup_{k \in K}\langle h, p_n k\rangle\}$ (the normal cone to $K_n$ at $x$). We have

$$\langle p_n g - A_n(t,x), w\rangle = \langle p_n g, w\rangle - \langle A_n(t,x), w\rangle$$
$$= \langle g, w\rangle - \langle A(t,x), w\rangle \quad \text{(note that } p_n w = w \text{ and recall the definition of } A_n)$$
$$= \langle g - A(t,x), w\rangle.$$

Since $w \in N_{K_n}(x)$, by definition we have

$$\langle w, p_n k\rangle \leq \langle w, x\rangle \quad \text{for all} \quad k \in K,$$
$$\Rightarrow \langle w, k\rangle \leq \langle w, x\rangle \quad \text{for all} \quad k \in K,$$
$$\Rightarrow w \in N'_K(x) = T'_K(x)^- \subseteq X.$$

Therefore we have that $\langle g - A(t,x), w\rangle \leq 0 \Rightarrow \langle p_n g - A_n(t,x), w\rangle \leq 0$ and since $w \in N_{K_n}(x)$ was arbitrary, we conclude that $p_n g - A_n(t,x) \in T_{K_n}(x)$. So indeed

$$p_n F(t,x) - A_n(t,x) \subseteq T_{K_n}(x).$$

Hence from Papageorgiou [9], we have that $S_n(K) = S_n$ and $S(K) = S$. Also from Theorem 5 we know that $S_n \xrightarrow{h} S$ and for each $n \geq 1$, $S_n$ is compact and connected in $L^p(T,H)$. Hence by Proposition 2, so is $S \subseteq L^p(T,H)$. $\square$

Finally using Theorem 4, together with a well known fixed point theorem of Eilenberg-Montgomery [4], we can establish the existence of periodic trajectories. More precisely, we consider the following problem:

$$\left\{ \begin{array}{c} \dot{x}(t) + A(t,x(t)) \in F(t,x(t)) \text{ a.e.} \\ x(0) = x(b). \end{array} \right\} \tag{7}$$

We will need the following hypotheses:

$\underline{H(K)}$: $K \subseteq H$ is a nonempty, bounded, closed and convex set such that $K_n = p_n(K) = K \cap H_n$, $n \geq 1$.

$\underline{H'_{\mathcal{T}}}$: for all $(t,x) \in T \times (K \cap X)$, we have that $[F(t,x) - A(t,x)] \cap T'_K(x) \neq \emptyset$.

**Theorem 8.** *If hypotheses $H(A)$, $H(F)$, $H(K)$ and $H'_\tau$ hold, then problem* (7) *admits a solution.*

PROOF: As in the proof of Theorem 7, we can show that for all $n \geq 1$, all $t \in T$ and all $x \in K_n$

$$[p_n F(t,x) - A_n(t,x)] \cap T_{K_n}(x) \neq \emptyset.$$

Then from Hu-Papageorgiou [5], we know that the Galerkin approximation $(1)_n$ has a nonempty set of solutions which remain in $K$ (i.e. $K$ is invariant with respect to $(1)_n$). Denote by $\widehat{S}(K_n)(x_0^n)$ this solution set. We know (cf. Hu-Papageorgiou [5]) that $\widehat{S}(K_n)$ is an $R_\delta$-compact in $L^p(T,H)$, in particular, then acyclic. Also $x_0 \to \widehat{S}(K_n)(x_0)$ is u.s.c. Therefore $\Gamma_n : K_n \to 2^{K_n} \setminus \{\emptyset\}$ defined by $\Gamma_n(v) = e_b \circ \widehat{S}(K_n)(v)$ (with $e_b(\cdot)$ being the evaluation at $b$ map; since $\widehat{S}(K_n)(v) \subseteq W_{pq}(T) \subseteq C(T,H)$, this map is well defined) is pseudo-acyclic. Apply the Eilenberg-Montgomery [4] fixed point theorem to get $v_n \in \Gamma_n(v_n)$, $n \geq 1$. Let $x_n(\cdot) \in C(T,H)$ be the trajectory for $(1)_n$ (with $x_0^n = v_n$) such that $x_n(0) = x_n(b)$. From the proof of Theorem 4 and since by hypothesis $H(K)$, $K$ is bounded, we have that $\{x_n\}_{n\geq 1}$ is bounded in $W_{pq}(T)$. So we may assume that $x_n \to x$ in $L^p(T,H)$. From Theorem 4, we know that $\dot{x}(t) + A(t,x(t)) \in F(t,x(t))$ a.e., $x(0) = x(b)$; i.e. $x(\cdot) \in W_{pq}(T)$ solves (7). $\square$

## 5. Example

We conclude this work with an example illustrating the applicability of our abstract results.

Let $Z \subseteq \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial Z = \Gamma$. We consider the following periodic multivalued distributed system ($p \geq 2$):

$$(8) \qquad \left\{ \begin{array}{c} \dfrac{\partial x}{\partial t} - \displaystyle\sum_{k=1}^{N} D_k(a_k(t,z)|D_k x|^{p-2} D_k x) = h(z) \ \text{ a.e. on } \ T \times Z \\ x(0,z) = x(b,z) \ \text{ a.e. on } \ Z, \ x|_{T\times\Gamma} = 0, \\ f_1(t,z,x(z)) \leq h(z) \leq f_2(t,z,x(z)) \ \text{ a.e.} \end{array} \right\}$$

We will need the following hypotheses on the data:

$\underline{H(a)}$: $a_k : T \times Z \to \mathbb{R}$ is a measurable function and $0 < \beta_1 \leq a_k(t,z) \leq \beta_2$ a.e. on $T \times Z$, $k \in \{1, \dots, N\}$.

$\underline{H(f)}$: $f_i : T \times Z \times \mathbb{R} \to \mathbb{R}$, $i = 1,2$, are functions such that $f_1 \leq f_2$ and

(1) $(t,z) \to f_i(t,z,x)$ is measurable,
(2) $x \to f_1(t,z,x)$ is l.s.c., while $x \to f_2(t,z,x)$ is u.s.c.,
(3) $|f_i(t,z,x)| \leq a(t,z) + c(z)|x|$ a.e., with $a(\cdot,\cdot) \in L^q(T, L^2(Z))$ and $c(\cdot) \in L^\infty(Z)$.

$\underline{H''_\tau}$: for every $x \in L^2(Z)$ with $\|x\|_{L^2(Z)} = r$, we have $\int_Z u(z)x(z)\,dz \leq \sum_{k=1}^N \int_Z a_k(t,z)|D_k x|^p\,dz$ for some $u \in L^2(Z)$, with $f_1(t,z,x(z)) \leq u(z) \leq f_2(t,z,x(z))$ a.e.

**Theorem 9.** *If hypotheses $H(a)$, $H(f)$ and $H''_T$ hold, then problem* (8) *admits a solution $x(\cdot,\cdot) \in C(T, L^2(Z)) \cap L^p(T, W_0^{1,p}(Z))$ such that $\frac{\partial x}{\partial t} \in L^q(T, W^{-1,q}(Z))$.*

PROOF: In this case the evolution triple consists of $X = W_0^{1,p}(Z)$, $H = L^2(Z)$ and $X^* = W^{-1,q}(Z)$. Then from the Sobolev embedding theorem, we know that all embeddings are continuous dense and compact.

Let $\widehat{a} : T \times X \times X \to \mathbb{R}$ be the time varying Dirichlet form defined by

$$\widehat{a}(t,x,y) = \int_Z \sum_{k=1}^{N} a_k(t,z)|D_k x|^{p-2} D_k x D_k y \, dz.$$

Applying Hölder's inequality, we get

$$\begin{aligned}|\widehat{a}(t,x,y)| &\leq \beta_2 \sum_{k=1}^{N} \Big(\int_Z |D_k x|^p \, dz\Big)^{1/q} \Big(\int_Z |D_k y|^p \, dz\Big)^{1/p} \\ &\leq \widehat{\beta}_2 \|x\|^{p-1} \|y\| \quad \text{for some} \quad \widehat{\beta}_2 > 0.\end{aligned}$$

We have just seen that $\|A(t,x)\|_* \leq \beta \|x\|^{p-1}$, $\beta > 0$. Recalling the basic inequality

$$2^{2-p}|\gamma - \delta|^p \leq (\gamma|\gamma|^{p-2} - \delta|\delta|^{p-2})(\gamma - \delta) \quad \gamma, \delta \in \mathbb{R},$$

we get that there exists $\theta > 0$ such that

$$\theta\|x - y\|^p \leq \widehat{a}(t,x,x-y) - \widehat{a}(t,y,x-y) = \langle A(t,x), x-y\rangle - \langle A(t,y), x-y\rangle.$$

Also it is clear that $x \to A(t,x)$ is continuous, while from Fubini's theorem we have that $t \to \langle A(t,x), y\rangle$ is measurable $\Rightarrow t \to A(t,x)$ is weakly measurable and since $X^* = W^{-1,q}(Z)$ is separable, from the Pettis measurability theorem, we conclude that $t \to A(t,x)$ is measurable.

Next let $F : T \times H \to P_{fc}(H)$ be defined by

$$F(t,x) = \{u \in L^2(Z) : f_1(t,z,x(z)) \leq u(z) \leq f_2(t,z,x(z)) \ \text{ a.e.}\}$$

**Claim #1.** $t \to F(t,x)$ is measurable.

Note that $GrF(\cdot,x) = \{(t,u) \in T \times H : \int_C f_1(t,z,x(z))\,dz \leq \int_C u(z)\,dz \leq \int_C f_2(t,z,x(z))\,dz$ for all $C \in B(Z) =$ Borel $\sigma$-field of $Z\}$. Recall that $B(Z)$ is countably generated, i.e. $B(Z) = \sigma(\{C_n\}_{n\geq 1})$. Let $\mathcal{L}$ be the field generated by $\{C_n\}_{n\geq 1}$. Then $\mathcal{L}$ is countable; i.e. $\mathcal{L} = \{\widehat{C}_n\}_{n\geq 1}$. So

$$\begin{aligned} GrF(\cdot,x) = \bigcap_{n\geq 1} \Big\{ &(t,u) \in T \times H : \int_{\widehat{C}_n} f_1(t,z,x(z))\,dz \leq \int_{\widehat{C}_n} u(z)\,dz \\ &\leq \int_{\widehat{C}_n} f_2(t,z,x(z))\,dz \Big\}, \\ &\Rightarrow GrF(\cdot,x) \in B(T) \times B(H) \quad \text{(Fubini's theorem)}, \\ &\Rightarrow t \to F(t,x) \ \text{ is measurable.} \end{aligned}$$

**Claim #2.** $GrF(t,\cdot) = \{[x,u] \in H \times H : u \in F(t,x)\}$ is sequentially closed in $L^2(Z) \times L^2(Z)_w$.

Let $[x_n,u_n] \xrightarrow{s\times w} [x_n,u_n] \in GrF(t,\cdot)$. We have

$$\int_C f_1(t,z,x_n(z))\,dz \le \int_C u_n(z)\,dz \le \int_C f_2(t,z,x_n(z))\,dz,\ n \ge 1,\ C \in B(Z).$$

Using hypothesis $H(f)$, together with Fatou's lemma, in the limit as $n \to \infty$, we get

$$\int_C f_1(t,z,x(z))\,dz \le \int_C u(z)\,dz \le \int_C f_2(t,z,x(z))\,dz,\ C \in B(Z),\ \Rightarrow u \in F(t,x).$$

**Claim #3.** $|F(t,x)| = \sup\{\|u\|_2 : u \in F(t,x)\} \le \widehat{a}(t) + \widehat{c}|x|^{2/q}$ a.e. with $\widehat{a} \in L^q(T)$, $\widehat{c} > 0$.

Indeed from hypothesis $H(f)$ (3), we have

$$\int_Z |f_i(t,z,x(z))|^2\,dz \le \int_Z 2a(t,z)^2\,dz + \int_Z 2c(z)^2|x(z)|^2\,dz$$
$$\le \widehat{a}(t)^2 + \widehat{c}^2|x|^2 \quad \text{with} \quad \widehat{a}(\cdot) \in L^q(T),\ \widehat{c} > 0$$
$$\Rightarrow |hf_i(t,x)(\cdot)| \le \widehat{a}(t) + \widehat{c}|x| \quad (\text{with} \quad \widehat{f_i}(t,x)(\cdot) = f_i(t,\cdot,x(\cdot))) \le \widehat{a}(t) + \widehat{c}_1|x|^{2/q}$$

(applying on the second summand of the right-hand side of the previous inequality, Young's inequality $ab \le \dfrac{\varepsilon}{p}a^p + \dfrac{1}{qe^q}b^q,\ \dfrac{1}{p} + \dfrac{1}{q} = 1,\ p = \dfrac{2}{q} > 1$).

So $F(t,x)$ satisfies hypothesis $H(F)$.

Next let $K = rB_H = \{u \in H : |v| \le r\}$. Then for each $x \in W_0^{1,p}(Z)$

$$T'_K = \begin{cases} W^{-1,q}(Z) & \text{if } |x| < r \\ \{v \in W^{-1,q}(Z) : \langle v,x\rangle \le 0\} & \text{if } |x| = r. \end{cases}$$

From hypothesis $H''_\tau$, we get that

$$[F(t,x) - A(t,x)] \cap T'_K(x) \ne \emptyset, \quad x \in K \cap X.$$

Rewrite (8) in the equivalent abstract form (6). Then apply Theorem 8 to get $x \in C(T, L^2(Z)) \cap L^p(T, W_0^{1,p}(Z))$, a solution of (8) with $\frac{\partial x}{\partial t} \in L^q(T, W^{-1,q}(Z))$. $\square$

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(*Received* February 3, 1994)