# A generalization of Magill's Theorem for non-locally compact spaces

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*Abstract.* In the theory of compactifications, Magill's theorem that the continuous image of a remainder of a space is again a remainder is one of the most important theorems in the field. It is somewhat unfortunate that the theorem holds only in locally compact spaces. In fact, if all continuous images of a remainder are again remainders, then the space must be locally compact. This paper is a modification of Magill's result to more general spaces. This of course requires restrictions on the nature of the function.

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## Introduction

The theorem known as Magill's theorem is the following:

**Theorem** [M1]**.** *For a locally compact space $X$, the following are equivalent:*

1. *$Y$ is a remainder of $X$.*
2. *$Y$ is the continuous image of a remainder of $X$.*
3. *$Y$ is a continuous image of $\beta X \setminus X$.*

This theorem is a cornerstone of the theory of compactifications. It is easy to see that any space $X$ having the property that each continuous image of $\beta X \setminus X$ is a remainder must have a one-point compactification. This forces the space to be locally compact. It is very difficult, however, to find hypotheses involving something weaker than local compactness and still have a robust theorem. The point of this paper is to try exactly that.

## Results

All spaces considered are at least Tychonoff. Let $X$, $Y$ be spaces. We will say that $Y$ is a remainder of $X$, if there exists a Hausdorff compactification $\alpha X$ of $X$ such that $Y = \alpha X \setminus X$. We denote by $\pi_{\beta\alpha}$ the natural map from $\beta X$ onto $\alpha X$. Because of the following easily established Proposition, we can without loss of generality restrict our considerations to perfect maps.

**Proposition 1.** *Let $Y = \alpha X \setminus X$ be a remainder of $X$. Then $\pi_{\beta\alpha}|_{\beta X\setminus X} : \beta X \setminus X \to Y$ is a (surjective) perfect map.*

**Definition.** For any map $f : X \to Y$, we put
$$M_f = \cup\{f^{-1}(p) : |f^{-1}(p)| > 1\}.$$
If $X$ is compact, as would be the case if $X$ were the remainder of a locally compact space, then $\overline{M_f}$ is compact. We will say that a closed continuous map $f$ is *supercompact* if $\overline{M_f}$ is compact.

Clearly every supercompact map has compact fibers so it is perfect. One can prove that every quotient map such that $\overline{M_f}$ is compact, is also closed, hence supercompact.

For any map $f : X \to Y$, we will say that $E \subset X$ is *saturated with respect to* $f$ if $f^{-1}(f(E)) = E$.

The following lemma uses arguments similar to [M2, Theorem 13].

**Lemma 2.** *Let $K$ be a compact Hausdorff space and let $q$ be a quotient map from $K$ onto a space $Z$. Let $F$ be a closed subset of $K$ such that $M_q \subset F$ and suppose that $q(F)$ is Hausdorff. Then $Z$ is Hausdorff.*

Proof: Let $x_1, x_2$ be distinct points of $Z$. The only nontrivial case is when $x_1, x_2$ are in $G = q(F)$. Since $F$ is saturated, $G$ is closed, in $Z$ and hence normal (since it is Hausdorff and compact). Let $U_1, U_2$ be open subsets of $G$ which contains $x_1, x_2$, respectively, and such that $\overline{U_1} \cap \overline{U_2} = \emptyset$. Let $V_i$ be an open subset of $Z$ such that $V_i \cap G = U_i$, $i = 1, 2$, and let $W_1, W_2$ be disjoint open subsets of $K$ such that $q^{-1}(\overline{U_i}) \subset W_i$. Put $S_i = W_i \cap q^{-1}(V_i)$. One has $S_i \cap F = q^{-1}(U_i)$. Since $q$ is injective on $K \setminus F$, the sets $S_i$ are saturated, hence $q(S_1)$ and $q(S_2)$ are disjoint open subsets of $Z$ which contain $x_1, x_2$ respectively. □

Let $Y$ be a Hausdorff space and let $f : \beta X \setminus X \to Y$ be a quotient map. We denote by $q_f$ the map $1_X \cup f : \beta X \to X \cup Y$ and by $\tau_f$ the quotient topology induced by $q_f$ on $X \cup Y$ (the union is assumed to be disjoint). Then $(X \cup Y, \tau_f)$ is a compact (not necessarily Hausdorff) space in which the image of $X$ is dense. Suppose $(X \cup Y, \tau_f)$ is Hausdorff. Then $q_f$ is closed and so that $q_f|_X$ is a homeomorphism. Then it follows that $Y$ is a remainder of $X$.

**Proposition 3.** *Let $Y$ be a Hausdorff space and let $f : \beta X \setminus X \to Y$ be a quotient map. Then $(X \cup Y, \tau_f)$ is Hausdorff if and only if $q_f(Cl_{\beta X}(M_f))$ is Hausdorff.*

Proof: If $q_f(Cl_{\beta X}(M_f))$ is Hausdorff, then we can apply Lemma 2, putting $K = \beta X$, $Z = X \cup Y$, $q = q_f$ and $F = Cl_{\beta X}(M_f)$. □

From Proposition 3 it easily follows that:

**Theorem 4.** *Let $X, Y$ be spaces and let $f$ be a supercompact map from $\beta X \setminus X$ onto $Y$. Then $Y$ is a remainder of $X$.*

In the proof of Theorem 3.7 in [T], it is proved that, if $\beta X \setminus X$ is realcompact and $C^*$-embedded in $\beta X$, then, for each compactification $\alpha X$ of $X$, $Cl_{\beta X\setminus X}(M_{\pi_{\beta\alpha}})$ must be compact. The proof of that theorem is based on the following fact

[T, Lemma 3.6]: if $\beta X \setminus X$ is $C^*$-embedded in $\beta X$ and $\alpha X$ is any compactification of $X$, then $M_{\pi_{\beta\alpha}}$ does not contain any copy of $\mathbf{N}$ which is $C$-embedded in $\beta X \setminus X$. Now suppose that $\beta X \setminus X$ is normal and every noncompact closed subset of $\beta X \setminus X$ is not pseudocompact (for instance, let $\beta X \setminus X$ be paracompact). Also, suppose again that $\beta X \setminus X$ is $C^*$-embedded in $\beta X$. Then $Cl_{\beta X \setminus X}(M_{\pi_{\beta\alpha}})$ must be compact. Otherwise, $M_{\pi_{\beta\alpha}}$ would contain a copy of $\mathbf{N}$ $C$-embedded in $Cl_{\beta X \setminus X}(M_{\pi_{\beta\alpha}})$, hence in $\beta X \setminus X$. Then one has:

**Corollary 5.** *Suppose $\beta X \setminus X$ is $C^*$-embedded in $\beta X$ and realcompact or paracompact. Then $Y$ is a remainder of $X$ if and only if $Y$ it is a supercompact image of $\beta X \setminus X$.*

**Corollary 6.** *Suppose $X$ is not locally compact. If $Z = \beta X \setminus X$ is $C$-embedded in $\beta X$ and realcompact or paracompact, then there is a space $Y$ and a perfect surjective map $f : Z \to Y$ which cannot be the restriction to $Z$ of any $\pi_{\beta\alpha}$ from $\beta X$ to a compactification $\alpha X$ of $X$.*

Proof: The hypotheses imply that $Z$ is not pseudocompact. Let $A = \{a_n\}_{n \in \mathbf{N}} \subset Z$ a $C^*$-embedded copy of $\mathbf{N}$. Let us identify $a_{2n-1}$ and $a_{2n}$ for every $n$ and let $Y$ be the quotient space and $f$ be the quotient mapping. Clearly $f$ is closed, because, for every closed subset $F$ of $Z$, $f^{-1}(f(F)) = F \cup B$, where $B$ is a subset of $A$, so it is closed. It is easy to see that $Y$ is Hausdorff, hence $f$ is perfect. But $Cl_Z(M_f)$ is not compact, so there is no compactification $\alpha X$ such that $f = \pi_{\beta\alpha}|_Z$. $\square$

Thus, although supercompactness seems unnecessarily strong, in some cases it is necessary. It is also true [T] that given any space $Y$ there is a space $X$ such that $\beta X \setminus X = Y$ and $\beta X \setminus X$ is $C^*$-embedded in $\beta X$.

Now we will give more applications of Lemma 1 and Proposition 3.

For a compactification $\alpha X$, we put $C_\alpha(X) = \{f \in C^*(X) : f \text{ extends to } \alpha X\}$. If $f \in C_\alpha(X)$, we denote by $f^\alpha$ the unique extension of $f$ to $\alpha X$. If $\mathcal{F} \subset C_\alpha(X)$, we denote by $\mathcal{F}^\alpha$ the set of the extensions.

We will say that a family $\mathcal{F} \subset C^*(X)$ *generates* the compactification $\alpha X$ if $\mathcal{F} \subset C_\alpha(X)$ and $\mathcal{F}^\alpha$ separates points of $\alpha X$. Every $\mathcal{F} \subset C^*(X)$ which separates points from closed sets of $X$ generates a compactification of $X$.

**Theorem 7.** *Let $Y$ be a Hausdorff space. Then $Y$ is a remainder of $X$ if and only if there is a continuous map $f$ from $\beta X \setminus X$ onto $Y$ such that the family $\mathcal{G}(f) = \{g \in C^*(X) \mid g^\beta \text{ is constant on the fibers of } f\}$ separates points from closed subsets of $X$.*

Proof: If $Y = \alpha X \setminus X$ is a remainder of $X$, then $\mathcal{G}(f) = C_\alpha(X)$ separates points from closed subsets of $X$.
Conversely we will prove that $(X \cup Y, \tau_f)$ is Hausdorff. Put $G = q_f(Cl_{\beta X}(\beta X \setminus X))$. It is easy to see that $G = Cl_{(X \cup Y, \tau_f)}(Y)$. By Lemma 2, it suffices to prove that $G$ is Hausdorff. First, let $y_1, y_2 \in Y$ and let $V_1, V_2$ be disjoint open subsets of $Y$ with $y_i \in V_i$. Put $H_i = Y \setminus V_i$. Then $Cl_{(X \cup Y, \tau_f)}(H_1), Cl_{(X \cup Y, \tau_f)}(H_2)$

are (closed) subsets of $G$, whose union is $G$, and such that $y_i \notin Cl_{(X\cup Y,\tau_f)}(H_i)$, $i = 1, 2$. This proves that $y_1, y_2$ are separated by disjoint open subsets of $G$.
Now, let $\alpha X$ be the compactification generated by $\mathcal{G}(f)$. Let $j : X \cup Y \to \alpha X$ be defined by $j(x) = \pi_{\beta\alpha}(q_f^{-1}(x))$. Clearly $\pi_{\beta\alpha}$ is constant on the fibers of $f$, hence of $q_f$, and so $j$ is well defined. Since $\pi_{\beta\alpha} = j \circ q_f$, $j$ is continuous and surjective. Let $y_1, y_2$ be points of $G$ such that at least one of them is not in $Y$. Then $j(y_1), j(y_2)$ are distinct points of $\alpha X$, hence they are separated by disjoint open subsets $W_1, W_2$ of $\alpha X$. Then $j^{-1}(W_i) \cap G$, $i = 1, 2$ are open subsets of $G$ which separates $y_1, y_2$. □

**Theorem 8.** *Let $f : \beta X \setminus X \to Y$ be a perfect surjective map such that $Cl_{\beta X}(M_f) \cap X = \{x_1, \dots, x_n\}$.*
*Then the following are equivalent:*

(i) *$Y$ is the remainder of a compactification $\alpha X$ of $X$ and $f = \pi_{\beta\alpha}|_{\beta X\setminus X}$.*
(ii) *There exist pairwise disjoint open sets $U_i$ in $Cl_{\beta X}(M_f)$, such that $x_i \in U_i$, $i = 1, dots, n$ and the $f(U_i \setminus X)$ are still pairwise disjoint.*

Proof: First, we prove that (i) implies (ii). If $Y = \alpha X \setminus X$, then we can take, in $\alpha X$, pairwise disjoint open neighborhoods $V_1, \dots, V_n$ of $x_1, \dots, x_n$, respectively, and put $U_i = \pi_{\beta\alpha}^{-1}(V_i) \cap Cl_{\beta X}(M_f)$, for each $i$.
Now, we will prove the converse. Let $q_f$, $(X \cup Y, \tau_f)$ be defined as in Proposition 3. We need only to prove that $q_f(Cl_{\beta X}(M_f))$ is Hausdorff. Put $q = q_f|_{Cl_{\beta X}(M_f)}$ and $S = Cl_{\beta X\setminus X}(M_f)$, so that $Cl_{\beta X}(M_f) = S \cup \{x_1, \dots, x_n\}$. Then $S$ is locally compact and, since $f|_S$ is perfect, then $T = f(S)$ is also locally compact. Clearly, one has $q(Cl_{\beta X}(M_f) = T \cup \{x_1, \dots, x_n\}$. Let $U_i' = U_i \cap S$, $i = 1, \dots, n$, and let $K = S \setminus \bigcup_{i=1}^n U_i'$, $K_1 = f^{-1}(f(K))$. $K$ is clearly compact, hence $K_1$ is also compact. If we put $W_i = U_i' \setminus K_1$, for each $i$, then $W_i = f^{-1}(f(W_i))$, so that the $f(W_i)$ are open and pairwise disjoint in $T$. Furthermore, the $f(W_i)$ are not relatively compact in $T$. Then $f(K), f(W_1), \dots, f(W_n)$ determine an $n$-point compactification $\kappa T = T \cup \{y_1, \dots, y_n\}$ where $f(W_i) \cup \{y_i\}$ is a neighborhood of $y_i$ for each $i$. We can suppose that $y_i = x_i$, i.e. that the underlying set of $\kappa T$ is $q_f(Cl_{\beta X}(M_f)$. We want to prove that the topology of $\kappa T$ is less fine than the (quotient) topology induced by $q$. Since the two topologies coincide on $T$, we need only to prove that, for each $i$ and for each basic neighborhood of $x_i$ in $\kappa T$, the inverse image with respect to $q$ is open. Let $H$ be any compact subset of $T$. One has $q^{-1}((f(W_i) \cup \{x_i\}) \setminus H) = (W_i \cup \{x_i\}) \setminus f^{-1}(H) = U_i \setminus (K_1 \cup f^{-1}(H))$, which is open. Then $q_f(Cl_{\beta X}(M_f)$, with the quotient topology, is Hausdorff. □

**Corollary 9.** *If $Z = \beta X \setminus X$ and $|Cl_{\beta X}(Z) \setminus Z| = 1$, then every perfect image of $Z$ is a remainder of a compactification $\alpha X$ of $X$ and $f = \pi_{\beta\alpha}|_{\beta X\setminus X}$.*

Note that Corollary 9 provides a new proof to Corollary 2.3 in [R].

Even if the space $X$ satisfies the hypotheses of Corollary 6, it can happen that every perfect image of $\beta X \setminus X$ is a remainder of $X$. For instance, if $\beta X \setminus X \cong \mathbf{N}$, every perfect image of $\beta X \setminus X$ is homeomorphic to $\mathbf{N}$. But, in general, there exist perfect images of $\beta X \setminus X$ which are not remainders of $X$ (see the following example).

**Example.** As we indicated earlier it is known that, for every Tychonoff space $Z$, there exists a space $X$ such that $Z \cong \beta X \setminus X$ and $\beta X \setminus X$ is $C^*$-embedded in $\beta X$. So suppose $\beta X \setminus X \cong \mathbf{R}$ and $C^*$-embedded in $\beta X$. Then $[0, +\infty)$ is a perfect image of $\beta X \setminus X$, but it is not a remainder of $X$, because it is not a supercompact image of $\beta X \setminus X$. In fact, suppose $f : \mathbf{R} \to [0, +\infty)$ is supercompact. Let $(a, b) \subset \mathbf{R}$ be an open interval containing $M_f$. Then $f(-\infty, a]$ and $f([b, +\infty)$ are closed, connected and noncompact and they must be disjoint. But this is not possible in $[0, +\infty)$.

**Example.** In Corollary 5 we cannot drop the hypothesis that $\beta X \setminus X$ is $C^*$-embedded. In fact, let $W$ be a space such that $\beta W \setminus W \cong \mathbf{I}$ and let $p \in \beta W \setminus W$ be the point corresponding to 1 in the homeomorphism. Let $X = W \cup \{p\}$, so that $Z = \beta X \setminus X \cong [0, 1)$. Clearly $|Cl_{\beta X}(Z) \setminus Z| = 1$, hence every perfect image of $[0, 1)$ is a remainder of $X$. But, clearly, there exist perfect images of $[0, 1)$ which are not supercompact images. For instance, let $Y = \{(x, 0) : 0 \leq x < 1\} \cup (\bigcup_{n \geq 2} \{(1 - 1/n, y) : 0 \leq y \leq 1/n\} \subset \mathbf{R}^2$. It is easy to see that $Y$ is a perfect image of $[0.1)$. Suppose $f : [0, 1) \to Y$ is supercompact and surjective. Let $z = (1 - 1/m, 0) \notin f(\overline{M_f})$, which is compact. Let $K$ be a compact and connected neighborhood of $z$ which misses $f(\overline{M_f})$ and $K_1 = f^{-1}(K)$. Then $f|_{K_1}$ is a homeomorphism from $K_1$ to $K$. Therefore $K_1$ must be a closed interval, while $K$ cannot be homeomorphic to an interval, a contradiction.

**Example.** Suppose $\beta X \setminus X$ is $C^*$-embedded in $\beta X$ and homeomorphic to $[0, \omega_1)$. Let $Y$ be the quotient space of $[0, \omega_1)$ obtained by identifying $\tau + 1$ and $\tau + 2$, for each limit ordinal $\tau < \omega_1$. Clearly the quotient space is again $[0, \omega_1)$ and the quotient map $f$ is perfect, so, by Corollary 9, there exists a compactification $\alpha X$ of $X$ with $\alpha X \setminus X = [0, \omega_1)$ and $f = \pi_{\beta\alpha}|_{\beta X \setminus X}$. Clearly $f$ is not supercompact but $\alpha X \setminus X$ is still a supercompact image of $\beta X \setminus X$.

Now we rephrase a theorem by Rayburn into a form more relevant to our needs.

**Theorem 10** [R, Theorem 1.3]**.** *Let $Z = \beta X \setminus X$ and let $f : Z \to Y$ be a perfect surjective map. Then the following are equivalent:*

(i) *$Y$ is the remainder of a compactification $\alpha X$ of $X$ and $f = \pi_{\beta\alpha}|_{\beta X \setminus X}$.*

(ii) *There exists a compactification $\gamma Y$ of $Y$ and an extension $\tilde{f} : Cl_{\beta X}(Z) \to \gamma Y$ such that $\tilde{f}|_{Cl_{\beta X}(Z) \setminus Z}$ is injective.*

Note that Corollary 9 also follows from Theorem 10. The hypothesis implies that $Z$ is locally compact. If $f : Z \to Y$ is perfect and surjective, then $Y$ is also locally compact. Then we can take as $\gamma Y$ the one-point compactification of $Y$. It is easy to see that the natural extension $\tilde{f}$ of $f$ to $Cl_{\beta X}(Z)$ is continuous and, obviously, injective on $Cl_{\beta X}(Z) \setminus Z$.

**Corollary 11.** *Let $Z = \beta X \setminus X$ be $C^*$-embedded in $\beta X$ and let $f : Z \to Y$ be a perfect surjective map. Then the following are equivalent:*

(i) *$Y$ is the remainder of a compactification $\alpha X$ of $X$ and $f = \pi_{\beta\alpha}|_{\beta X \setminus X}$.*

(ii) *The extension $\tilde{f} : Cl_{\beta X}(Z) \to \beta Y$ is injective on $Cl_{\beta X}(Z) \setminus Z$.*

Proof: It is easy to see that, if $f : Z \to Y$ is perfect and surjective and there exists a compactification $\gamma Y$ of $Y$ such that the extension of $f$ from $\beta Z$ to $\gamma Y$ is injective on the remainder, then $\gamma Y = \beta Y$. We can then use the above theorem. □

These results and the theorem of Rayburn (Theorem 9) suggest the following general problem: given a perfect surjective map $f : Z \to Y$ and a compactification $\kappa Z$ of $Z$, find conditions which ensure that there exists a compactification $\gamma Y$ of $Y$ and an extension of $f$, $\tilde{f} : \kappa Z \to \gamma Y$ such that $f|_{\kappa Z \setminus Z}$ is injective. In particular: find conditions on $f$ for $\tilde{f} : \beta Z \to \beta Y$ to be injective on $\beta Z \setminus Z$. This question is certainly central to our results.

**Lemma 12.** *Let $f : Z \to Y$ be a perfect surjective map and let $\tilde{f} : \beta Z \to \beta Y$ be its extension. Suppose that*

(i) *For all closed completely separated subsets $A$ and $B$ of $Z$, there exists a relatively compact open subset $U$ of $Z$ such that $f(A \setminus U)$ and $f(B \setminus U)$ are completely separated.*

*Then*

(ii) *$\tilde{f}|_{\beta Z \setminus Z}$ is injective.*

*If $Z$ is locally compact, then* (ii) *implies* (i).

Proof: For a closed set $A$ in $Z$, let us denote by $A^*$ the set $Cl_{\beta Z}(A) \setminus A$. Since $f$ is perfect, $\tilde{f}(A^*) \subset \beta Y \setminus Y$. Now, suppose $f$ satisfies (i) and let $p, q$ be distinct points of $\beta Z \setminus Z$. Let $p \in V$, $q \in W$, where $V, W$ are open in $\beta Z$ and have disjoint closures. If we put $A = Cl_{\beta Z}(V) \cap Z$ and $B = Cl_{\beta Z}(W) \cap Z$, then $A$ and $B$ are completely separated, $p \in A^*$, $q \in B^*$. Let $U$ be as in the hypothesis, so that $p \in (A \setminus U)^*$, $q \in (B \setminus U)^*$. Furthermore, $\tilde{f}((A \setminus U)^*) \cap \tilde{f}((B \setminus U)^*) \subset \tilde{f}(Cl_{\beta Z}(A \setminus U)) \cap \tilde{f}(Cl_{\beta Z}(B \setminus U)) = Cl_{\beta Y}(f(A \setminus U)) \cap Cl_{\beta Y}(f(B \setminus U)) = \emptyset$. Then $\tilde{f}(p) \neq \tilde{f}(q)$.
Now suppose $Z$ is locally compact, and $\tilde{f}|_{\beta Z \setminus Z}$ is injective. Let $A, B$ be closed and completely separated, so that, $\tilde{f}(A^*) \cap \tilde{f}(B^*) = \emptyset$. Then $F = \tilde{f}(Cl_{\beta Z}(A)) \cap \tilde{f}(Cl_{\beta Z}(B)$ is a compact subset of $Y$. If $K = f^{-1}(F)$, $K$ is also compact. Let $U$ be a relatively compact neighbourhood of $K$. Then $Cl_{\beta Y}(f(A \setminus U)) \cap Cl_{\beta Y}(f(B \setminus U)) = \tilde{f}(Cl_{\beta Z}(A \setminus U) \cap \tilde{f}(Cl_{\beta Z}(B \setminus U) = \emptyset$. □

Thus we have:

**Theorem 13.** *Let $Z = \beta X \setminus X$ be $C^*$-embedded in $\beta X$ and let $f : Z \to Y$ be perfect and surjective. Suppose that $f$ satisfies the condition* (i) *in the above* *lemma. Then there is a compactification $\alpha X$ of $X$ with $Y = \alpha X \setminus X$ and $f = \pi_{\beta\alpha}|_{\beta X\setminus X}$. If $Z$ is locally compact, the converse is also true.*

**Lemma 14.** *Let $Z$ be paracompact and let $f : Z \to Y$ be a perfect map. Then the following are equivalent:*

(i) *$f$ is supercompact.*
(ii) *For every pair $A, B$ of disjoint closed subsets of $Z$, $f(A) \cap f(B)$ is compact.*

Proof: If $f$ is supercompact and $A, B$ are closed and disjoint, then $f(A) \cap f(B)$ is closed and it is a subset of $f(Cl_Z(M_f))$, which is compact.
Conversely, suppose that $f$ satisfies (ii), but it is not supercompact. Then $M_f$ contains a copy $N$ of $\mathbf{N}$ which is $C$-embedded in $Cl_Z(M_f)$, hence closed in $Z$. Since the fibers of $f$ are compact, $N$ meets infinitely many fibers, so we can choose an infinite set $A = \{a_n\}_{n\in\mathbf{N}} \subset F$ such that $N \cap f^{-1}(f(a_n)) = \{a_n\}$ for each $n$. Clearly $A$ is closed in $Z$. Let $B = \{b_n\}_{n\in\mathbf{N}}$ such that, for each $n$, $f(b_n) = f(a_n)$ and $b_n \neq a_n$. Then $B$ is also closed. Suppose not and let $\{b_{n_\lambda}\}$ be an ultranet in $B$ converging to $z \in Z \setminus B$. The corresponding ultranet $\{a_{n_\lambda}\}$ is nontrivial, so it converges to a point $p \in \beta Z \setminus Z$. If $\tilde{f} : \beta Z \to \beta Y$ is the extension of $f$, one has $\tilde{f}(p) = \tilde{f}(z) \in Y$. This is impossible because $f$ is perfect. Then $A$ and $B$ are disjoint closed sets such that $f(A) \cap f(B) = f(A)$ is not compact, a contradiction. □

**Example.** The above lemma is not true for normal spaces. If $Z = [0, \omega_1)$, then every closed continuous map from $Z$ into a space $Y$ satisfies (ii). In fact, if $A$ and $B$ are closed and disjoint, one of them is compact. However we have earlier seen that there are mappings $f : [0, \omega_1) \to [0, \omega_1)$ which are not supercompact.

**Note.** In the proof of Lemma 12 we have seen that, if $f$ satisfies (ii), then, for $A$ and $B$ closed and completely separated, $f(A) \cap f(B)$ is compact. To prove that, we did not use local compactness.

By the above note, Corollary 5, and Lemma 14, one has:

**Theorem 15.** *Suppose $Z = \beta X \setminus X$ is paracompact and $C^*$-embedded in $\beta X$ and let $f : Z \to Y$ be a perfect surjective map. Then the following are equivalent:*

(i) *$Y$ is the remainder of a compactification $\alpha X$ of $X$ and $f = \pi_{\beta\alpha}|_Z$.*
(ii) *If $A, B$ are disjoint closed subsets of $Z$, then $f(A) \cap f(B)$ is compact.*

Now suppose $\mathcal{F} \subset C^*(X)$, where $X$ is locally compact. For each $h \in \mathcal{F}$ let $I_h$ be a closed interval containing the image of $h$. It is known that the smallest compactification to which every element of $\mathcal{F}$ extends, denoted by $\omega_{\mathcal{F}} X$, can be constructed as follows: put on the disjoint union $X \cup \prod_{h\in\mathcal{F}} I_h$ the topology generated by sets of the form $W \cup \left(e_{\mathcal{F}}^{-1}(W) \setminus F\right)$, where $W$ is a basic open set of $\prod_{h\in\mathcal{F}} I_h$, $e_{\mathcal{F}}$ is the diagonal map of $\mathcal{F}$ and $F$ is any compact subset of $X$; $\omega_{\mathcal{F}} X$ is the closure of $X$ in that space, which is compact [CFV, Proposition 1.2].

**Lemma 16.** *Let $f : Z \to Y$ be a perfect surjective map and let $\kappa Z$ be a compactification of $Z$. Put*

$$\mathcal{F} = \{g \circ f : g \in C^*(Y) \text{ and } g \circ f \in C_\kappa(Z)\}.$$

*Then*

(i) *There exists a compactification $\gamma Y$ of $Y$ and an extension of $f$, $\tilde{f} : \kappa Z \to \gamma Y$ such that $\tilde{f}|_{\kappa Z \setminus Z}$ is injective*

*implies*

(ii) *$\mathcal{F}^\kappa$ separates points of $\kappa Z \setminus Z$.*

*If $Z$ is locally compact, then the converse is also true.*

Proof: First we prove that (i) implies (ii). Let $p, q \in \kappa Z \setminus Z$ be distinct. Then $\tilde{f}(p) \neq \tilde{f}(q)$. Therefore there exists $g \in C_\gamma(Y)$ such that $g^\gamma(\tilde{f}(p)) \neq g^\gamma(\tilde{f}(q))$. But, clearly, $g^\gamma \circ \tilde{f}$ is an extension of $g \circ f$ to $\kappa Z$, then one has $g \circ f \in \mathcal{F}$ and $(g \circ f)^\kappa(p) \neq (g \circ f)^\kappa(q)$.
Now suppose that $Z$ is locally compact and that $\mathcal{F}^\kappa$ separates points of $\kappa Z \setminus Z$. Put $\mathcal{G} = \{g \in C^*(Y) : g \circ f \in \mathcal{F}\}$. Since $f$ is surjective, $g \mapsto g \circ f$ is an injective map from $\mathcal{G}$ onto $\mathcal{F}$. Let $\gamma Y$ be the smallest compactification to which every element of $\mathcal{G}$ extends. Then $\gamma Y$ can be obtained as the closure of $Y$ in $Y \cup \prod_{g \in \mathcal{G}} I_g$ with the topology described above. The hypothesis implies that $\kappa Z$ is the smallest compactification to which every element of $\mathcal{F}$ extends. Therefore $\kappa Z$ can be obtained as the closure of $Z$ in $Z \cup \prod_{g\circ f \in \mathcal{F}} I_{g \circ f} = Z \cup \prod_{g \in \mathcal{G}} I_g$. We can define $\hat{f} : Z \cup \prod_{g \in \mathcal{G}} I_g \to Y \cup \prod_{g \in \mathcal{G}} I_g$ by $\hat{f}(p) = f(p)$ if $p \in Z$ and $\hat{f}(p) = p$ otherwise. We will prove that $\hat{f}$ is continuous. Let $U = W \cup \left(e_{\mathcal{G}}^{-1}(W) \setminus F\right)$ a basic open subset of $Y \cup \prod_{g \in \mathcal{G}} I_g$. Then $\hat{f}^{-1}(U) = W \cup \left((e_{\mathcal{G}} \circ f)^{-1}(W) \setminus f^{-1}(F)\right)$. One has $e_{\mathcal{G}} \circ f = e_{\mathcal{F}}$ and, since $f$ is perfect, $f^{-1}(F)$ is compact. Then $\hat{f}^{-1}(U)$ is a basic open set. Clearly, $\hat{f}(\kappa Z) = \gamma Y$. Then we can take $\tilde{f} = \hat{f}|_{\kappa Z}$. □

**Note.** If $\mathcal{F}$ is defined as in the above theorem, then it is easy to see that $\mathcal{F} = \{h \in C_\kappa(Z) : h \text{ is constant on the fibers of } f\}$.

The following is true for any $Z$.

**Lemma 17.** *Let $f : Z \to Y$ be a perfect surjective map and let $\tilde{f} : \beta Z \to \beta Y$ be its extension. Put*

$$\mathcal{F} = \{h \in C^*(Z) : h \text{ is constant on the fibers of } f\}.$$

*Then the following are equivalent:*

(i) *$\tilde{f}|_{\beta Z \setminus Z}$ is injective.*

(ii) *$\mathcal{F}^\beta$ separates points of $\beta Z \setminus Z$.*

PROOF: We need only to prove that (ii) implies (i). Let us identify $\beta Y$ with $\overline{e(Y)}$, where $e : Y \to \prod_{g\in C^*(Y)} I_g$ is the diagonal map of $C^*(Y)$. Let $\tilde{f} : \beta Z \to \prod_{g\in C^*(Y)} I_g$ be the extension of $f$ to $\beta Z$, which is defined by $\tilde{f}(x) = \{(g \circ f)^\beta(x)\}_{g\in C^*(Y)}$. Then $\tilde{f}(\beta Z) = \beta Y$. Furthermore, since $\{(g\circ f)^\beta : g \in C^*(Y)\} = \mathcal{F}^\beta$ separates points of $\beta Z \setminus Z$, $\tilde{f}|_{\beta Z\setminus Z}$ is injective. □

**Theorem 18.** *Let $f : Z = \beta X \setminus X \to Y$ be perfect and surjective. Put*

$$\mathcal{E} = \{h \in C(Cl_{\beta X\setminus X}(Z)) : h \textit{ is constant on the fibers of } f\}.$$

*If $Y$ is the remainder of a compactification $\alpha X$ of $X$ and $f = \pi_{\beta\alpha}|_Z$, then $\mathcal{E}$ separates points of $Cl_{\beta X}(Z)\setminus Z$. If $Z$ is locally compact, the converse is also true.*

**Theorem 19.** *Let $f : Z = \beta X \setminus X \to Y$ be perfect and surjective. Also, suppose that $Z$ is $C^*$-embedded in $\beta X$. Put*

$$\mathcal{F} = \{h \in C^*(Z) : h \textit{ is constant on the fibers of } f\}.$$

*Then the following are equivalent:*

(i) *$Y$ is the remainder of a compactification $\alpha X$ of $X$ and $f = \pi_{\beta\alpha}|_Z$.*
(ii) *$\mathcal{F}^\beta$ separates points of $\beta Z \setminus Z$.*

**Lemma 20.** *Let $f : Z \to Y$ be a perfect surjective map, and let $\kappa Z$ be a compactification of $Z$. Then the following are equivalent:*

(i) *Each $p \in \kappa Z \setminus Z$ has a local basis $\mathcal{W}_p$ such that for each $W \in \mathcal{W}_p$, $W \cap Z$ is saturated with respect to $f$.*
(ii) *There exists a compactification $\gamma Y$ of $Y$ and an extension $\tilde{f}$ of $f$, $\tilde{f} : \kappa Z \to \gamma Y$ such that $\tilde{f}|_{\kappa Z\setminus Z}$ is injective.*

PROOF: First we prove that (i) implies (ii). Put $S = \kappa Z \setminus Z$ and consider the disjoint union $Y \cup S$ with the quotient topology induced by $\tilde{f} = f \cup 1_S$. We need only to prove that space is Hausdorff. First let $p, q \in S$ and let $W \in \mathcal{W}_p$, $W' \in \mathcal{W}_q$ be disjoint. Then they are saturated with respect to $\tilde{f}$, hence their images are disjoint open neighborhoods of $p$ and $q$, respectively, in $Y \cup S$. Now, suppose $p \in S$ and $q \in Y$. Since $f$ is perfect, $f^{-1}(q)$ is closed in $\kappa Z$. Let $V$ and $V'$ be disjoint open subsets of $\kappa Z$ which contain $p$ and $f^{-1}(q)$ respectively. Let $W \in \mathcal{W}_p$ such that $W \subset V$. Since $f$ is closed, $V' \cap Z$ contains an open subset $U$ of $Z$, containing $f^{-1}(q)$, which is saturated with respect to $f$. Let $W'$ be an open subset of $\kappa Z$ such that $U = W' \cap Z$. We can choose $W'$ contained in $V'$. Therefore, both $W$ and $W'$ are saturated with respect to $\tilde{f}$, and their images contain $p$ and $q$ respectively. Finally, if $p, q \in Y$, then we can take disjoint open subsets $V, V'$ of $\kappa Z$ which contain $f^{-1}(p)$ and $f^{-1}(q)$ respectively. As before, $V$ and $V'$ contain open subsets of $\kappa Z$, saturated with respect to $\tilde{f}$, which contain $f^{-1}(p)$ and $f^{-1}(q)$ respectively.

Conversely, let $p \in \kappa Z \setminus Z$ and let $V$ be a neighborhood of $p$ in $\kappa Z$. We need to prove that there exists a neighborhood $W$, of $p$ such that $p \in W \subset V$ and $W \cap Z$ is saturated with respect to $f$. Put $F = \kappa Z \setminus V$, $G = \tilde{f}(F)$. Then $G$ is closed and $\tilde{f}(p) \notin G$. Then we can take $W = \kappa Z \setminus \tilde{f}^{-1}(G)$. □

And finally we have:

**Theorem 21.** *Let $f : Z = \beta X \setminus X \to Y$ be perfect and surjective. Then the following are equivalent:*

1. *$Y$ is the remainder of a compactification $\alpha X$ of $X$ and $f = \pi_{\beta\alpha}|_Z$.*
2. *For every $p \in Cl_{\beta X}(Z) \setminus Z$ there is a local basis $\mathcal{W}_p$ of $p$ in $Cl_{\beta X}(Z)$ such that, for each $W \in \mathcal{W}_p$, $W \cap Z$ is saturated with respect to $f$.*

## References

[CFV] Caterino A., Faulkner G.D., Vipera M.C., *Two applications of singular sets to the theory of compactifications*, Rend. Ist. Mat. Univ. Trieste **21** (1989), 248–258.
[E] Engelking R., *General Topology*, Heldermann, Berlin, 1989.
[M1] Magill K.D., *A note on compactifications*, Math. Z. **94** (1966), 322–325.
[M2] ———, *The lattice of compactifications of a locally compact space*, Proc. London Math. Soc. **18** (1968), 231–244.
[R] Rayburn M.C., *On Hausdorff compactifications*, Pacific J. of Math. **44** (1973), 707–714.
[T] Fu-Chien Tzung, *Sufficient conditions for the set of Hausdorff compactifications to be a lattice*, Ph.D. Thesis, North Carolina State University.

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