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On uniformly smoothing stochastic operators

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Abstract. We show that a stochastic operator acting on the Banach lattice $L^1(m)$ of all m-integrable functions on $(X, \mathcal{A})$ is quasi-compact if and only if it is uniformly smoothing (see the definition below).

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Let $(X, \mathcal{A}, m)$ be a σ-finite measure space. By $\mathcal{D}$ we denote the set of all densities from $L^1(m)$, i.e. m integrable positive functions f such that $\int_X f\,dm = 1$. A linear operator $P\colon L^1(m) \to L^1(m)$ is said to be stochastic if $P(\mathcal{D}) \subseteq \mathcal{D}$.

Stochastic operators have broad applications. The reader may find appropriate references in [LM]. Among other properties, usually the asymptotic behaviour of the iterates P^n is studied. In the middle of the eighties Komornik and Lasota introduced to the theory of stochastic operators the concept of smoothness. Namely, P is said to be smoothing if

(S) there exist a set $F \in \mathcal{A}$ of finite measure and constants $0 < \eta < 1$, $0 < \delta$ such that for any $f \in \mathcal{D}$ and $E \in \mathcal{A}$ with $m(E) \leq \delta$ we have

$$\overline{\lim_{n\to\infty}} \int_{E \cup F^c} P^n f\,dm \leq \eta\,,$$

where F^c stands here and in the sequel for the complementation $X \setminus F$.

Smoothing stochastic operators have nice asymptotic properties. It is proved in [KL] that any smoothing stochastic operator P is asymptotically periodic i.e. there exist pairwise orthogonal densities $g_1, \dots, g_r$, positive functionals $\Lambda_1, \dots, \Lambda_r$ and a permutation α of the set $\{1, \dots, r\}$ such that $\lim_{n\to\infty} \| P^n f - \sum_{i=1}^r \Lambda_i(f) g_{\alpha^n(i)} \| = 0$ and $Pg_i = g_{\alpha(i)}$ $i = 1, 2, \dots, r$. In particular, for some constant d the sequence P^{nd} converges in the strong operator topology to $\sum_{i=1}^r \Lambda_i \otimes g_i$. The most general result in this direction was finally obtained by Komornik. Namely, it was

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proved in [K] that any power bounded positive, and linear operator on $L^1(m)$ is asymptotically periodic.

In this note we discuss the uniform version of (S). Following [B3] (see Problem 3, page 57) we adapt here:

Definition. Let $0 < \eta < 1$. A stochastic operator P is said to be uniformly η-smoothing if there are $F \in \mathcal{A}$ with $m(F) < \infty$, and a constant $0 < \delta$ such that for some natural n_0

$$\text{(US–}\eta\text{)} \qquad \sup_{f\in\mathcal{D}} \int\limits_{E\,\cup\,F^c} P^{n_0} f\,dm \le \eta$$

for all $E \in \mathcal{A}$ satisfying $m(E) \le \delta$.

We will show that operators satisfying (US–η), are quasi-compact. Let us recall that an operator P is quasi-compact if $\| P^n - K \| < 1$ for some compact operator K and natural n. It is known (see for instance [B2]) that quasi-compact stochastic operators P are exactly those which satisfy $\| P^{nd} - \sum\limits_{i=1}^{r} \Lambda_i \otimes g_i \| \xrightarrow[n\to\infty]{} 0$, for suitable d, r, Λ_i, and g_i. We will exploit here the characterization of quasi-compact operators obtained in [B1]. In particular we shall apply some of the results from the mentioned paper to Markov operators acting on the Banach lattice $C(\Delta)$ of all continuous functions on Δ, where Δ stands for the set of all linear and multiplicative functionals on $L^\infty(m)$ equipped with the *-weak topology, so Hausdorff and compact. We recall that a linear operator $T\colon C(\Delta) \to C(\Delta)$ is Markov if $T\mathbf{1} = \mathbf{1}$ and $Tf \ge 0$ for $f \ge 0$. The dual space to $C(\Delta)$ is identified with Radon, finite (signed) measures on Δ. The *-weak compact (nonempty) set of all probability measures μ on Δ such that $T^*\mu = \mu$ is denoted by $P_T(\Delta)$. Clearly the adjoint to P operator $T = P^*$ is markovian.

A linear operator R acting on a Banach space $\mathcal{X}$ is said to be strongly ergodic if for all $x \in X$ the Cesaro means $n^{-1}(I + R + \cdots + R^{n-1})x$ are convergent in the norm of $\mathcal{X}$. Sine's mean ergodic theorem (see [S]) provides necessary and sufficient conditions for strong ergodicity. Namely, it holds if and only if R-invariant vectors separate R^*-invariant ones. It is easy to verify that R^*-invariant vectors always separate R-invariant ones. In [B1] it is proved that a Markov operator T on $C(\Delta)$ is quasi-compact if T^* is strongly ergodic and the topological support $S(\mu)$ of any μ from $P_T(\Delta)$ is non-meager. Finally we notice that the quasi-compactness of P is equivalent to the quasi-compactness of its adjoint P^*.

Theorem. *Let P be a stochastic operator on $L^1(m)$. Then P is quasi-compact if and only if P is η-uniformly smoothing for some (for all) $0 < \eta < 1$.*

Proof: Assume that P is η-uniformly smoothing with F, n_0, η, δ as in (US–η), and let $X = \bigcup\limits_{j=1}^{\infty} X_j$ where X_j are pairwise disjoint with positive finite measure.

We assume that $X_1 = F$. Now let us define a probability measure

$$m_0 = \sum_{j=1}^{\infty} t_j m|_{X_j} \quad \text{where} \quad \sum_{j=1}^{\infty} t_j m(X_j) = 1\,, \quad \text{and} \quad t_j > 0.$$

Clearly m_0 and m are equivalent, so $L^\infty(m_0) = L^\infty(m)$. The measure m_0 may be transported on Δ by the Gelfand transform $\widehat{\ }$. Then, for any $f \in L^\infty$ we have

$$\int_X f dm_0 = \int_\Delta \widehat{f} d\widehat{m}_0$$

where $\widehat{f} \in C(\Delta)$ is the image of f by $\widehat{\ }$. By $\widetilde{\ }$ let us denote the inverse operation to $\widehat{\ }$.

First we show that measures from $P_T(\Delta)$ are absolutely continuous with respect to $\widehat{m}_0$. Since $T^* L^1(\widehat{m}_0) \subseteq L^1(\widehat{m}_0)$, it is sufficient to show that any $\widehat{\nu} \in P_T(\Delta)$ has a nonzero absolutely continuous with respect to $\widehat{m}_0$ component. If not, let us suppose that for some $\widehat{\nu} \in P_T(\Delta)$ one has $\widehat{\nu} \perp \widehat{m}_0$. Then there exists a clopen set $\widehat{U} \subseteq \Delta$ so that

$$(\star) \qquad \widehat{m}_0(\widehat{U}) < t_1 \delta \quad \text{with} \quad \widehat{\nu}(\widehat{U}) = 1.$$

Let $\widehat{f} \in C(\Delta)$ be such that $\int \widehat{f} d\widehat{m}_0 = 1$ and $T^{*n_0}(\widehat{fm_0})(\widehat{U}) > \dfrac{1}{2} + \dfrac{\eta}{2}$. We get $\int_U P^{n_0} \frac{d(\widehat{f m_0})^{\sim}}{dm} dm > \dfrac{1}{2} + \dfrac{\eta}{2} > \eta$. This implies $m(U \cap F) > \delta$, so $m_0(U \cap F) > t_1 \delta$, and finally contradicting $(\star)$ we get $\widehat{m}_0(\widehat{U}) \geq \widehat{m}_0(\widehat{U} \cap \widehat{F}) > t_1 \delta$. Therefore $P_T(\Delta) \subseteq L^1(\widehat{m}_0)$, which easily implies that the topological support of $\nu \in P_T(\Delta)$ is non-meager.

Applying Sine's mean ergodic from [S] we notice that the operator T is strongly ergodic. In particular, $A_n^* \nu = n^{-1}(I^* + T^* + \cdots + T^{*(n-1)})\nu$ is *-weak convergent. Since Δ has the Grothendieck property (*-weak convergent sequences from $C(\Delta)^*$ are weakly convergent) thus $A_n^* \nu$ is weakly convergent. But weakly convergent Cesaro means are norm convergent. Therefore, T^* is strongly ergodic. Using results of [B1] we easily obtain quasi-compactness of $T = P^*$. By Theorem 2 from [B2], there is a natural d such that P^{*nd} is convergent in the operator norm to a finite dimensional projection. This is equivalent to the norm convergence of P^{nd}, and P is quasi-compact.

To prove the opposite let us assume that a stochastic operator P is quasi-compact. For some d we have $\lim_{n\to\infty} P^{nd} = \sum_{i=1}^{r} \Lambda_i \otimes g_i$, where $g_i \in \mathcal{D}$ are pairwise orthogonal (i.e. $g_i \cdot g_j = 0$ m a.e. for $i \neq j$) and $\Lambda_i(f) = \int f\, h_i\, dm$ where $\| h_i \|_\infty \leq 1$. For a given $0 < \eta < 1$ we choose a set $F \in \mathcal{A}$ of finite measure and positive δ that if $m(E) < \delta$ then $\int_{E \cup F^c} \sum_{j=1}^{r} g_j dm < \frac{\eta}{2}$. If n is such that $\| P^{nd} - \sum_{j=1}^{r} \Lambda_j \otimes g_j \| < \frac{\eta}{2}$, then we have

$$\int_{E \cup F^c} P^{nd} f\, dm = \int_{E \cup F^c} \left(P^n f - \sum_{j=1}^{r} \lambda_j(f) g_j \right) dm + \sum_{j=1}^{r} \lambda_j(f) \int_{E \cup F^c} g_j\, dm$$

$$\leq \frac{\eta}{2} + \int_{E \cup F^c} \sum_{j=1}^{r} g_j\, dm \leq \eta$$

where f is an arbitrary density. □

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