The fixed point set of open mappings on extremally disconnected spaces

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Abstract. We give an example of an extremally disconnected compact Hausdorff space with an open continuous selfmap such that the fixed point set is nonvoid and nowhere dense, resp. that there is exactly one nonisolated fixed point.

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1. Introduction

Let X be an extremally disconnected compact Hausdorff space and let $f : X \longrightarrow X$ be a continuous selfmap. We are interested in the fixed point set $Fix = \{x \in X : f(x) = x\}$. At first we will collect some known facts about Fix.

For arbitrary $T \subseteq X$, we are looking for the smallest closed $T^{\#} \supseteq T$ such that $X \backslash T^{\#}$ is invariant. We proceed by induction:

$$\begin{aligned} T_0 &= \overline{T}, \\ T_{\alpha+1} &= T_\alpha \cup f^{-1}[T_\alpha], \\ T_\alpha &= \overline{\bigcup_{\beta<\alpha} T_\beta} \qquad \text{for } \alpha \text{ a limit number.} \end{aligned}$$

There is a κ where the procedure is terminating, i.e. $T_\kappa = T_{\kappa+1}$. Put $T^{\#} = T_\kappa$. Surely, $T^{\#}$ will be the smallest closed subset of X containing T and its complement being invariant. By induction we easily prove the following:

(i) If T is invariant then $T^{\#}$ is invariant too.

(ii) If T is open then $T^{\#}$ is open and closed.

(iii) If $Fix \subseteq T^{\#}$ and T is closed then $Fix \subseteq T$.

(For proving (ii) and (iii) use the fact that X is extremally disconnected.)
Abramovich, Arenson, and Kitover [AAK] showed

(iv) $Fix^{\#}$ is open and closed.

Define now

$$X_1 = X \backslash (Fix^{\#})$$
$$X_2 = (int(Fix))^{\#}$$
$$X_3 = X \backslash (X_1 \cup X_2)$$

X_1 is invariant by the definition of the operator $^{\#}$. From (i) we infer that X_2 is invariant too. $Fix^{\#}$ is invariant by (i), $X \backslash (int(Fix))^{\#}$ is invariant by the definition of $^{\#}$, so $X_3 = Fix^{\#} \cap X \backslash (int(Fix))^{\#}$ is also invariant. $Fix^{\#}$ is open and closed by (iv), so X_1 is open and closed. X_2 is open and closed by (ii). It follows that X_3 is also open and closed. We have got a partition of X into three invariant open and closed components with

in X_1	$Fix = \emptyset$	
in X_2	Fix is open and closed	$Fix^{\#} = X_2$
in X_3	Fix is nowhere dense	$Fix^{\#} = X_3$.

As a result of this considerations we notice the following. When we are examining our situation, i.e. an extremally disconnected compact Hausdorff space with a continuous mapping on it, we can assume that Fix has one of these three properties.

In 1968 Frolík [F] proved that the fixed point set of a continuous selfmap of a compact extremally disconnected Hausdorff space is open under the assumption that f is 1-1. That means that under this condition $X_3 = \emptyset$. The natural question whether this holds true also for f is open, was posed by Abramovich, Arenson, and Kitover [AAK] and asked also by Vermeer [V]. We will give here a counterexample, namely a compact extremally disconnected Hausdorff space X and an open continuous mapping f on it such that its fixed point set Fix is nowhere dense and nonvoid. Furthermore, we construct under additional set-theoretic assumptions an example of this situation with exactly one fixed point.

It remains an open question to me, to what extent these examples are typical.

2. Preliminaries

(a) The Ellentuck space

Recall the definition [E] of the Ellentuck topology on $Y = [\omega]^{\omega}$, generated by the base

$$\mathcal{E} = \{[s, A] \colon s \in [\omega]^{<\omega}, A \in [\omega]^{\omega}, \max s < \min A\}$$

where $[s, A] = \{M \in [\omega]^{\omega} \colon s \subset M \subseteq s \cup A\}$. Note that $\mathcal{E}' = \{[s, A] \in \mathcal{E} : s \neq \emptyset\}$ is already a base. The basic open sets $[s, A]$ are open and closed.

In Y, the following strengthening of the Galvin-Prikry lemma holds. We will quote it only in a weaker form needed here.

Lemma (Ellentuck [E]). *Let $\{S_i\}_{i<n}$ be a finite cover of Y by Borel sets S_i (Borel in the sense of the Ellentuck topology). Then there is an $A \in [\omega]^{\omega}$ and an $i < n$ such that $[A]^{\omega} \subseteq S_i$.*

(b) The lemma of Krawczyk and Steprāns

Furthermore, we need the following lemma, which generalizes the lemma of the three sets from the discrete case to zerodimensional spaces.

Lemma (Krawczyk, Steprāns [KS]). *Let X be a zerodimensional compact Hausdorff space and let $f : X \longrightarrow X$ be continuous. $Fix = \emptyset$ iff there is a partition (W_1, W_2, W_3) of X into open and closed sets with $W_i \cap f[W_i] = \emptyset$.*

(c) The absolute

Let us recall the notion of the absolute. The absolute $E(Y)$ of a topological space Y is defined as the Stone space of the Boolean algebra of all regular open subsets of Y. For every regular open subset U of Y we define $E(U) = \{x \in E(Y) \colon U \in x\}$. The $E(U)$'s form a base in $E(Y)$.

A set $\mathcal{B}$ of non-void open sets is called π-base iff for any non-void open set U there is $V \in \mathcal{B}$ such that $V \subseteq U$. If $\mathcal{B}$ is a π-base of regular open sets in Y then $\{E(U) : U \in \mathcal{B}\}$ is a π-base in $E(Y)$.

Let $g : Y \longrightarrow Y$ be open and continuous. We define the absolute $E(g) : E(Y) \longrightarrow E(Y)$ of g as follows: let $E(g)(x)$ be the ultrafilter generated by $\{int(\overline{g[U]}) \colon U \in x\}$. $E(g)$ is open and continuous, too.

(d) Selective ultrafilters (see e.g. [CN])

Call a uniform ultrafilter $\mathcal{U}$ on a cardinal κ selective iff for any $\{A_\alpha\}_{\alpha<\kappa} \subset \mathcal{U}$ there is $A \in \mathcal{U}$ such that for all $\alpha, \beta \in A$, $\alpha < \beta$ it holds $\beta \in A_\alpha$. The existence of selective ultrafilters is independent from ZFC. Selective ultrafilters are κ-complete, i.e. for any $\beta < \kappa$, $\{A_\alpha\}_{\alpha<\beta} \subset \mathcal{U}$ we have $\bigcap\{A_\alpha\}_{\alpha<\beta} \in \mathcal{U}$.

3. The main construction

We are ready to formulate our main result:

Theorem 1. *There is an example of an extremally disconnected compact Hausdorff space X with an open continuous selfmap $f : X \longrightarrow X$ such that the fixed point set Fix is nowhere dense and not empty.*

PROOF: Let $Y = [\omega]^\omega$ be equipped with the Ellentuck topology. Define a map $g : Y \longrightarrow Y$ by $g(M) = M \backslash \{\min M\}$. g is continuous since $g^{-1}[[s, A]] = \bigcup\{[\{n\} \cup s, A] \colon n < \min s\}$ and open since $g[[s, A]] = [s \backslash \{\min s\}, A]$ for any $[s, A] \in \mathcal{E}'$.

Take the absolute $X = E(Y)$ of Y and the absolute $f = E(g) : X \longrightarrow X$ of the open mapping g. We have constructed a compact extremally disconnected Hausdorff space X with an open continuous map f on it.

Since $E(U) \cap f[E(U)] = E(U) \cap E(int(\overline{g[U]})) = E(U \cap int(\overline{g[U]})) = E(\emptyset) = \emptyset$ for $U \in \mathcal{E}'$ and these $E(U)$'s form a π-base, we conclude that Fix is nowhere dense.

It remains to show that $Fix \neq \emptyset$. Assume by contradiction that $Fix = \emptyset$. By the lemma of Krawczyk and Steprāns, this is equivalent with the existence of a clopen partition (W_1, W_2, W_3) of X with $W_i \cap f[W_i] = \emptyset$. Translate this to the space Y and find there regular open U_1, U_2, U_3 with $W_i = E(U_i)$. The union of the U_i's is dense in Y and $U_i \cap g[U_i] = \emptyset$. So, $\overline{U}_1 \cup \overline{U}_2 \cup \overline{U}_3$ is a finite cover of $Y = [\omega]^\omega$ by Borel sets. By the theorem of Ellentuck, there is $A \in [\omega]^\omega$ with $[A]^\omega$ contained in one element of the cover, say $[A]^\omega \subseteq \overline{U}_{i_o}$. Since $[A]^\omega = [\emptyset, A]$ is open and U_{i_0} is regular open, we have $[A]^\omega \subseteq U_{i_0}$. But now, $g[[A]^\omega] \subset [A]^\omega$ and $U_{i_0} \cap g[U_{i_0}] = \emptyset$, a contradiction. □

4. Single fixed point

We are now going to construct a compact extremally disconnected Hausdorff space and an open continuous map on it, such that there is exactly one nonisolated fixed point. The assumption of the existence of a selective uniform ultrafilter $\mathcal{U}$ on a cardinal κ is needed. We proceed along the line of the main example. Equip $Y = [\kappa]^\omega$ with the topology generated by

$$\mathcal{E}_{\mathcal{U}} = \{[s, A] : s \in [\kappa]^{<\omega},\ A \in \mathcal{U},\ \max s < \min A\}$$

where $[s, A] = \{M \in [\kappa]^\omega : s \subset M \subseteq s \cup A\}$.

We continue with two lemmas for the space Y.

Lemma 1. *Let $\mathcal{U}$ be a uniform selective ultrafilter on κ, $U \subseteq Y$ and assume that there is no $A \in \mathcal{U}$ such that $[\emptyset, A] \subseteq U$. Then there is a $C \in \mathcal{U}$ such that there are no $t \in [C]^{<\omega}$, $D \in \mathcal{U}$ with $D \subseteq C$, $\max t < \min D$ and $[t, D] \subseteq U$.*

Proof: For $s \in [\kappa]^{<\omega}$, define

$$B_s = \{\alpha < \kappa : \exists A_{s,\alpha} \in \mathcal{U} : \max s < \alpha < \min A_{s,\alpha}\ \&\ [s \cup \{\alpha\}, A_{s,\alpha}] \subseteq U\}.$$

Case 1: $B_s \in \mathcal{U}$

Since $\mathcal{U}$ is selective, we find an $A_s \in \mathcal{U}$, $A_s \subseteq B_s$ such that if $\alpha, \beta \in A_s$, $\alpha < \beta$ then $\beta \in A_{s,\alpha}$.

Case 2: $B_s \notin \mathcal{U}$

Put $A_s = \kappa \backslash B_s$. Then $A_s \in \mathcal{U}$.

For $\alpha < \kappa$, define now $A_\alpha = \bigcap\{A_s : s \in [\kappa]^{<\omega}, \max s \leq \alpha\}$. Since $\mathcal{U}$ is κ-closed, $A_\alpha \in \mathcal{U}$. Once more we can apply that $\mathcal{U}$ is selective. We find $A \in \mathcal{U}$ such that for all $\alpha, \beta \in A$, $\alpha < \beta$ we have $\beta \in A_\alpha$. Put $C = A \backslash \{\min A\}$. Surely, $C \in \mathcal{U}$.

We claim that C is as desired, i.e. there are no $t \in [C]^{<\omega}$, $D \in \mathcal{U}$ with $D \subseteq C$, $\max t < \min D$ and $[t, D] \subseteq U$. Suppose by contradiction the existence of such t, D. We can assume that t is of minimal length. $t \neq \emptyset$ by the assumption of the lemma. Put $\gamma = \max t$ and $t' = t \backslash \{\gamma\}$. If $t' \neq \emptyset$ then put $\gamma' = \max t'$ else $\gamma' = \min A$. We have $[t' \cup \{\gamma\}, D] \subseteq U$, so $\gamma \in B_{t'}$. On the other hand, $\gamma, \gamma' \in A$, $\gamma > \gamma'$ and therefore $\gamma \in A_{\gamma'} \subseteq A_{t'}$. It follows that Case 1 applies when $s = t'$.

If $E \subset \kappa$ then $E(> \alpha)$ denotes the set $\{\beta \in E : \beta > \alpha\}$. We have for any $\alpha \in D$:

$$\alpha \in D = D(> \gamma') \subseteq C(> \gamma') \subseteq A(> \gamma') \subseteq A_{\gamma'} \subseteq A_{t'} \subseteq B_{t'}$$

and therefore

$$D(>\alpha) \subseteq C(>\alpha) \subseteq A(>\alpha) \subseteq A_\alpha(>\alpha) \subseteq A_{t'}(>\alpha) \subseteq A_{t',\alpha}.$$

It follows

$$[t', D] = \bigcup\{[t' \cup \{\alpha\}, D(>\alpha)] : \alpha \in D\} \subseteq \bigcup\{[t' \cup \{\alpha\}, A_{t',\alpha}] : \alpha \in D\} \subseteq U.$$

But this is in contradiction with the minimality of the length of t. $\square$

Lemma 2. *Let $\mathcal{U}$ be a selective ultrafilter on κ, U an open subset of Y. Then there is $A \in \mathcal{U}$ such that $[\emptyset, A] \subseteq U$ or $[\emptyset, A] \cap U = \emptyset$.*

Proof: Assume that there is no $A \in \mathcal{U}$ such that $[\emptyset, A] \subseteq U$. By Lemma 1, we find $C \in \mathcal{U}$ such that there are no $t \in [C]^{<\omega}$, $D \in \mathcal{U}$ with $D \subseteq C$, $\max t < \min D$ and $[t, D] \subseteq U$. We claim that $[\emptyset, C] \cap U = \emptyset$. Assume by contradiction that $[\emptyset, C] \cap U \neq \emptyset$. U is of the form $\bigcup_{i \in I}[s_i, A_i]$ with $s_i \in [\kappa]^{<\omega}$, $A_i \in \mathcal{U}$. So there is $i \in I$ such that $[\emptyset, C] \cap [s_i, A_i] \neq \emptyset$. But then $s_i \in [C]^{<\omega}$, $A_i \cap C \in \mathcal{U}$ and $[s_i, A_i \cap C] \subseteq U$ – in contradiction to our assumption. $\square$

We are now able to prove the following theorem:

Theorem 2. *If there exists a selective uniform ultrafilter $\mathcal{U}$ on a cardinal κ then there is an example of an extremally disconnected compact Hausdorff space X with an open continuous selfmap $f : X \longrightarrow X$ such that there is exactly one fixed point which is not isolated.*

Proof: Let $Y = [\kappa]^\omega$ be equipped with the topology $\mathcal{E}_\mathcal{U}$ as above. Define $g : Y \longrightarrow Y$ by $g(M) = M \backslash \{\min M\}$. g is open and continuous by the same arguments as above. Take again the absolutes $X = E(Y)$ and $f = E(g)$. Once more, we have got a compact extremally disconnected Hausdorff space and an open continuous mapping on it, such that the fixed point set is nowhere dense.

Define $Z = \bigcap\{E([\emptyset, A]) : A \in \mathcal{U}\}$. Z is an intersection of a centered system of closed invariant sets. Hence it is nonvoid and invariant. We claim $Fix \subseteq Z$. To see this, note that for any $A, B \in \mathcal{U}$, $s \in [\kappa]^{<\omega}$ we have

$$f^{|s|}[E([s, B])] \cap E[\emptyset, A] = E([\emptyset, B]) \cap E([\emptyset, A]) = E([\emptyset, A \cap B]) \neq \emptyset.$$

This is equivalent to $E([s, B]) \cap f^{-|s|}E([\emptyset, A]) \neq \emptyset$ and therefore $E([\emptyset, A])^\# = X$. From this and the assertion (iii) of the introduction, it follows that $Fix \subseteq E([\emptyset, A])$ for any $A \in \mathcal{U}$, i.e. $Fix \subseteq Z$.

We will now show that $Z = \{x_0\}$. From this, it will be clear that x_0 is the only fixed point. Take an arbitrary regular open subset U of Y. By Lemma 2 we find $A \in \mathcal{U}$ such that $[\emptyset, A] \subseteq U$ or $[\emptyset, A] \cap U = \emptyset$. But that means in the first case $Z \subseteq E([\emptyset, A]) \subseteq E(U)$ and in the second case $Z \cap E(U) \subseteq E([\emptyset, A]) \cap E(U) = \emptyset$. Since the $E(U)'s$ form a base of X, we get $Z = \{x_0\}$. $\square$

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