On a method for a-posteriori error estimation of approximate solutions to parabolic problems

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Abstract. The aim of the paper is to derive a method for the construction of a-posteriori error estimate to approximate solutions to parabolic initial-boundary value problems. The computation of the suggested error bound requires only the computation of a finite number of systems or linear algebraic equations. These systems can be solved parallelly. It is proved that the suggested a-posteriori error estimate tends to zero if the approximation tends to the true solution.

Keywords: parabolic problem, a-posteriori error estimate

Classification: 65M

Introduction

In this note we deal with the linear parabolic initial-boundary value problem

$$\begin{aligned} u_t - \Delta u &= f_0 \text{ in } \Omega \times I, \\ u &= 0 \text{ on } \partial\Omega \times I, \\ u(0;x) &= 0 \text{ on } \Omega. \end{aligned} \tag{1}$$

We are interested in the construction of a-posteriori error estimate to the approximate solution to this problem. In the literature, there are two approaches to this problem. The first one [1] is especially designed for the estimation of the error of a finite-element approximation to (1). The second approach [2] is based on the construction of the conjugate problem to (1) and approximation of the solution to the conjugate problem. The approximations to the solution to the conjugate problem are constructed independently on the computed approximation to the original problem.

The method presented in this paper is also based on the construction of the conjugate problem but it exploits the approximate solution to the original problem in order to construct an a-posteriori error estimate. A similar construction was used in [4] in order to derive a-posteriori error estimate to some nonlinear elliptic boundary value problems. Problem (1) is considered only for the sake of simplicity. Generalization to more complex parabolic problems is sketched in Remark 4.

Notations, preliminaries, problem formulation

We suppose that $\Omega \subset R^2$ is a simply connected bounded domain with lipschitzian boundary and $f_0 \in L_2(\Omega)$. We shall adopt the following notations: $T \in R$, $T > 0$, $I = (0,T)$, $U = L_2(0,T;W_0^{1,2}(\Omega))$, $\mathbf{H} = L_2(0,T;L_2^2(\Omega))$, $V_0 = \{v \in W^{1,2}(\Omega) \mid \int_\Omega v \,\mathrm{d}x = 0\}$, $V = L_2(O,T;V_0)$. The norms in U, V, are defined by

$$\|u\| = \|\nabla u\|_{L_2(0,T;L_2^2)(\Omega)},$$

where ∇ denotes the gradient with respect to the "space" variable. The inner product in $\mathbf{H}$ will be denoted by $[.,.]$ and the duality pairing between U^* and U or V^* and V by $\langle .,. \rangle$. The adjoint of a linear continuous operator B will be denoted by B^*. Let $f \in U^*$ be the functional defined by $\langle f,u \rangle = \int_\Omega f_0 u \,\mathrm{d}x$. Let us introduce the operators

$$K \in L(U,\mathbf{H}), Ku = \nabla u,$$
$$M : \{u \in U \mid u' \in U^*, u(0) = 0\}, Mu = u',$$
$$N : \mathbf{H} \to \mathbf{H}, N\mathbf{h} = \mathbf{h} - \mathbf{w},$$

where u' is the time derivative of u in the sense of distributions and $\mathbf{w} \in \mathbf{H}$ is an arbitrary (but fixed) element satisfying $K^*\mathbf{w} = f$. Such an element can be constructed e.g. by the formula

$$\mathbf{w} = \left(-\int_0^{x_1} f_0(s,x_2)\,\mathrm{d}s, 0 \right).$$

Under the above notations, problem (1) can be formulated in the weak sense:

Find $u \in U$ satisfying

$$u' + K^*NKu = 0,\ u(0) = 0,\ \text{i.e. } u' + K^*Ku = f, u(0) = 0. \tag{2}$$

It is well known [3] that the solution to (2) exists and is unique. Moreover it can be approximated e.g. by the Galerkin-Rothe method, i.e. a sequence $u_n \in U, n = 1,2,\dots$ satisfying $u_n \to u$ in U can be constructed.

Our problem is now (as usual by the a-posteriori error estimation) to construct a sequence $c_n \in R, n = 1,2,\dots$ satisfying

$$\|u_n - u\| \le c_n \to 0, \text{ for } n \to \infty$$

if $u_n \to u$ in U. The estimate c_n is allowed to be dependent on the computed approximation u_n.

Results

From [2] we can derive the following inequality which will be the basis of our method.

Assertion 1 [2]. *For arbitrary* $u_n \in U, \mathbf{z}_n \in \mathbf{H}$ *it holds*

$$\|u_n - u\|_U \leq \|Mu_n + K^*\mathbf{z}_n\|_{U^*} + \|NKu_n - \mathbf{z}_n\|_{\mathbf{H}}. \tag{3}$$

Our problem is thus reduced to the problem of construction of a sequence $\mathbf{z}_n$ satisfying

$$\|Mu_n + K^*\mathbf{z}_n\|_{U^*} + \|NKu_n - \mathbf{z}_n\|_{\mathbf{H}} \to 0$$

if $u_n \to u$ in U. In the paper [2] we can find the following suggestion:

Assertion 2 [2]. *If* $\mathbf{z}_n \in \mathbf{H}, \mathbf{z}_n \to \mathbf{z}_0$ *in* $\mathbf{H}$ *where* $\mathbf{z}_0$ *is the solution to the problem*

$$\mathbf{z}_0 + \mathbf{w} + K\int_0^t K^*\mathbf{z}_0\, ds = 0,\ \mathbf{z}_0 \in \mathbf{H},\ \int_0^t K^*\mathbf{z}_0\, ds \in U, \tag{4}$$

then the right hand side in (3) *tends to zero.*

Introducing a new variable $\mathbf{v}(t) = \int_0^t \mathbf{z}\, \mathrm{d}s$ it is easy to show that $\mathbf{z}_0 = \mathbf{v}'$ where $\mathbf{v}$ is the solution to the problem

$$\begin{gathered} \mathbf{v}' + \mathbf{w} + KK^*\mathbf{v} = 0, \\ \mathbf{v}(0) = \mathbf{0},\ K^*\mathbf{v} \in U,\ \mathbf{v}' \in \mathbf{H}. \end{gathered} \tag{5}$$

Thus in order to estimate the error of u_n we have to approximate the solution to (4) or (5) which is again an initial-boundary value problem (i.e. of the same rate of difficulty as the original problem). Moreover proceeding in this way we do not exploit the computed approximation of u. In the paper [2] it is required that $K^*\mathbf{z}_n \in \mathbf{H}$ and $K^*\mathbf{z}_n \to Mu$ in $\mathbf{H}$ and the $\|.\|_{U^*}$ term in (3) is estimated by the more comfortable $\mathbf{H}$ norm. Here we shall proceed in another way. Our first requirement is to choose $\mathbf{z}_n$ so that the uncomfortable $\|.\|_{U^*}$ term in (3) vanishes. There are many possibilities how to satisfy this requirement. Our choice is the following one: Put $\mathbf{z}_n = NK\tilde{u}_n$, where $\tilde{u}_n \in U$ is the (unique) solution to the problem

$$K^*NK\tilde{u}_n = -Mu_n, \tag{6}$$

i.e.

$$K^*K\tilde{u}_n = K^*\mathbf{w} - Mu_n. \tag{7}$$

This $\mathbf{z}_n$ clearly satisfies $K^*\mathbf{z}_n + Mu_n = 0$ and thus we have

$$\|u_n - u\| \leq \|NKu_n - \mathbf{z}_n\|_{\mathbf{H}}.$$

The continuity of M and $(K^*K)^{-1}$ together with the uniqueness of the solution of the problem $K^*K\tilde{u} = K^*\mathbf{w} - Mu$ imply now that the right hand side in (3) tends to zero (for $\mathbf{z}_n = NK\tilde{u}_n$).

However, (7) is a linear elliptic equation in an infinite-dimensional space (more precisely for each $t \in I$ it is an elliptic equation in an infinite-dimensional space). Thus $\tilde{u}_n$ cannot be computed exactly. If we replace $\tilde{u}_n$ by its approximation $\bar{u}_n$ and put $\mathbf{z}_n = NK\bar{u}_n$ then $K^*\mathbf{z}_n + Mu_n \neq 0$ and the $\|.\|_{U^*}$ term in (3) will not vanish.

Now we shall avoid this difficulty. Let $\mathbf{w}_n \in \mathbf{H}$ be the element

$$\mathbf{w}_n = \left(\int_0^{x_1} u_n'(t; s, x_2) \,\mathrm{d}s, 0 \right)$$

satisfying the equation $K^*\mathbf{w}_n = -Mu_n$. If we denote $L \in L(V, \mathbf{H})$ the operator

$$Lv = \text{curl } v = \left(-\frac{\partial v}{\partial x_2}, \frac{\partial v}{\partial x_1} \right),$$

then we have $ImK = KerL^*$ and $ImL = KerK^*$ and thus it holds $K^*(\mathbf{w}_n + Lv) = K^*\mathbf{w}_n + K^*Lv = -Mu_n$ for arbitrary $v \in V$. Thus if $\mathbf{z} \in \mathbf{H}$ has the form $\mathbf{z} = \mathbf{w}_n + Lv$ for some $v \in V$ then

$$\|u_n - u\|_U \leq \|NKu_n - \mathbf{z}_n\|_{\mathbf{H}} = \|NKu_n - (\mathbf{w}_n + Lv)\|_{\mathbf{H}}. \tag{8}$$

Our aim is now to minimize the right hand side of (8) i.e. to choose v_n so that

$$v_n = \text{argmin}_{v \in V} \|NKu_n - (\mathbf{w}_n + Lv)\|^2_{\mathbf{H}}.$$

Let us compute

$$\begin{gathered} \|NKu_n - (\mathbf{w}_n + Lv)\|^2_{\mathbf{H}} = \|NKu_n - \mathbf{w}_n\|^2_{\mathbf{H}} - 2[Lv, NKu_n - \mathbf{w}_n] + \|Lv\|^2_{\mathbf{H}} = \\ \langle L^*Lv, v\rangle - 2\langle L^*(NKu_n - \mathbf{w}_n), v\rangle + \|NKu_n - \mathbf{w}_n\|^2_{\mathbf{H}} = \\ \langle L^*Lv, v\rangle + 2\langle L^*(\mathbf{w} + \mathbf{w}_n), v\rangle + \|NKu_n - \mathbf{w}_n\|^2_{\mathbf{H}}. \end{gathered}$$

Thus v_n is the (unique) solution to the problem

$$v \in V, \quad L^*Lv = -L^*(\mathbf{w} + \mathbf{w}_n). \tag{9}$$

Problem (9) is in fact conjugate problem in the sense of [2] to the problem (7) and thus it holds $Lv + \mathbf{w}_n = NK\tilde{u}_n (= \mathbf{z}_n)$. It means we can write

$$\|u_n - u\|_U \leq \|NKu_n - (\mathbf{w}_n + Lv_n)\|_{\mathbf{H}} \to 0. \tag{10}$$

Equation (9) is again a set of elliptic equations in an infinite-dimensional space. The aim of the following theorem is to show that it is sufficient to solve a set of elliptic equations in a finite-dimensional space.

Theorem 1. *Let $V_n, n = 1, 2, \dots$ be a sequence of finite-dimensional subspaces of V satisfying*

$$\lim_{n\to\infty} \inf_{y\in V_n} \|v - y\| = 0.$$

for $v \in V$. Put $\mathbf{z}_n = \mathbf{w}_n + L\tilde{v}_n$, where $\tilde{v}_n \in V_n$ is the (unique) solution to the (finite-dimensional) problem

$$\langle L^* L\tilde{v}_n, \phi\rangle = -\langle L^*(\mathbf{w} + \mathbf{w}_n), \phi\rangle \text{ for all } \phi \in V_n.$$

Then

$$\|u_n - u\|_U \le \|NKu_n - (\mathbf{w}_n + L\tilde{v}_n)\|_{\mathbf{H}} \to 0.$$

PROOF: Due to the continuity of $(L^*L)^{-1}$ the sequence v_n tends in V to the element $v_0 \in V$, which is the unique solution to the problem $L^*Lv_0 = -L^*(\mathbf{w} + \mathbf{w}_0)$, $\mathbf{w}_0$ being the solution to $K^*\mathbf{w}_0 = -Mu$. Since $\tilde{v}_n$ can be viewed as the Galerkin approximation to v_n we have (e.g. from [3, Theorem III.3.3])

$$\|v_n - \tilde{v}_n\| \le C\|v_n - P_nv_n\| \le C(\|P_nv_n - P_nv_0\| + \|P_nv_0 - v_0\| + \|v_0 - v_n\|) \to 0,$$

where $P_n \in L(V, V_n)$ is the orthogonal projector onto V_n. The assertion of the theorem follows now from the fact that $\|P_n\| = 1$ and from (10). □

Remark 2. If $\phi_i, i = 1, 2, ..., m$ is the basis of V_n then

$$\tilde{v}_n = \sum_{i=1}^{m} c_i\phi_i,$$

where $c_i \in R$, $i = 1, 2, \dots, m$ is the unique solution to the system of linear algebraic equations

$$\sum_{i=1}^{m}[L\phi_i, L\phi_j]c_i = -[\mathbf{w} + \mathbf{w}_n, L\phi_j],\ j = 1, \dots, m. \tag{11}$$

Remark 3. Let us emphasize that (11) represents a system of linear algebraic equations for each $t \in I$. However, if the approximation u_n of u is piecewise linear i.e. if it has the Rothe form

$$u_n(t; x) = \frac{(t_i - t)}{t_i - t_{i-1}}u_n(t_{i-1}; x) + \frac{(t - t_{i-1})}{t_i - t_{i-1}}u_n(t_i; x),$$

where $t_i = iT/p, i = 1, 2, \dots, p$ for some $p \in N$, then the right hand side of (11) is constant on (t_{i-1}, t_i) and thus it is sufficient to solve (11) only for a finite set of “parameters” $t_i \in I, i = 1, 2, \dots, p$. Moreover these equations are independent each of other and thus they can be solved parallelly.

Remark 4. The proposed method can be straightforwardly generalized for parabolic problems

$$u_t - \nabla A \nabla u = f,$$

where $A = [a_{ij}]_{i,j=1,2}, a_{ij} \in L_\infty(\Omega)$ is the matrix of coefficients which is uniformly positive definite. In the case of nonsymmetric A we can use the theory of conjugate problems for nonpotential operators. We obtain

$$\|u_n - u\| \leq \|AKu_n - \mathbf{w} - (\mathbf{w}_n + L\tilde{v}_n)\|_{\mathbf{H}} \to 0,$$

where $\tilde{v}_n \in V_n$ is the solution to the problem

$$\langle L^* A_0^{-1} L\tilde{v}_n, \phi\rangle = -\langle L^* A_0^{-1}(\mathbf{w} + \mathbf{w}_n), \phi\rangle, \text{ for all } \phi \in V_n.$$

Nonhomogeneous initial and boundary conditions can be treated by the same way.

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(*Received* July 26, 1993)