On superpositional measurability of semi-Carathéodory multifunctions

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Abstract. It is shown that product weakly measurable lower weak semi-Carathéodory multifunction is superpositionally measurable.

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1. Introduction

This note is closely related to the author's previous paper [10]. Here we deal with multifunctions of Carathéodory type but we use a concept of a weaker measurability than in [10]. This allows us to establish new facts about the relation between being product weakly measurable and being superpositionally weakly measurable for such multifunctions.

We assume the reader is familiar with basic notions concerning multifunctions. In case of need the necessary information can be found in [1]-[5] and [9].

Throughout this note:

$(T, \mathcal{A})$	is a measurable space,
X	is Polish space (i.e. X is separable and metrizable by a complete metric),
$\mathcal{B}(X)$	is a σ-field of Borel subsets of X,
$\mathcal{A} \otimes \mathcal{B}(X)$	is a product σ-field on $T \times X$ (i.e. the minimal σ-field containing all products $A \times B$ with $A \in \mathcal{A},\ B \in \mathcal{B}(X)$),
Y	is a topological space.

Let a multifunction $F : T \times X \to 2^Y$ (2^Y denotes the family of all subsets of Y) be given. We say that F is *product weakly measurable* if it is weakly $\mathcal{A} \otimes \mathcal{B}(X)$-measurable and F is *superpositionally weakly measurable* if for every weakly $\mathcal{A}$-measurable multifunction $G : T \to 2^X$ with nonempty values a multifunction $F_G : T \to 2^Y$ defined by the superposition $F_G(t) = F(t, G(t))$ is weakly $\mathcal{A}$-measurable where $F(t, G(t))$ denotes the sum of sets $F(t, x)$ when $x \in G(t)$. Further F is called *a lower* (resp. *upper*) *weak semi-Carathéodory* multifunction if $F(\cdot, x)$ is weakly $\mathcal{A}$-measurable for fixed $x \in X$ and $F(t, \cdot)$ is lower (resp. upper) semicontinuous for fixed $t \in T$.

2. Main result

The following lemma plays a key role in the proof of the main result of this note.

Lemma. *Let a multifunction $F : T \times X \to 2^Y$ be product weakly measurable and let a function $g : T \to X$ be $\mathcal{A}$-measurable. Then for every open set $U \subset Y$ the set $\{t \in T : F(t, g(t)) \cap U \neq \emptyset\}$ belongs to $\mathcal{A}$.*

Proof: Define $\varphi : T \to T \times X$ setting $\varphi(t) = (t, g(t))$. We claim that $\varphi^{-1}(M) \in \mathcal{A}$ for any $M \in \mathcal{A} \otimes \mathcal{B}(X)$. In fact, if $A \in \mathcal{A}$, $B \in \mathcal{B}(X)$, then

$$\varphi^{-1}(A \times B) = \{t \in T : (t, g(t)) \in A \times B\} = A \cap g^{-1}(B) \in \mathcal{A}.$$

Hence $\varphi^{-1}(M) \in \mathcal{A}$ for any $M \in \mathcal{A} \otimes \mathcal{B}(X)$. So, if $U \subset Y$ is an open set we get

$$\begin{aligned} F_g^-(U) &= (F \circ \varphi)^-(U) = \{t \in T : F(t, g(t)) \cap U \neq \emptyset\} \\ &= \{t \in T : (t, g(t)) \in F^-(U)\} = \varphi^{-1}(F^-(U)) \in \mathcal{A} \end{aligned}$$

as $F^-(U) \in \mathcal{A} \otimes \mathcal{B}(X)$. Thus F_g is superpositionally weakly measurable. □

Now we can prove the following

Theorem. *Every lower weak semi-Carathéodory and product weakly measurable multifunction $F : T \times X \to 2^Y$ is superpositionally weakly measurable.*

Proof: First we notice that the lower semicontinuity of $F(t, \cdot)$ implies the equality

$$\{t \in T : F(t, B) \cap A \neq \emptyset\} = \{t \in T : F(t, \bar{B}) \cap A \neq \emptyset\}$$

for every $B \subset X$ and every open $A \subset Y$ (the bar over B denotes the closure of B). Let a weakly $\mathcal{A}$-measurable multifunction $G : T \to 2^X$ with nonempty values be given. We claim that the superposition $F_G : T \to 2^Y$ is weakly $\mathcal{A}$-measurable. Indeed, because G is weakly $\mathcal{A}$-measurable then obviously the closedness of G, i.e. the multifunction $\bar{G} : T \to 2^X$ defined by $\bar{G}(t) = \overline{G(t)}$, $t \in T$, is weakly $\mathcal{A}$-measurable too (see [4, Proposition 2.6]).
Let $g_n : T \to X$, $n \in \mathbb{N}$, be a Castaing representation of $\bar{G}$ i.e. every function g_n is $\mathcal{A}$-measurable and $\bar{G}(t) = \overline{\{g_1(t), g_2(t), \dots\}}$. Fix an open set $A \in Y$. By Lemma the sets $\{t \in T : F(t, g_n(t)) \cap A \neq \emptyset\}$, $n \in \mathbb{N}$, belong to $\mathcal{A}$. Thus we have

$$\begin{aligned} F_G^-(A) &= \{t \in T : F(t, G(t)) \cap A \neq \emptyset\} = \{t \in T : F(t, \overline{G(t)}) \cap A \neq \emptyset\} \\ &= \bigcup_{n \in \mathbb{N}} \{t \in T : F(t, g_n(t)) \cap A \neq \emptyset\} \in \mathcal{A} \end{aligned}$$

and therefore the proof of weakly $\mathcal{A}$-measurability of F_G is complete. The multifunction F is superpositionally weakly measurable because G is an arbitrary multifunction. □

Final remarks

A similar result to our theorem was proved by A. Spakowski [8] under the following stronger assumptions: $F : T \times X \to 2^Y$ has relatively compact values, $F(t, \cdot)$ is (topologically) continuous and Y is a metric space. Such a multifunction F is lower weak semi-Carathéodory and product weakly measurable. The first statement is obvious. For proving the second one let us observe that the multifunction $\bar{F}$ having compact values and being topologically continuous with respect to x is also metrically continuous in x (see [2] or [5]). Hence by [6, Proposition 2.3] (or [7, Theorem 3.3]) $\bar{F}$ is weakly $\mathcal{A} \otimes \mathcal{B}(X)$-measurable and so is F. Therefore F is product weakly measurable.

The following example shows that our Theorem is not valid for upper weak semi-Carathéodory and product weakly measurable multifunction.

Example. Let T be the unit interval $[0, 1]$ equipped with the σ-field of Borel sets. Let $\mathcal{N}$ be the irrationals and let B be a closed subset of $T \times \mathcal{N}$ such that the projection $\mathrm{proj}_T(B)$ is not Borel. Recall that $\mathcal{N}$ is Polish. The multifunction $F : T \times \mathcal{N} \to 2^{\mathbb{R}}$ is defined as follows:

$$F(t,x) = \begin{cases} [0,1] & \text{if } (t,x) \in B, \\ \{0\} & \text{if } (t,x) \notin B. \end{cases}$$

It is not difficult to verify that F is weakly $\mathcal{B}(T) \otimes \mathcal{B}(\mathcal{N})$-measurable and $F(t, \cdot)$ is upper weak semicontinuous.
Let $G : T \to 2^{\mathcal{N}}$ be defined by $G(t) = \mathcal{N}$, $t \in T$. Then $\mathrm{gr}\, G$ (i.e. the graf of G) is the whole set $T \times \mathcal{N}$. For $A = (0, 1)$ we have $F^-(A) = B$ and

$$\begin{aligned} F_G^-(A) &= \{t \in T : F_G(t) \cap A \neq \emptyset\} = \{t \in T : F(t, G(t)) \cap A \neq \emptyset\} \\ &= \mathrm{proj}_T(F^-(A) \cap \mathrm{gr}\, G) = \mathrm{proj}_T(B \cap (T \times \mathcal{N})) = \mathrm{proj}_T(B) \notin \mathcal{B}(T). \end{aligned}$$

Thus the superposition F_G is not weakly $\mathcal{B}(T)$-measurable and therefore F is not superpositionally weakly measurable.

Let us observe that the triple $(T, \mathcal{B}(T); \mathcal{N})$ in the above example does not have the projection property. We say that a triple $(T, \mathcal{A}; X)$, where $(T, \mathcal{A})$ is a measurable space and X is a topological space, has the projection property if for each $A \in \mathcal{A} \otimes \mathcal{B}(X)$ $\mathrm{proj}_T(A)$ belongs to $\mathcal{A}$. If $(T, \mathcal{A}; X)$ has the projection property then for every upper weak semi-Carathéodory product weakly measurable multifunction $F : T \times X \to 2^Y$ and for every weakly measurable multifunction $G : T \to 2^X$ with nonempty closed values the superposition $F_G; T \to 2^Y$ is weakly measurable. The proof of this fact is identical to the proof of Theorem 1 [10].

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