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An approach to covering dimensions*

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Abstract. Using certain ideas connected with the entropy theory, several kinds of dimensions are introduced for arbitrary topological spaces. Their properties are examined, in particular, for normal spaces and quasi-discrete ones. One of the considered dimensions coincides, on these spaces, with the Čech-Lebesgue dimension and the height dimension of posets, respectively.

Keywords: Čech-Lebesgue dimension, height dimension of posets, dyadic expansion, rigged finite open covers, partition dimension

Classification: 54F45, 06A10

Using certain ideas connected with an approach to the entropy, we introduce several versions of the covering dimensions for arbitrary topological spaces. One of these dimensions, called the partition dimension, is shown to embrace both the Čech-Lebesgue dimension of normal spaces and the height dimension of posets, i.e. of quasi-discrete T_0-spaces.

Most of the dimensions in question are introduced by means of dyadic expansions (see 1.7). For every dyadic expansion $\mathscr{S}$ of a space, we define, first, numbers $\gamma(\mathscr{S})$ and $\Gamma(\mathscr{S})$. Taking the infimum of these numbers for all $\mathscr{S}$ refining a given finite open cover or a rigged (see 2.4) finite open cover, and then the supremum of these values for all (rigged) finite open covers, we obtain the considered dimension. Further versions are obtained if a simple restriction is imposed on the dyadic expansions. The dimensions obtained in this way can also be expressed by means of partitions refining (rigged) finite open covers. We also consider certain modifications of the Čech-Lebesgue dimension introduced directly by means of refinements of covers.

The main results are as follows. From a general point of view: the possibility to introduce dimensions of topological spaces by means of a device used in some questions of the entropy theory, and the usefulness of normal (see 2.15) partitions in the dimension theory. As for concrete results: the existence, already mentioned, of a dimension defined in a natural way for all topological spaces and coinciding (see 5.8), for normal spaces and posets, with well-known dimensions, characterization theorems (see 4.5, 4.7 and 5.5) for dimensions of posets, and the

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characterization (6.7) of the partition dimension (see 2.15) by means of mappings onto finite posets.

Some basic properties (including monotonicity as well as addition and sum formulas) are investigated for arbitrary spaces. We concentrate on two classes, though: normal spaces and quasi-discrete T_0-spaces, i.e. posets.

All the dimensions considered coincide for hereditarily normal spaces (see 3.6). For an arbitrary normal spaces, the question of coincidence remains partly open, though most of the dimensions do coincide. As for completely regular spaces, we only give an example for which two of the dimensions are distinct.

For posets, we present, for most of the dimensions, a complete characterization by means of properties of order. In the case of posets, it seems that the number of dimensions deserving a further detailed investigation reduces to three or four, for the behavior of the remaining ones is not too good.

The paper is organized as follows. Section 1 contains preliminaries, Section 2 the basic definitions. In Section 3, we present some results on the behavior of dimensions on arbitrary spaces.

Section 4 and 5 are devoted to dimensions of posets. Many results contained in these sections are known, at least partly, though perhaps in a different form. Therefore, some proofs or their parts are omitted. On the other hand, some statements or proofs provide a connection between the dimension theory of normal spaces and that of posets or seem to be new. In these cases, the proofs are given in full.

In Section 6, we consider questions concerning the characterization of a dimension by means of mappings onto finite posets. Finally, in Section 7, we add some remarks connected with the Dushnik-Miller dimension.

1.

1.1 Notation and conventions. The letter N denotes the set $\{0, 1, 2, \dots\}$; the letters i, j always denote natural numbers. The cardinality of a set X is denoted by $|X|$. Topological spaces will be called simply "spaces". The terminology concerning spaces is that of [EGT] and [EDT], except that the Čech-Lebesgue dimension is defined, in the usual way, for arbitrary spaces (not only for normal ones). — The letter S, possibly with subscripts, will always denote a space. If $X \subset S$, then $\operatorname{cl}_S X$ or $\operatorname{cl} X$ will denote the closure of X in S, and $\operatorname{op}_S X$ or $\operatorname{op} X$ will denote the intersection of all open $G \supset X$. — Partially ordered sets will be referred to as posets. — A space $\langle X, \tau \rangle$ or a poset $\langle Y, \leq \rangle$, etc., will be denoted, as a rule, by the same letter as its underlying set.

1.2. An indexed collection, say $\mathscr{X} = (x_k : k \in K)$, is simply called a collection. If $x_k \neq x_h$ for $h, k \in K$, $h \neq k$, we say that $\mathscr{X}$ is a collection without repetition. If $\mathscr{Y} = (y_m : m \in M)$ is another collection and there is a bijection $f : K \to M$ such that $x_k = y_{fk}$ for all $k \in K$, we say that $\mathscr{X}$ and $\mathscr{Y}$ are equal up to indexation and write $\mathscr{X} \equiv \mathscr{Y}$. If $\mathscr{X} = (x_k : k \in K)$ is a collection and $\mathscr{Y} = (x_k : k \in H)$, where $H \subset K$, we call $\mathscr{Y}$ a subcollection of $\mathscr{X}$ and write $\mathscr{Y} \subset \mathscr{X}$. — If

$\mathscr{X} = (X_k : k \in K)$ and $\mathscr{Y} = (Y_m : m \in M)$ are collections of sets and, for every $k \in K$, there is $m \in M$ with $X_k \subset Y_m$, we write $\mathscr{X} \prec \mathscr{Y}$ and say that $\mathscr{X}$ refines $\mathscr{Y}$. If $\mathscr{X} = (X_k : k \in K)$ is a collection of sets and A is a set, we put $\mathscr{X} \cap A = (X_k \cap A : k \in K)$. — Covers and partitions of a set are defined in the usual way. — If $f : X \to Y$ is a mapping and $\mathscr{U} = (U_k : k \in K)$ is a collection of subsets of Y, then $f^{-1}(\mathscr{U})$ denotes the collection $(f^{-1}(U_k) : k \in K)$. — A set is often considered as a collection without repetitions.

1.3. (A) Recall that a space is called quasi-discrete or A-discrete (or else finitely generated), if the intersection of an arbitrary system of open sets is open. These spaces were considered already in the thirties, see [A 35] and [A 37]. — (B) The following facts are well known. If $S = \langle S, \leq \rangle$ is a poset, then $\langle S, \tau \rangle$, where τ consists of all $G \subset S$ such that $x \in G$, $y \in S$, $x \leq y$ implies $y \in G$, is a quasi-discrete T_0-space, which will be denoted by S^t. If $S = \langle S, \tau \rangle$ is a quasi-discrete T_0-space, then $\langle S, \leq \rangle$, where $x \leq y$ iff $x \in \mathrm{cl}\{y\}$, is a poset, which will be denoted by S^p. Clearly, $(S^p)^t = S$ for every quasi-discrete T_0-space S, and $(S^t)^p = S$ for every poset S.

1.4. A poset will be always considered as a quasi-discrete T_0-space, and vice versa.

1.5. Let S be a poset, $x \in S$. The height of x, which will be denoted by $\mathrm{ht}_S(x)$ or $\mathrm{ht}(x)$, can be defined as follows: $\mathrm{ht}(x)$ is the least $n \in N \cup \{\infty\}$ such that, for every finite chain $C \subset S$ satisfying $\max C = x$, we have $|C| \leq n + 1$. We put $\mathrm{ht}(S) = \sup(\mathrm{ht}(x) : x \in S)$. This value is called the height of S. We will use the notation $\mathrm{ht}(S)$ or $\mathrm{hdim}(S)$ and the name <u>height</u> or <u>height dimension</u> of S.

1.6. In some articles (see e.g. [K 92] and [K 93]) of the author concerning entropies and related notions, the concept of dyadic expansions has been widely used, e.g. for the case of metrized probability spaces. In the present paper, this concept, transferred to topological spaces, will be used to introduce certain kinds of dimensions.

1.7. We recall the definition of dyadic expansions in a fairly general form (cf. [K 92, 1.11] and [K 93, 1.6]), Let S be a set or a set endowed with a structure, in particular a topological space. Let $\mathscr{D}$ be the system of all finite $D \subset \bigcup(\{0,1\}^n : n \in N)$ such that, for every $a \in D$, all $b \prec a$ ($b \prec a$ means that a is the concatenation of b and some c) are in D and either $\{a0, a1\} \subset D$ or $\{a0, a1\} \cap D = \emptyset$. For $D \in \mathscr{D}$, we put $D' = \{a \in D : a0 \in D\}$, $D'' = D \setminus D'$. A collection $(S_a : a \in D)$, where $D \in \mathscr{D}$, is called a dyadic expansion (abbreviated d.e.) of S, if $S_{a0} \cup S_{a1} = S_a$ for all $a \in D'$ and $S_\emptyset = S$ (provided $D \neq \emptyset$); $\mathscr{S}''$ will denote the collection $(S_a : a \in D'')$. — Observe that, in [K 92] and [K 93], the definition of $\mathscr{D}$ is slightly different, namely, $\emptyset$ is not in $\mathscr{D}$. In fact, we will use the void d.e. only if $S = \emptyset$.

1.8 Notation. In [K 92], there have been introduced, among other things, "Hartley values" of partitions (S_a, S_b) of sets equipped with a finite measure and a metric. We now introduce their analogues for the case of topological spaces. If S is a space, $S_a \subset S$, $S_b \subset S$, we put (1) $\gamma(S_a, S_b) = 0$ if S_a and S_b are separated (i.e. $\overline{S}_a \cap S_b = \emptyset = S_a \cap \overline{S}_b$), $\gamma(S_a, S_b) = 1$ if not, (2) $\Gamma(S_a, S_b) = 0$ if S_a and S_b can be separated by open sets (i.e. there are open G_a and G_b with $G_a \supset S_a$, $G_b \supset S_b$, $G_a \cap G_b = \emptyset$), $\Gamma(S_a, S_b) = 1$ if not. In a manner analogous to that used in [K 92, 1.17], we introduce $\gamma(\mathscr{S})$ and $\Gamma(\mathscr{S})$. If $\mathscr{S} = (S_a : a \in D)$ is a dyadic expansion of a topological space S, we put $\gamma(\mathscr{S}) = \sum(\gamma(S_{a0}, S_{a1}) : a \in D')$, $\Gamma(\mathscr{S}) = \sum(\Gamma(S_{a0}, S_{a1}) : a \in D')$, unless $\mathscr{S} = (S_a : a \in \emptyset) = \emptyset$, in which case we put $\gamma(\mathscr{S}) = \Gamma(\mathscr{S}) = -1$.

1.9 Notation. Let $\mathscr{X} = (X_k : k \in K)$ be a finite cover of a space S. Assume $S \neq \emptyset$. — (A) $\dim \mathscr{X}$ will denote the least $n \in N$ such that, for some open $U_k \supset X_k$, we have $\bigcap(U_k : k \in H) = \emptyset$ whenever $H \subset K$, $|H| > n+1$. Evidently, if X_k are open, i.e. $\mathscr{X}$ is a finite open cover (abbreviated f.o.c.), then $\dim \mathscr{X}$ has its usual value. — (B) γ-dim$\mathscr{X}$ and Γ-dim$\mathscr{X}$ denote, respectively, the least $n \in N$ such that there are $K_i \subset K$, $i \leq n$, for which $\bigcup K_i = K$ and $\gamma(X_h, X_k) = 0$ ($\Gamma(X_h, X_k) = 0$) whenever $h \neq k$ and $h, k \in K_i$ for some $i \leq n$. — If $S = \emptyset$, we put $\dim \mathscr{X} = \gamma$-dim$\mathscr{X} = \Gamma$-dim$\mathscr{X} = -1$.

2.

2.1 Notation and definition. Let S be a space. If $\mathscr{U}$ is a cover of S, then $\gamma(S, \mathscr{U})$ and $\Gamma(S, \mathscr{U})$ denote, respectively, the infimum of all $\gamma(\mathscr{S})$ and that of all $\Gamma(\mathscr{S})$, where $\mathscr{S}$ is a dyadic expansion of the S such that $\mathscr{S}''$ refines $\mathscr{U}$. The supremum of all $\gamma(S, \mathscr{U})$ (of all $\Gamma(S, \mathscr{U})$), where $\mathscr{U}$ is a finite open cover of S, will be denoted by γ-dim$S$ (by Γ-dim$S$). The classical Čech-Lebesgue dimension of S will be denoted, as usual, by $\dim S$.

2.2. It will be seen that γ-dim and Γ-dim have some nice properties; thus, for hereditarily normal spaces, they coincide with dim. On the other hand, their behavior on posets is not quite good. In particular, Γ-dim$S = \gamma$-dim$S = 0$ for every finite poset S with just one minimal element. Therefore, we are looking for some related kind of dimensions giving reasonable values also for posets. Dimensions of such sort can be obtained by considering a modified (enriched) concept of finite open cover. This is performed below.

2.3. First we introduce an auxiliary concept. A subset X of a space S will be called <u>quasi-closed</u>, if $\operatorname{cl} X \subset \bigcup(\operatorname{cl}\{x\} : x \in X)$. — The following assertions are evident: (1) every closed set is quasi-closed, (2) in a T_1-space, every quasi-closed set is closed, (3) if $X \subset Y \subset S$, X is quasi-closed in Y and Y is quasi-closed in S, then X is quasi-closed in S, (4) if X is quasi-closed in S and $T \subset S$, then $T \cap X$ is quasi-closed in T, (5) a space is quasi-discrete iff every $X \subset S$ is quasi-closed, (6) if X and Y are quasi-closed in S, then so is $X \cup Y$.

2.4 Definitions. (A) A pair $\langle \mathscr{U}, \mathscr{A} \rangle$ will be called a <u>rigged finite open cover</u> (abbreviated r.f.o.c.) of a space S, if (1) $\mathscr{U} = (U_k : k \in K)$ is a finite open cover of S and $\mathscr{A} = (A_k : k \in K)$ is a finite cover of S, (2) for every $k \in K$, $A_k \subset U_k$ and A_k is quasi-closed. — (B) If $\mathscr{F} = \langle \mathscr{U}, \mathscr{A} \rangle$ is a rigged finite open cover of S,

$\mathscr{U} = (U_k : k \in K)$, $\mathscr{A} = (A_k : k \in K)$, and $\mathscr{B} = (B_m : m \in M)$ is a finite cover of S, then we say that $\mathscr{B}$ refines $\mathscr{F}$ and write $\mathscr{B} \prec \mathscr{F}$, if $\mathscr{B}$ refines $\mathscr{U}$ and, for every $k \in K$, $A_k \subset \bigcup(B_m : m \in M, B_m \subset U_k)$.

2.5 Definition. A f.o.c. $\mathscr{U}$ of S will be called good, if there exists a rigged finite open cover of the form $\langle \mathscr{U}, \mathscr{A} \rangle$. — It is easy to see that each finite open cover of a normal space or of a quasi-discrete space is good.

2.6 Fact. If $\mathscr{F} = \langle \mathscr{U}, \mathscr{A} \rangle$ is a rigged finite open cover of a T_1-space S, then there exists a finite open cover $\mathscr{V}$ such that $\mathscr{W} \prec \mathscr{F}$ whenever $\mathscr{W}$ is a finite open cover refining $\mathscr{V}$.

PROOF: Let $\mathscr{U} = (U_k : k \in K)$, $\mathscr{A} = (A_k : k \in K)$. Since S is a T_1-space, all A_k are closed. If $\emptyset \neq L \subset K$, put $V(L) = \bigcap(U_k : k \in K) \setminus \bigcup(A_k : k \in K \setminus L)$. Clearly, $(V(L) : 0 \neq L \subset K)$ is a f.o.c. of S. Let $\mathscr{W} = (W_m : m \in M)$ be a f.o.c., $\mathscr{W} \prec \mathscr{V}$. Let $k \in K$. If $x \in A_k$, choose $m \in M$ and $L \subset K$ such that $x \in W_m \subset V(L)$. Evidently, $k \in L$, $W_m \subset U_k$. Hence $\bigcup(V_m : m \in M, V_m \subset U_k) \supset A_k$. $\square$

2.7 Notation and definitions. Let S be a space and let $\mathscr{F} = \langle \mathscr{U}, \mathscr{A} \rangle$ be a rigged finite open cover of S. Then $\gamma(S, \mathscr{F})$ and $\Gamma(S, \mathscr{F})$ denote, respectively, the infimum of all $\gamma(\mathscr{S})$ and that of all $\Gamma(\mathscr{S})$, where $\mathscr{S}$ is dyadic expansion of S such that $\mathscr{S}''$ refines $\mathscr{F}$. — Let S be a space. Then (A) γ-Dim$S$, respectively Γ-Dim$S$ denote the supremum of all $\gamma(S, \mathscr{F})$ and that of all $\Gamma(S, \mathscr{F})$, where $\mathscr{F}$ is a rigged finite open cover of S; (B) γ-dim$^*S$ and Γ-dim$^*S$ denote respectively the supremum of all $\gamma(S, \mathscr{U})$ and that of all $\Gamma(S, \mathscr{U})$, where $\mathscr{U}$ is a good finite open cover of S; (C) $\operatorname{Dim} S$ denotes the least $n \in \{-1\} \cup N \cup \{\infty\}$ such that every rigged f.o.c. of S is refined by some f.o.c. $\mathscr{U}$ with $\dim \mathscr{U} \leq n$; (D) $\dim^* S$ denotes the least $n \in \{-1\} \cup N \cup \{\infty\}$ such that, for every good finite open cover, there is a finer finite open cover $\mathscr{V}$ with $\dim \mathscr{V} \leq n$.

2.8 Fact. Let S be normal or quasi-discrete. Then Γ-dim$^*S = \Gamma$-dim$S$, $\dim^* S \equiv \dim S$, γ-dim$^*S = \gamma$-dim$S$. — This is a consequence of the fact that every finite open cover of S is good.

2.9. We are going to show that Γ-Dim, Γ-dim and Γ-dim* as well as γ-Dim, γ-dim and γ-dim* can be defined by means of refinements of (rigged) finite open covers, thus avoiding dyadic expansions. Nevertheless, the approach based on these expansions retains a certain value, if only for demonstrating a new aspect of the connections between entropy and dimension.

2.10 Lemma. *Let $S \neq \emptyset$ be a space. Let φ stand for γ or Γ. If $\mathscr{S}$ is a dyadic expansion of S, then φ-dim $\mathscr{S}'' \leq \varphi(\mathscr{S})$. If $\mathscr{X}$ is a finite partition of S, then there exists a dyadic expansion $\mathscr{S}$ of S such that $\mathscr{S}'' \equiv \mathscr{X}$ and $\varphi(\mathscr{S}) = \varphi$-dim$\mathscr{X}$.*

PROOF: We will consider only the case of γ; that of Γ is quite analogous. — I. We are going to prove that, for every $n \in N$,

(∗) if $\gamma(\mathscr{S}) \leq n$, then γ-dim$\mathscr{S}'' \leq n$.

This is evident for $n = 0$. Assume that $(*)$ holds for n. Let $\mathscr{S} = (S_a : a \in A)$ be a d.e. of S, $\gamma(\mathscr{S}) \le n+1$. Choose $b \in A'$ such that $\gamma(S_{b0}, S_{b1}) = 1$, whereas $\gamma(S_{a0}, S_{a1}) = 0$, whenever $a \in B \cap A'$, where $B = \{a \in A : b \prec a, b \ne a\}$. Put $\mathscr{T} = (S_a : a \in A \backslash B)$. Clearly, $\mathscr{T}$ is a d.e., $\gamma(\mathscr{T}) \le n$ and therefore γ-dim$\mathscr{T}'' \le n$. We have $\mathscr{T}'' = (S_a : a \in (A'' \setminus B) \cup \{b\})$, γ-dim$(S_a : a \in A'' \cap B) = 0$, which implies γ-dim$\mathscr{S}'' \le n+1$. This proves that $(*)$ holds for all $n \in N$. — II. Let $\mathscr{X}$ be a finite partition of S. We are going to prove that, for every $n \in N$,

$(**)$ if γ-dim$\mathscr{X} \le n$, then there exists a d.e. $\mathscr{S}$ of S such that $\mathscr{S}'' \equiv \mathscr{X}$ and $\gamma(\mathscr{S}) \le n$.

Clearly, this holds for $n = 0$. Assume that $(**)$ holds for n. Let $\mathscr{X} = (X_k : k \in K)$ be a partition of S, γ-dim$\mathscr{X} \le n+1$. Then there is a set $H \subset K$ such that γ-dim$(X_k : k \in H) \le n$, γ-dim$(X_k : k \in K \setminus H) \le 0$. Put $T = \bigcup(X_k : k \in H)$. There exists a d.e. $\mathscr{T} = (T_b : b \in B)$ of T such that $\mathscr{T}'' \equiv (X_k : k \in H)$ and $\gamma(\mathscr{T}) \le n$, as well as a d.e. $\mathscr{Z} = (Z_c : c \in C)$ of $S \setminus T$ such that $\mathscr{Z}'' \equiv (X_k \setminus T : k \in K \setminus H)$. Define a d.e. $\mathscr{V} = (V_a : a \in A)$ of S as follows: A consists of $\emptyset$, all $0b$, where $b \in B$, and all $1c$, where $c \in C$; $V_\emptyset = S$, $V_{0b} = T_b$, $V_{1c} = Z_c$. It is easy to see that $\gamma(\mathscr{V}) \le n+1$ and $\mathscr{V}'' \equiv \mathscr{X}$. □

2.11 Fact. Let φ stand for γ or for Γ. Let $\mathscr{B}$ be a finite cover of S. Then there exists a partition $\mathscr{X} \prec \mathscr{B}$ such that φ-dim$\mathscr{X} \le \varphi$-dim$\mathscr{B}$.

Proof: Let $\mathscr{B} = (B_k : k \in K)$, φ-dim$\mathscr{B} = n$. Then there are disjoint $K_i \subset K$, $i \le n$, such that $\bigcup K_i = K$ and φ-dim$(B_k : k \in K_i) = 0$ for $i \le n$. For $i \le n$, put $M_i = \bigcup(K_j : j < i)$, $T_i = \bigcup(B_k : k \in M_i)$. For $k \in K_i$, put $X_k = B_k \setminus T_i$. Clearly $\mathscr{X} = (X_k : k \in K) \prec \mathscr{B}$ and φ-dim$\mathscr{X} \le n$. □

2.12 Proposition. *Let φ stand for γ or for Γ; let $n \in N$. For every space S, (1) φ-dim$S \le n$ (respectively, φ-dim$^*S \le n$) iff, for every finite open cover (every good finite open cover) $\mathscr{U}$ of S, there is a partition $\mathscr{B} \prec \mathscr{U}$ satisfying φ-dim$\mathscr{B} \le n$, (2) φ-Dim$S \le n$ iff, for every r.f.o.c. $\mathscr{F}$ of S, there is a partition $\mathscr{B} \prec \mathscr{F}$ satisfying φ-dim$\mathscr{B} \le n$. In (1), "partition" can be replaced by "cover".*

Proof: The assertion involving partitions are easy consequences of 2.10. The last assertion follows from 2.11. □

Remark. It is an open question whether "cover" can replace "partition" also in (2).

2.13. We obtain two other kinds of dimensions, if we impose a fairly natural topological restriction (see 2.14) onto dyadic expansions. These dimensions can also be introduced by means of normal partitions (see 2.15, 2.16 and 2.17).

2.14 Notation. A dyadic expansion $\mathscr{S} = (S_x : x \in A)$ of S will be called normal if, for every $x \in A'$, at least one of the sets S_{x0}, S_{x1} is open in S_x. If $\mathscr{U}$ is a finite open cover (respectively, $\mathscr{F}$ is a rigged finite open cover) of S, then $\widehat{\gamma}(S, \mathscr{U})$ and $\widehat{\gamma}(S, \mathscr{F})$ will denote the infimum of all $\gamma(\mathscr{S})$, where $\mathscr{S}$ is normal, $\mathscr{S}'' \prec \mathscr{U}$, and that of all $\gamma(\mathscr{S})$, where $\mathscr{S}$ is normal, $\mathscr{S}'' \prec \mathscr{F}$. The supremum of all $\widehat{\gamma}(S,\mathscr{U})$, where $\mathscr{U}$ is a finite open cover of S, and that of all $\widehat{\gamma}(S,\mathscr{F})$, where $\mathscr{F}$ is a rigged finite open cover, will be denoted, respectively, by $\widehat{\gamma}$-dim$S$ and $\widehat{\gamma}$-Dim$S$ (this notation is preliminary, cf. 2.15 and 2.17).

In addition, we introduce (2.15) two dimensions more. However, they will be shown (see 2.17) to coincide, respectively, with $\widehat{\gamma}$-dim$S$ and $\widehat{\gamma}$-Dim$S$.

2.15 Definitions. A cover (in particular, a partition) $\mathscr{B} = (B_k : k \in K)$ of a space S will be called <u>normal of order</u> $\leq n$, if there are disjoint $K_i \subset K$, $i \leq n$, such that $\bigcup K_i = K$, γ-dim$(B_k : k \in K_i) = 0$ and $\bigcup(B_k : k \in \bigcup(K_j : j \leq i))$ is closed in S for every $i \leq n$. If S is a space, then p-dim$S$ (respectively, p-Dim$S$) will denote the least $n \in \{-1\} \cup N \cup \{\infty\}$ such that every finite open cover (every rigged finite open cover) is refined by a normal partition of order $\leq n$. — We will call p-Dim$S$ the <u>partition dimension</u> of S.

2.16 Lemma. *Let S be a space. If $\mathscr{S}$ is a normal dyadic expansion of S, then $\mathscr{S}''$ is a normal partition of S of order $\leq \gamma(\mathscr{S})$. If $\mathscr{X}$ is a normal partition of S of order $\leq n$, then there exists a normal dyadic expansion $\mathscr{S}$ of S such that $\mathscr{S}'' \equiv \mathscr{X}$, $\gamma(\mathscr{S}) \leq n$.*

Proof: I. We are going to show that

(∗) if $\mathscr{S} = (S_a : a \in K)$ is normal, $\gamma(\mathscr{S}) \leq n$, then $\mathscr{S}''$ is a normal partition of order $\leq n$.

This is evident for $n = 0$. Assume that (∗) holds for some $n \in N$ and prove that it holds for $n+1$. It is easy to see that there is an $a = (a(0),\dots,a(p)) \in A'$ such that, with $b(0) = \emptyset$, $b(i) = (a(0),\dots,a(i-1))$ for $0 < i < p$, we have (1) $S_{b(i+1)}$ is closed in $S_{b(i)}$ for $i < p$, (2) if $x \in A'$, $a \prec x$, then $\gamma(S_{x0}, S_{x1}) = 0$, (3) $\gamma(S_{b0}, S_{b1}) = 1$, where $b = b(p)$. Put $Z = \{x \in A'' : a \prec x\}$, $\mathscr{Z} = (S_x : x \in Z)$, $T = S \setminus \bigcup(S_x : x \in X)$. Clearly, all S_x, $x \in Z$, are closed (in S) and γ-dim$\mathscr{Z} = 0$. For every $x \in A$, put $T_x = T \cap S_x$; put $\mathscr{T} = (T_x : x \in A)$. We have $\gamma(\mathscr{T}) \leq n$, hence $\mathscr{T}$ is a normal partition of order $\leq n$. Together with the fact that S_x, $x \in Z$, are closed and γ-dim$\mathscr{Z} = 0$, this implies that $\mathscr{S}''$ is a normal partition of order $\leq n+1$. — II. The proof of the second assertion is easy and can be omitted. □

2.17 Fact and convention. It follows easily from 2.16 that $\widehat{\gamma}$-dim$S = p$-dim$S$, $\widehat{\gamma}$-Dim$S = p$-Dim$S$ for every space S. — In what follows, we always write p-dim instead of $\widehat{\gamma}$-dim, p-Dim instead of $\widehat{\gamma}$-Dim.

2.18. We have introduced eleven dimensions, namely Γ-dim, dim and γ-dim in 2.1, Γ-Dim, Dim, γ-Dim, Γ-dim*, dim* and γ-dim* in 2.7, p-dim $= \widehat{\gamma}$-dim and p-Dim $= \widehat{\gamma}$-Dim in 2.14 and 2.15 (see also 2.17). Many of these dimensions coincide both on normal spaces and on quasi-discrete ones (see 2.8, 3.5 and 5.7). In addition, we omit p-dim in what follows. Thus, in Section 4 and 5, devoted to posets, as well as in Section 6, we will deal with only six dimensions. In fact, it turns out that, on posets, only Γ-Dim, Dim, γ-Dim and p-Dim (which possibly differs from γ-Dim on normal spaces) have sufficiently good properties.

2.19 Remark. Using dyadic expansions, we can introduce a dimension coinciding with dim for every space. We will not go into these matters here and only point out that the approach just mentioned involves the concept of a balanced cover, i.e. a finite cover $\mathscr{B}$ such that γ-dim$\mathscr{B} = \dim \mathscr{B}$.

3.

In this section, we present some inequalities for dimensions of arbitrary spaces, as well as some equalities valid in special cases, e.g. for normal spaces. We also examine questions of monotonicity and of validity of addition and sum formulas. However, we do not deal with the question of which equalities are valid on particular classes of spaces except those of normal ones and of posets (see Section 4 and 5).

3.1 Fact. Let $\mathscr{F} = \langle \mathscr{U}, \mathscr{A} \rangle$ be a rigged finite open cover of S. Let $\mathscr{B} = (B_m : m \in M)$ and $\mathscr{C} = (C_t : t \in T)$ be finite covers of S. If $\mathscr{C} \prec \mathscr{B} \prec \mathscr{F}$ and every B_m is a union of some C_t, then $\mathscr{C} \prec \mathscr{F}$.

Proof: Let $\mathscr{U} = (U_k : k \in K)$, $\mathscr{A} = (A_k : k \in K)$. For every $k \in K$, there is a $M' \subset M$ such that $A_k \subset \bigcup(B_m : m \in M')$. For every $m \in M$, $B_m = \bigcup(C_t : t \in T_m)$ for some $T_m \subset T$, and therefore $A_k \subset \bigcup(C_t : t \in \bigcup(T_m : m \in M')) \subset U_k$. □

3.2 Notation. If $\mathscr{B} = (B_m : m \in M)$ is a finite cover of S, then at$\mathscr{B}$ will denote the family of all atoms of the Boolean algebra generated by S and the sets B_m.

3.3 Fact. If $\mathscr{B} = (B_m : m \in M)$ is a finite cover of S, then at$\mathscr{B}$ refines $\mathscr{B}$ and every B_m is a union of some atoms. If $\mathscr{B}$ is an open cover, then at$\mathscr{B}$ is a normal partition of order $\leq \dim \mathscr{B}$ and all atoms are locally closed.

Proof: We are going to prove that at$\mathscr{B}$ is a normal partition of order $\leq \dim \mathscr{B}$ provided $\mathscr{B}$ is a f.o.c. The remaining assertions are evident. — If $\emptyset \neq H \subset M$, put $C_H = \bigcap(B_m : m \in H) \setminus \bigcup(B_m : m \in M \setminus H)$; put $\widehat{M} = \{H \subset M : H \neq \emptyset, C_H \neq \emptyset\}$. We have at$\mathscr{B} = \{C_H : H \in \widehat{M}\}$. Put $n = \dim \mathscr{B}$, $\widehat{M_i} = \{H \in \widehat{M} : |H| = i+1\}$, $i \leq n$. It is easy to show that, due to the fact that B_m are open, we have γ-dim$(C_H : H \in \widehat{M_i}) \equiv 0$ for $i \leq n$ and therefore γ-dim(at$\mathscr{B}$) $\leq n$. Clearly, if $x \in \operatorname{cl} C_H \setminus C_H$, then $x \in C_G$ for some $G \in \widehat{M}$, $G \neq H$, $|G| < |H|$. It follows that every $T_i = \bigcup(C_H : |H| = i+1)$ is closed in $\bigcup(T_j : i \leq j)$. We have shown that at$\mathscr{B}$ is a normal partition of order $\leq n$. □

3.4 Proposition. (A) *For every space S,* Γ-dim$S \geq \dim S \geq p$-dim$S \geq \gamma$-dim$S$, Γ-Dim$S \geq \operatorname{Dim} S \geq p$-Dim$S \geq \gamma$-Dim$S$, Γ-dim$^*S \geq \dim^* S \geq \gamma$-dim$^*S$, Γ-Dim$S \geq \Gamma$-dim$^*S \leq \Gamma$-dim$S$, $\operatorname{Dim} S \geq \dim^* S \leq \dim S$, γ-Dim$S \geq \gamma$-dim$^*S \leq \gamma$-dim$S$. — (B) *If every finite open cover of S is good, in particular if S is a normal space or a poset, then* Γ-dim$^*S = \Gamma$-dim$S$, $\dim^* S = \dim S$, γ-dim$^*S = \gamma$-dim$S$,

Γ-Dim$S \geq \Gamma$-dim$S$, $\dim S \geq \dim S$, γ-Dim$S \geq \gamma$-dim$S$. — (C) *If S is a T_1-space, then* Γ-dim$S \geq \Gamma$-Dim$S$, $\dim S \geq \operatorname{Dim} S$, γ-dim$S \geq \gamma$-Dim$S$.

Proof: The assertions in (A) are proved in a straightforward way using the definitions and, in some cases, 3.1 and 3.3. — The equalities in (B) follow from the definitions, whereas the inequalities are proved, e.g. for dim and Dim, as follows: if $\mathscr{U}$ is a f.o.c. of S, take a rigged f.o.c. $\mathscr{F} = \langle \mathscr{U}, \mathscr{A} \rangle$ and a f.o.c. $\mathscr{V}$ such that $\mathscr{V} \prec \mathscr{F}$ and $\dim \mathscr{V} \leq \operatorname{Dim} S$. By the definition, $\mathscr{V}$ refines $\mathscr{U}$. The statement in (C) follows from 2.6. □

3.5 Proposition. *Let S be a normal space. Then* Γ-dim$S = \Gamma$-Dim$S$, $\dim S = \operatorname{Dim} S$, γ-dim$S = \gamma$-Dim$S$, p-dim$S = p$-Dim$S$, Γ-dim$S = \dim S$, p-dim$S = p$-Dim$S = \dim S$. *Thus, for a normal space, all the dimensions coincide, possibly except the dimension* γ-dim$S = \gamma$-Dim$S$.

Proof: The first four equalities follow from 2.6. — It is known (in a different formulation) that, for a normal S, $\dim S \leq n$ implies Γ-dim$S \leq n$; see [K 52, Lemma 1.7] and [O 71]. — We are going to show that, for a normal S, p-dim$S \leq n$ implies $\dim S \leq n$. Clearly, it suffices to prove that, for every $n \in N$, the following assertion holds: (P_n) if $\mathscr{U} = (U_k : k \in K)$ is a f.o.c., $\mathscr{B} = (B_m : m \in M)$ is a normal partition of S of order $\leq n$, and $\mathscr{B} \prec \mathscr{U}$, then there is a f.o.c. $\mathscr{V} \prec \mathscr{U}$ such that Γ-dim$\mathscr{V} \leq n$. Clearly, (P_0) is valid. We will assume (P_n) and prove (P_{n+1}). Let $\mathscr{B}$ and $\mathscr{U}$ have the properties stated in (P_{n+1}). There are disjoint $M_i \subset M$, $i \leq n+1$, such that $\bigcup M_i = M$, γ-dim$(B_m : m \in M_i) = 0$ for $i \leq n+1$ and, with $T_i = \bigcup (B_m : m \in M_i)$, $i \leq n+1$, each T_i is closed in $\bigcup (T_j : i \leq j \nleq n+1)$. Since the sets B_m, $m \in M_0$, are, due to γ-dim$(B_m : m \in M_0) = 0$, closed in T_0, and $\mathscr{B}$ refines $\mathscr{U}$, there are disjoint open $G_m \supset B_m$, $m \in M_0$, such that $(G_m : m \in M_0) \prec \mathscr{U}$. Since $T_0 \subset G = \bigcup G_m$, there is an open set Z such that $T_0 \subset Z \subset \operatorname{cl} Z \subset G$. Then $\mathscr{B}' = (B_m \setminus Z : m \in M)$ is a normal partition of order $\leq n$ of the normal space $S \setminus Z$, and therefore there exists a f.o.c. $\mathscr{W} = (W_c : c \in C)$ of $S \setminus Z$ satisfying Γ-dim$\mathscr{W} \leq n$, $\mathscr{W} \prec \mathscr{U}$. Clearly, the sets G_m, $m \in M_0$, and $W_c \setminus \operatorname{cl} Z$, $c \in C$, form f.o.c. $\mathscr{V}$ of S satisfying $\mathscr{V} \prec \mathscr{U}$ and Γ-dim$\mathscr{V} \leq n+1$.

3.6 Proposition. *For a hereditarily normal space, all the considered dimensions coincide.*

This follows from 3.5 and the fact that, for a hereditarily normal S, γ-dim$\mathscr{X} = \Gamma$-dim$\mathscr{X}$ for every finite cover $\mathscr{X}$.

3.7. As already mentioned, we omit a detailed examination of which equalities hold between various dimensions on particular classes of spaces. Nevertheless, one counterexample will be given. By 3.4, $\dim S \geq \dim^* S$ for every S, and it is easy to see that $\dim^* S \geq \dim \beta S$ whenever S is completely regular. In the example below, S is completely regular, $\dim S = 1$, $\dim^* S = \dim \beta S = 0$.

Let S' be the product $[0, \omega_1] \times [0, \omega_2]$ of spaces of ordinals. Put $z = (\omega_1, \omega_2)$, $S = S' \setminus \{z\}$, $A_1 = [0, \omega_1) \times \{\omega_2\}$, $A_2 = \{\omega_1\} \times [0, \omega_2)$. The sets A_1 and A_2 are closed in S, and if $U_i \supset A_i$ are open, then $u_1 \cap U_2 \neq \emptyset$. This implies $\dim S > 0$, and it is easy to see that $\dim S = 1$. Since $\beta S = S'$, we have $\dim \beta S = 0$. We are going to prove $\dim^* S = 0$.

I. We will show that

(∗) if $F_i \subset G_i \subset S$, $i = 1, 2$, F_i are closed, G_i are open, $F_1 \cup F_2 = S$, then there are open $H_i \subset G_i$ with $H_1 \cup H_2 = S$, $H_1 \cap H_2 = \emptyset$.

From (∗), it is easy to deduce that

(∗∗) if $(G_k : k \in K)$ is a f.o.c. of S and there are closed $F_k \subset G_k$ with $\bigcup F_k = S$, then there are open $H_k \subset G_k$, $k \in K$, with $\bigcup H_k = S$ and $H_i \cap H_j = \emptyset$ for $i, j \in K$, $i \neq j$.

By the definition (2.7), (∗∗) means that $\dim^* S = 0$. Thus, (∗) implies $\dim^* S = 0$. — II. Let F_i, G_i satisfy the conditions in (∗). For $j = 1, 2$, we have either $|A_j \setminus G_1| < \omega_j$ or $|A_j \setminus G_2| < \omega_j$, since otherwise the intersection of closed sets $A_j \setminus G_1$ and $A_j \setminus G_2$ would be non-void, which contradicts $G_1 \cup G_2 = S$. Hence there are $a_1 < \omega_1$ and $a_2 < \omega_2$ such that $a_1 < x < \omega_1$ implies $(x, \omega_2) \in G_{k(1)}$ and $a_2 < y < \omega_2$ implies $(\omega_1, y) \in G_{k(2)}$, where $k(1), k(2) \in \{1, 2\}$. — III. If $k(1) = k(2) = i$, then $G_i \cup \{z\}$ is a neighborhood of $z \in S'$; due to $\dim S' = 0$, this implies the existence of H_1, H_2 with the properties from (∗). — IV. Consider the case $k(1) \neq k(2)$; we can assume $k(1) = 1$, $k(2) = 2$. We have $(x, \omega_2) \in G_1$ for $a_1 < x < \omega_1$, $(\omega_1, y) \in G_2$ for $a_2 < y < \omega_2$. Put $B_1 = A_1 \setminus G_2$, $B_2 = A_2 \setminus G_1$; clearly, $B_1 \subset S \setminus F_2$, $B_2 \subset S \setminus F_1$. We are going to prove that either $|B_1| < \omega_1$ or $|B_2| < \omega_2$. Suppose $|B_i| = \omega_i$. Let S^* consist of all (x, y) such that either $(x, \omega_2) \in B_1$, $y \leq \omega_2$, or $(\omega_1, y) \in B_2$, $x \leq \omega_1$. Put $C_1 = \{x : (x, \omega_2) \in B_1\} \cup \{\omega_1\}$, $C_2 = \{y : (\omega_1, y) \in B_2\} \cup \{\omega_2\}$. Let $f_i : C_i \to \{x : x \leq \omega_i\}$ be isomorphisms. For $(x, y) \in S^*$ put $g(x, y) = (f_1 x, f_2 y)$. Clearly, $g : S^* \to S$ is a homeomorphism, $g(B_i) = A_i$. Since $S \setminus F_2 \supset B_1$, $S \setminus F_1 \supset B_2$, we get $g(S \setminus F_2) \supset A_1$, $g(S \setminus F_1) \supset A_2$. Since $F_i \cup F_2 = S$, we have $g(S \setminus F_1) \cap g(S \setminus F_2) = \emptyset$, which is a contradiction, for A_1 and A_2 cannot be separated by open sets. — V. We have shown that either $|B_1| < \omega_1$ or $|B_2| < \omega_2$. It suffices to consider e.g. $|B_1| < \omega_1$. Then there is c_1 such that $c_1 < x < \omega_1$ implies $(x, \omega_2) \in G_2$. By II, $G_2 < y < \omega_2$ implies $(\omega_1, y) \in G_2$. We now make use of the following fact: if $G \subset S$ is open and contains all (x, ω_2) and all (ω_1, y) for sufficiently large $x < \omega_1$ and $y < \omega_2$, then $G \cup \{z\}$ is a neighborhood of z in S'. Thus, $G_2 \cup z$ is a neighborhood of z, from which it follows (cf. III) that there are H_1 and H_2 with the properties stated in (∗). □

3.8. Monotonicity for arbitrary subspaces does not hold for the considered dimensions (i.e. Γ-dim, dim, γ-dim, Γ-Dim, Dim, γ-Dim, Γ-dim*, dim*, γ-dim*, p-dim, p-Dim), not even for hereditarily normal spaces. In fact, there is (see [PP 79]) a hereditarily normal S with subspaces S_n, $n \in N$, such that $\dim S = 0$, $\dim S_n = n$. On the other hand, monotonicity for closed subspaces is valid without any restriction.

3.9 Proposition. *Let φ be any of the considered dimension. Then $\varphi(T) \leq \varphi(S)$ for every S and every closed $T \subset S$.*

This is proved in a similar way for all the dimensions in question. Therefore we consider only one case, that of Dim. Put $\varphi(S) = n$. Let $\langle \mathscr{U}, \mathscr{A} \rangle$ be a rigged f.o.c. of T, $\mathscr{U} = (U_k : k \in K)$, $\mathscr{A} = (A_k : k \in K)$. Let V_k be open in S, $V_k \cap T = U_k$. Take some z non $\in K$ and put $V_z = S$, $A_z = S$, $K' = K \cup \{z\}$. Then $\mathscr{F} = \langle (V_k : k \in K'), (A_k : k \in K') \rangle$ is a rigged f.o.c. of S. Since $\dim S = n$, there is a f.o.c. $\mathscr{W} = (W_c : c \in C)$ of S which refines $\mathscr{F}$ and satisfies $\dim \mathscr{W} \leq n$. It is easy to see that $\mathscr{W}^* = (W_c \cap T : c \in C)$ refines $\langle \mathscr{U}, \mathscr{A} \rangle$ and that $\dim \mathscr{W}^* \leq n$.

3.10. It is well known [Z 63] that, for $\varphi = \dim$, the addition formula $\varphi(S) \leq \varphi(X) + \varphi(Y) + 1$, where $S = X \cup Y$, is valid provided S is normal. However, it fails, in general, for Γ-dim, Γ-Dim, dim and Dim on finite posets. An elementary example: K is a finite set, $|K| \geq 3$; S consists of K and all $\{k\} \in K$ and is ordered by inclusion; $X = \{K\}$, $Y = S \setminus \{K\}$. — On the other hand, for γ-dim and γ-Dim, the formula is valid without restrictions.

3.11 Proposition. *Let $\varphi = \gamma$-dim or $\varphi = \gamma$-Dim. Then, for every space S, $\varphi(S) \leq \varphi(X) + \varphi(Y) + 1$ if $S = X \cup Y$.*

Proof: Let $\varphi = \gamma$-Dim; the other case is analogous. Put $m = \varphi(X)$, $n = \varphi(Y)$. Let $\mathscr{F} = \langle \mathscr{U}, \mathscr{A} \rangle$ be a rigged f.o.c. of S. Then $\mathscr{F}_X = \langle \mathscr{U} \cap X, \mathscr{A} \cap X \rangle$ and $\mathscr{F}_Y = \langle \mathscr{U} \cap Y, \mathscr{A} \cap Y \rangle$ are, respectively, rigged f.o.c. of X and Y. Hence there exist finite covers $\mathscr{X} = (X_m : m \in M)$ and $\mathscr{Y} = (Y_t : t \in T)$ of X and Y such that $\mathscr{X}$ and $\mathscr{Y}$ refine $\mathscr{F}_X$ and $\mathscr{F}_Y$, respectively, and $\gamma\text{-}\dim \mathscr{X} \leq m$, $\gamma\text{-}\dim \mathscr{Y} \leq n$. Let $\mathscr{Z}$ consist of all X_m and all Y_t. It is easy to see that $\mathscr{Z}$ refines $\mathscr{F}$. Clearly, $\gamma\text{-}\dim \mathscr{Z} \leq m + n + 1$. □

3.12. For normal spaces S and the dimensions under consideration, with a possible exception of γ-Dim $= \gamma$-dim, the sum formula does hold. This is an immediate consequence of 3.4 and of the well-known sum formula for dim on normal spaces. As for γ-Dim $= \gamma$-dim, the questions as to whether it coincides with dim and whether it satisfies the sum formula remain open.

For completely regular spaces, we have no definite results. Possibly, the sum formula fails for all the dimensions considered. In this connection, observe that it fails for the dimension defined for completely regular spaces S as $\dim \beta S$; see [P 79].

4.

In the case of posets, there are seven dimensions to be considered, namely Γ-Dim, Dim, γ-Dim, Γ-dim, dim, γ-dim, and p-Dim (for the remaining ones see 3.4 and 2.18). It will be seen later (5.7) that, in addition, γ-Dim and p-Dim do coincide for posets.

4.1. The terms (for subsets of posets) chain, antichain, bounded from above (from below) have the usual meaning. We call an antichain A strong from above (respectively, from below), if no $X \in A$ with $|X| > 1$ is bounded from above (from below). The set of all minimal elements of a poset S will be denoted by $\mathrm{Min}(S)$.

A collection $(X_k : k \in K)$ of subsets of S will be called antichain-like, if $k, h \in K$, $x \in X_h$, $v \in X_k$, $x \leq y$ imply $h = k$.

4.2. It is easy to see that, for any poset S, (1) $A \subset S$ is an antichain iff $\gamma\text{-dim}(\{x\} : x \in A) = 0$, (2) every antichain is locally closed, (3) a collection $\mathscr{X} = (X_k : k \in K)$ is antichain-like iff $\gamma\text{-dim}\mathscr{X} = 0$, (4) an antichain $A \subset S$ is strong from above iff $\Gamma\text{-dim}(\{x\} : x \in A) = 0$.

4.3 Fact. Let S be a finite poset. Put $\mathscr{W}_S = (\mathrm{op}\{x\} : x \in \mathrm{Min}(S))$, $\mathscr{W}^*_S = (\mathrm{op}\{x\} : x \in S)$, $\mathscr{F}_S = \langle \mathscr{W}^*_S, \mathscr{W}^*_S \rangle$. Then (1) if a finite open cover $\mathscr{U}$ refines $\mathscr{W}_S$, then, for some $\mathscr{V}$, $\mathscr{W}_S \equiv \mathscr{V} \subset \mathscr{U}$; (2) if a finite open cover $\mathscr{U}$ refines $\mathscr{F}_S$, then, for some $\mathscr{V}$, $\mathscr{W}^*_S \equiv \mathscr{V} \subset \mathscr{U}$; (3) if a partition $\mathscr{B}$ refines $\mathscr{F}_S$, then $\mathscr{B}$ consists of singletons (and, possibly, the void set).

Proof: Assertion (1): for every $x \in \mathrm{Min}(S)$, choose $k(x) \in K$ with $x \in U_{k(x)}$; for some $y \in S$, we have $U_{k(x)} \subset \mathrm{op}\{y\}$, which implies $x = y$, $U_{k(x)} = \mathrm{op}\{x\}$. — Assertion (2): it is easy to see that, for every $x \in S$, $\mathrm{op}\{x\} = \bigcup(U_k : k \in K, U_k \subset \mathrm{op}\{x\})$ and therefore, if $x \in U_h$, then $\mathrm{op}\{x\} = U_h$. — Assertion (3): let $\mathscr{B} = (B_m : m \in M)$ and let $m \in M$, $x \in B_m$, $y \in B_m$; due to $\mathscr{B} \prec \langle \mathscr{W}^*_S, \mathscr{W}^*_S \rangle$, we have $\{x, y\} \subset \mathrm{op}\{x\}$, $\{x, y\} \subset \mathrm{op}\{y\}$, hence $x = y$. $\square$

4.4 Fact. Let S be a finite poset. Put $\mathscr{W}_S = (\mathrm{op}\{x\} : x \in \mathrm{Min}(S))$, $\mathscr{W}^*_S = (\mathrm{op}\{x\} : x \in S)$. Then (1) $\Gamma\text{-dim}S = \Gamma\text{-dim}\mathscr{W}_S$, (2) $\dim S = \dim \mathscr{W}_S$, (3) $\Gamma\text{-Dim}S = \Gamma\text{-dim}\mathscr{W}^*_S$, (4) $\mathrm{Dim}\, S = \dim \mathscr{W}^*_S$, (5) $\gamma\text{-Dim}S = \gamma\text{-dim}(\{x\} : x \in S)$.

Proof: By 4.3 (1), $\mathscr{W}_S$ is the finest f.o.c. of S; this implies (1) and (2). — From 4.3 (2), it follows that $\mathscr{W}^*_S$ refines every r.f.o.c. of S, and if a f.o.c. $\mathscr{U}$ refines $\mathscr{F}_S$, then $\Gamma\text{-dim}\mathscr{U} \geq \Gamma\text{-dim}\mathscr{W}_S$, $\dim \mathscr{U} \geq \dim \mathscr{W}_S$. This proves (3) and (4). — Clearly, $\gamma\text{-Dim}S$ is equal (see 2.11) to the least n such that every r.f.o.c. of S is refined by some partition $\mathscr{B}$ with $\gamma\text{-dim}\mathscr{B} \leq n$. Hence, by 4.3 (1), $\gamma\text{-dim}S \geq \gamma\text{-dim}\mathscr{B}_S$, where $\mathscr{B}_S = (\{x\} : x \in S)$. Since $\mathscr{B}_S$ refines every r.f.o.c., we have $\gamma\text{-dim}S = \gamma\text{-dim}\mathscr{B}_S$. $\square$

4.5 Proposition. *Let S be a finite poset. Let $n \in N$. Then* (1) $\Gamma\text{-dim}S \leq n$ *iff* $\mathrm{Min}(S)$ *is a union of at most* $n+1$ *strong (from above) antichains,* (2) $\dim S \leq n$ *iff* $|X| \leq n+1$ *for every* $X \subset \mathrm{Min}(S)$ *bounded from above,* (3) $\gamma\text{-dim}S \leq n$ *iff there are antichain-like collections* $(X_{ik} : k \in K_i)$, $i \leq n$, *such that every* X_{ik} *is bounded from below and* $\bigcup(X_{ik} : i \leq n, k \in K_i) = S$.

Proof: Assertions (1) and (2) follow easily from 4.4 (1). If $\gamma\text{-dim}S \leq n$, then there is (see 2.11) a partition $\mathscr{B}$ refining $\mathscr{W}_S = (\mathrm{op}\{x\} : x \in \mathrm{Min}(S))$ and satisfying $\gamma\text{-dim}\mathscr{B} \leq n$. Hence there are $X_{ik} \subset S$, $i \leq n$, $k \in K_i$, such that $\gamma(X_{ik} : k \in K_i) = 0$, $i \leq n$, $\bigcup(X_{ik} : i \leq n, k \in K_i) = S$, and every X_{ik} is contained in some $\mathrm{op}\{x\}$, $x \in \mathrm{Min}(S)$. By 4.2 (4), this implies the condition in question. — If this condition is satisfied, then $\gamma\text{-dim}(X_{ik} : k \in K_i) = 0$, $i \leq n$, and we get $\gamma\text{-dim}(X_{ik} : i \leq n, k \in K_i) \leq n$, $(X_{ik} : i \leq n, k \in K_i) \prec \mathscr{W}_S$ which proves $\gamma\text{-dim}S \leq n$. $\square$

4.6. Recall the well-known fact that the height dimension of a poset is equal to the least $n \in N$ such that S is a union of $n+1$ antichains; if there is no such $n \in N$, then the dimension is infinite.

4.7 Proposition. *Let S be a finite poset. Then, for any $n \in N$,* (1) Γ-Dim$S \le n$ *iff S is a union of at most $n+1$ strong (from above) antichains;* (2) Dim $S \le n$ *iff $|X| \le n+1$ for every $X \subset S$ bounded from above;* (3) *for γ-Dim$S \le n$, each of the following conditions is necessary and sufficient:* (a) *S is a union of at most $n+1$ antichains;* (b) *$|C| \le n+1$ for every chain $C \subset S$.*

PROOF: Assertions (1) and (2) follows easily from 4.4 (3) and (4). — If γ-Dim$S \le n$, then, by 4.4 (5), γ-dim$(\{x\} : x \in S) \le n$, hence there are $A_i \subset S$, $i \le n$, such that $\bigcup A_i = S$ and γ-dim$(\{x\} : x \in A_i) = 0$ for $i \le n$. By 4.2 (1), A_i are antichains. — By 4.6, (a) and (b) are equivalent. — Assume that $|C| \le n+1$ for every chain $C \subset S$. Put $A_i = \{x \in S : ht(x) = i\}$. Clearly, $A_i = \emptyset$ if $i > n$. Hence γ-dim$(\{x\} : x \in S) \le n$ and therefore, by 4.4, γ-dim$S \le n$. □

4.8 Proposition. *For every finite poset S, p-Dim$S = \gamma$-Dim$S$.*

PROOF: We have to show that p-Dim$S \le \gamma$-Dim$S$. Let γ-Dim$S = n$. By 4.7, $|C| \le n+1$ for every chain $C \subset S$. Let A_i, $1 \le i \le n+1$, consist of all $x \in S$ such that $ht(x) = i$. It is easy to see that, for every $i \le n$, A_i is closed in S, hence (see 4.2 (1)), $(\{x\} : x \in S)$ is a normal partition of order $\le n$ (see 2.15). This proves p-Dim$S \le n$. □

4.9. Clearly, in the case of finite posets, Γ-dim, dim and γ-dim are not monotonic, whereas Γ-Dim, Dim and γ-Dim are (this follows from 4.7). — The sum formula for finite posets holds (see 4.9) for Dim and γ-Dim. For Γ-dim, dim and γ-dim, it fails already for the poset described in 3.10; for Γ-Dim it fails, if $|K| \ge 5$, for the set $\{X \subset K : 0 < |X| \le 2\}$ ordered by inclusion.

4.10 Fact. Let $\varphi =$ Dim or $\varphi = \gamma$-Dim. Let S be a finite poset. If X and Y are closed, $S = X \cup Y$, then $\varphi(S) = \max(\varphi(X), \varphi(Y))$.

PROOF: For $\varphi = \gamma$-Dim, this follows from 4.7 (3). If $\varphi =$ Dim and $\varphi(X) \le n$, $\varphi(Y) \le n$, let $T \subset S$ be bounded from above. Let $b \in S$, $t \le b$ for all $t \in T$. We have $t \subset X$ or $t \in Y$, hence, X and Y being closed, $T \subset X$ or $T \subset Y$; thus $|T| \le n+1$. We have shown that $|T| \le n+1$ for every $T \subset S$ bounded from above. By 4.7, this implies Dim $S \le n$. □

4.11. The results stated in 3.10, 4.9 and 4.10 can be summarized as follows:

	Γ-dim	dim	γ-dim	Γ-Dim	Dim	γ-Dim
monotonicity	−	−	−	+	+	+
addition formula	−	−	+	−	−	+
sum formula	−	−	−	−	+	+

Remark. For arbitrary posets, we obtain the same table, see 5.15.

4.12. Observe that the dimensions listed in 4.11 are distinct. This is clear from the table and the fact that e.g. Γ-dim$S = 2$, $\dim S = 1$ if $S = \{X \subset \{1,2,3\} : 0 < |X| < 3\}$.

5.

5.1. In what follows, we will consider, as an auxiliary concept, classes $\mathfrak{A}$ which satisfy the following conditions: (1) each element of $\mathfrak{A}$ is of the form (∗) $\langle \mathscr{U}, \mathscr{A}, \mathscr{V}, S\rangle$, where S is a space, $\langle \mathscr{U}, \mathscr{A}\rangle$ is a rigged finite open cover of S, and $\mathscr{V}$ is a finite cover of S; (2) if $\mathscr{F} = \langle \mathscr{U}, \mathscr{A}, \mathscr{V}, S\rangle \in \mathfrak{A}$ and $X \subset S$ is finite, then $\mathscr{F} \cap X \in \mathfrak{A}$, where $\mathscr{F} \cap X = \langle \mathscr{U} \cap X, \mathscr{A} \cap X, \mathscr{V} \cap X, X\rangle$; (3) for every space S, there is a system $\mathscr{Z} = \mathscr{Z}_S$ of its finite subsets such that (i) if $X \subset S$ is finite, then $X \subset Z$ for some $Z \in \mathscr{Z}$, (ii) if $\mathscr{F} = \langle \mathscr{U}, \mathscr{A}, \mathscr{V}, S\rangle$ is of the form (∗) and $\mathscr{F} \cap X \in \mathfrak{A}$ for every $X \in \mathscr{Z}_S$, then $\mathscr{F} \in \mathfrak{A}$.

5.2 Lemma. *Let $\mathfrak{A}$ satisfy* (1)–(3) *from* 5.1, *and let $\mathscr{Z}_S$ possess the properties stated in* 5.1 (3). *Let S be a poset. Assume that, for every finite set K, there is a finite set $M = M(K)$ such that the following holds: if $X \in \mathscr{Z}$ and $\mathscr{P} = (P_k : k \in K)$ is a finite open cover of X, then $\langle \mathscr{P}, \mathscr{P}, \mathscr{V}, X\rangle \in \mathfrak{A}$ for some cover $\mathscr{V} = (V_m : m \in M(K))$ of X. Then, for every finite open cover $\mathscr{U} = (U_k : k \in K)$ of S, there exists a cover $\mathscr{B} = (B_m : m \in M(K))$ such that $\langle \mathscr{U}, \mathscr{U}, \mathscr{B}, S\rangle \in \mathfrak{A}$.*

Proof: Consider the compact space M^S, where $M = M(K)$. For every $X \in \mathscr{Z}_S$, let $F(X)$ consist of all $f \in M^S$ such that $\langle \mathscr{U} \cap X, \mathscr{U} \cap X, \mathscr{V}, X\rangle \in \mathfrak{A}$, where $\mathscr{V} = (f^{-1}(m) \cap X : m \in M)$. By the assumption, $F(X) \neq \emptyset$. From (2) in 5.1, it follows that $F(X) \supset F(Y)$ for $X, Y \in \mathscr{Z}$, $X \subset Y$. This implies that $\{F(X) : X \in \mathscr{Z}_S\}$ is a base of a filter. Put $F(S) = \bigcap(\operatorname{cl} F(X) : X \in \mathscr{Z}_S)$. Since M^S is compact, $F(S) \neq \emptyset$. Let $g \in F(S)$ and put $\mathscr{B} = (g^{-1}(m) : m \in M)$. Then, for every finite $X \in \mathscr{Z}$, $\langle \mathscr{U}, \mathscr{U}, \mathscr{B}, S\rangle \cap X \in \mathfrak{A}$ and therefore, by (3) in 5.1, $\langle \mathscr{U}, \mathscr{U}, \mathscr{B}, S\rangle \in \mathfrak{A}$. □

Remark. Instead of $\langle \mathscr{P}, \mathscr{P}, \dots\rangle$, we could consider $\langle \mathscr{P}, \mathscr{T}, \dots\rangle$, where $\langle \mathscr{P}, \mathscr{T}\rangle$ is a rigged f.o.c. However, it is sufficient to consider only $\langle \mathscr{P}, \mathscr{P}, \dots\rangle$, since if S is a poset, then (1) for every f.o.c. $\mathscr{P}$, $\langle \mathscr{P}, \mathscr{P}\rangle$ is a rigged f.o.c., (2) a rigged f.o.c. $\langle \mathscr{P}, \mathscr{T}\rangle$ is refined by $\mathscr{V}$ whenever $\langle \mathscr{P}, \mathscr{P}\rangle$ is.

5.3 Fact. Let S be a space and let $n \in N$. Let $\mathscr{F} = \langle \mathscr{U}, \mathscr{A}\rangle$ be a rigged f.o.c. of S, $\mathscr{U} = (U_k : k \in K)$. If $\mathscr{V}$ is a finite cover of S, $\mathscr{V} \prec \mathscr{F}$ and γ-dim$\mathscr{V} \leq n$ (respectively, Γ-dim$\mathscr{V} \leq n$), then there is a finite cover $\mathscr{W}$ of the form $\mathscr{W} = (W_m : m \in K \times \{0, \dots, n\})$ such that (1) for every (k, i), $k \in K$, $i \leq n$, $W_{ki} \subset V_k$, (2) for $i \leq n$, γ-dim$\mathscr{W}_i \leq 0$ (respectively, Γ-dim$\mathscr{W}_i \leq 0$), where $\mathscr{W}_i = (W_{ki} : k \in K)$, (3) if $\mathscr{V}$ is a normal partition of order $\leq n$, then so is $\mathscr{W}$. — The proof is easy and may be omitted.

5.4 Proposition. *Let S be a poset, let $n \in N$ and let φ be one of the dimensions* Γ-Dim, Dim, γ-Dim. *Then $\varphi(S) \leq n$ if and only if $\varphi(X) \leq n$ for every finite $X \subset S$.*

Proof: I. Consider the case $\varphi = \Gamma$-Dim. Assume that $\varphi(X) \le n$ whenever $X \subset S$ is finite. Let $\mathscr{U} = (U_k : k \in K)$ be a f.o.c. of S. Put $M = K \times \{i : i \le n\}$. Let $\mathfrak{A}$ be the class of all $\mathscr{F} = \langle \mathscr{V}, \mathscr{V}, \mathscr{C}, T\rangle$ which are of the form $(*)$ from 5.1 and satisfy the following conditions: $T \subset S$, $\mathscr{C} = (C_m : m \in K \times \{i : i \le n\})$, $C_{ki} \subset U_k$ for all k, i, Γ-dim$(C_{ki} : k \in K) \le 0$ for all $i \le n$, $\mathscr{C} \prec \langle \mathscr{V}, \mathscr{V}\rangle$. Clearly, $\mathfrak{A}$ satisfies (1) and (2) from 5.1. It also satisfies (3) with $\mathscr{Z}_i = \{Y : Y \subset T,\ Y \text{ finite}\}$, since if Γ-dim$(C_{ki} \cap X : k \in K) \le 0$ for every finite $X \subset Z$, then Γ-dim$(C_{ki} : k \in K) \le 0$. This follows from the fact that if $P \subset T$ and $P \cap X$ is open in X for every finite $X \subset T$, then P is open in T. From 5.3, it follows that the assumptions in 5.2 are satisfied. Hence, by 5.2, there is a finite cover $\mathscr{B} = (B_m : m \in M)$ of S such that $\langle \mathscr{U}, \mathscr{U}, \mathscr{B}, S\rangle \in \mathfrak{A}$ and therefore $\mathscr{B} \prec \langle \mathscr{U}, \mathscr{U}\rangle$, Γ-dim$\mathscr{B} \le n$. This proves Γ-dim$S \le n$. — If Γ-Dim$S \le n$, then, by 3.9, Γ-Dim$X \le n$ for every finite $X \subset S$. — II. The case $\varphi = \gamma$-Dim is completely analogous to that of Γ-Dim. — III. Consider the case $\varphi =$ Dim. Let $\operatorname{Dim} X \le n$ for every finite $X \subset S$. Let $\mathfrak{A}$ be the class of all $\mathscr{F} = \langle \mathscr{V}, \mathscr{V}, \mathscr{C}, T\rangle$ which are of the form $(*)$ from 5.1 and satisfy the following conditions: $T \subset S$, $\mathscr{C} = (C_k : k \in K)$, C_k are open, $C_k \subset U_k$, $\mathscr{C}$ refines $\langle \mathscr{V}, \mathscr{V}\rangle$. It is easy to see that $\mathfrak{A}$ satisfies (1)–(3) from 5.1 (with $Z_T = \{Y : Y \subset T,\ Y \text{ finite}\}$) and that the assumptions in 5.2 are satisfied. Therefore, for every given f.o.c. $\mathscr{U}$ of S, there is a f.o.c. $\mathscr{B} = (B_k : k \in K)$ of S which refines $\langle \mathscr{V}, \mathscr{V}\rangle$ and satisfies $\dim \mathscr{B} \le n$. Thus, $\operatorname{Dim} S \le n$. — By 3.9, $\operatorname{Dim} S \le n$ implies $\operatorname{Dim} X \le n$ for finite $X \subset S$. □

5.5 Theorem. *Let S be a poset. Then, for any $n \in N$,*

(1) Γ-Dim$S \le n$ *if and only if S is a union of at most $n+1$ strong (from above) antichains;*
(2) $\operatorname{Dim} S \le n$ *if and only if $|X| \le n+1$ for every $X \subset S$ bounded from above;*
(3) *each of the following conditions is necessary and sufficient for γ-Dim $\le n$:*
   (a) *S is a union of at most $n+1$ antichains,*
   (b) *$|C| \le n+1$ for every chain $C \subset S$.*

Proof: I. If Γ-Dim$S \le n$, then, by 5.4 and 4.6, every finite $X \subset S$ is a union of at most $n+1$ antichains which are strong from above. Put $M = \{0, \dots, n\}$ and consider the compact space M^S. For every finite $X \subset S$, let $F(X)$ consist of all $f \in M^S$ such that every $f^{-1}(m)$, $m \in M$, is a strong from above (in X) antichain. It is easy to see that $F(S) = \bigcap(\operatorname{cl} F(X) : X \subset S \text{ finite}) \ne \emptyset$. Let $g \in F(S)$. For every finite $X \subset S$, each $g^{-1}(m) \cap X$, $m \in M$, is a strong from above (in X) antichain. This implies that every $g^{-1}(m)$ is a strong from above (in S) antichain. — If the condition in (1) is fulfilled for S, then, clearly, it is fulfilled for every $X \subset S$, and therefore, by 4.6 and 5.4, Γ-Dim$S \le n$. — II. For γ-Dim, the proof is analogous to that for Γ-Dim. — III. If $\operatorname{Dim} S \le n$, then, by 5.4 and 4.7, we have $|X| \le n+1$ for every $X \subset S$ bounded from above. — If this condition holds for S, then it holds for every $T \subset S$ and therefore, by 4.7 and 5.4, we have $\operatorname{Dim} S \le n$. □

5.6 Proposition. *Let S be a poset; let $n \in N$. Then*

(1) p-Dim$S \le n$ *iff* p-Dim$X \le n$ *for every finite* $X \subset S$,
(2) p-Dim$S = \gamma$-Dim$S$.

The first assertion is proved the same way as the analogous assertions (see 5.4) concerning Γ-Dim, etc. The second statement follows from (1) and 4.7.

5.7 Proposition. *For every poset S, p-Dim$S$ and γ-Dim$S$ are equal to the height dimension* hdim S.

This follows from 5.6 and 5.5 (3).

5.8 Theorem. *The partition dimension p-Dim coincides with the Čech-Lebesgue dimension on normal spaces and with the height dimension on posets.*

5.9 Definition. Let S be a poset. A subspace $T \subset S$ will be called relatively bounded from below (in S), if every finite $X \subset T$ bounded from below in S is bounded from below in T.

5.10 Proposition. *Let S be a poset. Every finite $X \subset S$ is contained in a finite $T \subset S$ relatively bounded from below.*

Proof: Let $\mathscr{Y}$ consist of all subsets of X bounded from below in S and let $\mathscr{Y}^*$ consist of all sets which are maximal in $\mathscr{Y}$. For every $Y \in \mathscr{Y}^*$ choose $p = p(Y) \in S$ such that $p \le y$ for all $y \in T$. Let P be the set of all $p(Y)$. Put $T = X \cup P$. — We are going to prove that T is relatively bounded from below. Let $A \subset T$ be bounded from below in S. Then A contains at most one $p \in P$. Indeed, suppose that $p(Y_1), p(Y_2) \in A$ and $b \le p(Y_i)$, $i = 1, 2$. Then $Y_1 \cup Y_2$ is bounded in S and therefore $Y_1 = Y_2$. If $p(Y) \in A$, then $Y \cup A$ is bounded in S and therefore $A \subset Y$, hence A is bounded from below in T. If $A \subset X$, then $A \subset Y$ for some $Y \in \mathscr{Y}^*$, hence A is bounded from below in T. □

5.11 Proposition. *Let S be a poset. Let φ stand for* Γ-dim, dim *or* γ-dim. *If $T \subset S$ is finite and relatively bounded from below, then $\varphi(T) \le \varphi(S)$.*

Proof: Put $B = \mathrm{Min}(T)$. For every $b \in B$, put $V_b = \mathrm{op}_S(\mathrm{cl}_S\{b\})$; for some a non $\in B$, put $V_a = S \setminus \mathrm{cl}_S B$. Clearly, $\mathscr{V} = (V_b : b \in B \cup \{a\})$ is a f.o.c. of S, $V_a \cap T = \emptyset$. We are going to show that, for every $b \in B$, $V_b \cap T = \mathrm{op}_T\{b\}$. Let $x \in V_b \cap T$. Then there is $y \in S$ such that $y \le b$, $y \le x$. Thus $\{b, x\}$ is bounded from below in S, hence in T. Let $z \in T$, $z \le b$, $z \le x$. Since $b \in \mathrm{Min}(T)$, we get $z = b$, $x \in \mathrm{op}_T\{b\}$. We have shown that $\mathscr{V} \cap T$ consists of all $\mathrm{op}_T\{b\}$, $b \in B$, and of $\emptyset$. — Let $\varphi(S) = n \in N$. Then there exists a finite cover $\mathscr{W}$ of S refining $\mathscr{V}$ and such that, as the case may be, (1) $\mathscr{W}$ is a f.o.c. and Γ-dim$\mathscr{W} < n$, (2) $\mathscr{W}$ is a f.o.c. and dim $\mathscr{W} \le n$, (3) γ-dim$\mathscr{W} \le n$. Evidently $\mathscr{W} \cap T$ refines $(\mathrm{op}_T\{b\} : b \in B)$ and (1)–(3) hold with $\mathscr{W}$ replaced by $\mathscr{W} \cap T$. Since $(\mathrm{op}_T\{b\} : b \in B)$ is the finest f.o.c. of T, we see that $\varphi(T) \le n$. □

5.12 Theorem. *Let S be a poset. Let φ be one of the dimensions* Γ-dim, dim γ-dim. *Then, for any $n \in N$, $\varphi(S) \leq n$ if and only if $\varphi(X) \leq n$ for every relatively bounded (from below) finite $T \subset S$.*

We omit the proof, since it is analogous to that of 5.4. The only substantial difference consists in putting $\mathscr{X}_T = \{X : X \subset T,\ X \text{ finite relatively bounded from below}\}$.

5.13. In 5.5, we have given necessary and sufficient conditions, expressed in terms of order relation, for $\varphi(S) \leq n$, where S is a poset and φ is one of the dimensions Γ-Dim, Dim, γ-Dim. For Γ-dim, etc., conditions of this kind can also be easily given. However, it is difficult to present them in a sufficiently simple form. Therefore, and in view of not too good properties of these dimensions, we omit the corresponding propositions.

5.14. In 4.11, we have summarized the results in monotonicity of dimensions etc., for finite posets. For arbitrary posets, we have only to check which of the positive assertions remain true. For γ-dim, the addition formula holds by 3.10. For Γ-Dim and Dim, we have monotonicity by 5.4. The sum formula for Dim is proved in the same way as in 4.10. For γ-Dim, see 3.10 and 5.4. Thus, we obtain the same table as in 4.11.

6.

In this section, we consider continuous mappings onto finite posets. The main result (6.6) concerns the characterization of the partition dimension p-Dim by means of these mappings. We present similar results for other dimensions.

6.1 Notation. Let $\mathscr{B} = (B_m : m \in M)$, where no B_m is void, be a partition of S. We put $f_{\mathscr{B}}(x) = m$ for $x \in B_m$. The set M with the quotient topology induced by the mapping $f_{\mathscr{B}}$ will be denoted by $T(S, \mathscr{B})$. If $\mathscr{U}$ is a finite open cover of S, then we often write $f_{\mathscr{U}}$ instead of $f_{\mathrm{at}\,\mathscr{U}}$ (cf. 3.2 and 3.3) and $T(S, \mathscr{U})$ instead of $T(S, \mathrm{at}\,\mathscr{U})$.

6.2 Facts. (A) If T is a finite poset, then $(\{x\} : x \in T)$ is a normal partition. — (B) If X and Y are spaces, $f : X \to Y$ is continuous and $\mathscr{B}$ is a normal partition of Y of order $\leq n$, then $f^{-1}(\mathscr{B})$ is a normal partition of X of order $\leq n$.

Proof: Put $A_0 = \mathrm{Min}(T)$; if A_i is defined, put $A_{i+1} = \mathrm{Min}(T \setminus \bigcup(A_j : j \leq i))$. Obviously, $A_n = \emptyset$ for some $n \leq |T|$. It is easy to see that, for $j \leq n$, $\bigcup(A_i : i \leq j)$ is closed and γ-dim$A_j \leq 0$. — The second statement is evident. □

6.3 Lemma. *Let $\mathscr{B} = (B_m : m \in M)$, where no B_m is void, be a partition of a space S. The following properties of $\mathscr{B}$ are equivalent:*

(1) *$T(S, \mathscr{B})$ is a T_0-space (hence a poset).*
(2) *$\mathscr{B}$ is normal,*
(3) *for $p \in N$, $p \geq 1$, there are no distinct $m(i) \in M$, $i \leq p$, such that $B_{m(i)} \cap \mathrm{cl}\, B_{m(i+1)} \neq \emptyset$ for $i < p$, $B_{m(p)} \cap \mathrm{cl}\, B_{m(0)} \neq \emptyset$,*
(4) *there is a finite open cover $\mathscr{U}$ of S such that $\mathscr{B} \equiv \mathrm{at}\,\mathscr{U}$.*

If $\mathscr{B}$ is normal, then it is of order $\leq n$ iff γ-Dim$T(S,\mathscr{B}) \leq n$.

Proof: It is easy to see that, in $T = T(S,\mathscr{B})$, $a \in \mathrm{cl}\{b\}$, $a \neq b$, iff for some $p \in N$, $p \geq 1$, there are $m(i) \in M$, $i \leq p$, such that $a = m(0)$, $b = m(p)$, $B_{m(i)} \cap \mathrm{cl}\, B_{m(i+1)}$ for $i < p$. Hence, if $a \in \mathrm{cl}\{b\}$, $b \in \mathrm{cl}\{a\}$, then (3) fails. We have shown that (3) implies (1). — If T is a T_0-space, then, by 6.2, $\mathscr{B} = f_{\mathscr{B}}^{-1}(\{t\} : t \in T)$ is normal. Thus, (1) implies (2). — If (2) holds, then there are disjoint $M_i \subset M$, $i \leq n$, such that $\bigcup M_i = M$, γ-dim$(B_m : m \in M_i) \leq 0$ for $j \leq n$, and every $\bigcup(B_m : m \in \bigcup(M_j : j \leq i))$, $i \leq n$, is closed in S. Clearly, if $a, b \in M$, $a \neq b$, $a \in M_i$, $b \in M_j$, $a \in \mathrm{cl}\{b\}$, then $i < j$. This implies (3). — If (1) holds, then $\mathscr{B}$ is normal by 6.2. — The assertion concerning the order of $\mathscr{B}$, follows, for $\mathscr{B}$ normal, from $\mathscr{B} = f^{-1}(\{t\} : t \in T)$.

If $\mathscr{U}$ is a f.o.c. of S, then, by 3.3, at$\,\mathscr{U}$ is a normal partition. Thus, it remains to be shown that if $\mathscr{B}$ is normal, then $\mathscr{B} \equiv$ at$\,\mathscr{U}$ for some f.o.c. $\mathscr{U}$. Since (2) implies (1), $T = T(S,\mathscr{B})$ is a finite poset. Put $\mathscr{W}_T^* = (\mathrm{op}\{t\} : t \in T)$, $\mathscr{V} = f_{\mathscr{B}}^{-1}(\mathscr{W}_T^*)$. Clearly, at$\,\mathscr{U} = f_{\mathscr{B}}^{-1}(\mathrm{at}\,\mathscr{W}_T^*) = \mathscr{B}$. □

6.4 Lemma. *If $\mathscr{U}$ is a finite open cover of a space S, then $T = T(S,\mathscr{U})$ is a finite poset, p-Dim$T \leq \dim \mathscr{U}$. If $\mathscr{U}$ is irreducible,* (*i.e. no $\mathscr{V} \subset \mathscr{U}$, $\mathscr{V} \neq \mathscr{U}$, is a cover*), *then* $\dim T = \dim \mathscr{U}$, Γ-dim$T = \Gamma$-dim$\mathscr{U}$.

Proof: The partition $\mathscr{B} =$ at$\,\mathscr{U}$ is normal of order $\leq \dim \mathscr{U}$, by 3.3. As an easy consequence, we get p-Dim$T = \gamma$-Dim$T \leq \dim \mathscr{U}$. Assume that $\mathscr{U} = (U_k : k \in K)$ is irreducible. Put $\mathscr{V} = (\mathrm{op}\{t\} : t \in \mathrm{Min}(T))$. By 4.4, we have $\dim T = \dim \mathscr{V}$, Γ-dim$T = \Gamma$-dim$\mathscr{V}$. For every $z \in \{0,1\}^K$ put $B_z = \bigcap(W_{zk} : k \in K)$, where $W_{zk} = U_k$ if $z(k) = 1$, $W_{zk} = S \setminus U_k$ if $z(k) = 0$. Put $Z = \{z \in \{0,1\}^K : B_z \neq \emptyset\}$. Clearly, at$\,\mathscr{U} \equiv \mathscr{B} = (B_z : z \in Z)$. From the irreducibility of $\mathscr{U}$, it follows easily that (1) Min(T) consists of all $z_h \in \{0,1\}^K$, where $z_h(k) = 1$ if $k = h$, $z_h(k) = 0$ if $k \neq h$ (2) $f^{-1}(\mathrm{op}\{z_h\}) = U_h$ for every $h \in K$. Hence $\dim \mathscr{V} = \dim \mathscr{U}$, Γ-dim$\mathscr{V} = \Gamma$-dim$\mathscr{U}$. This implies $\dim T = \dim \mathscr{U}$, Γ-dim$T = \Gamma$-dim$\mathscr{U}$.

□

6.5 Definition. Let X and Y be spaces and let $f : X \to Y$ be a mapping. Let $\mathscr{H}$ be a finite open cover or a rigged finite open cover of X. We will say that (1) f is <u>strongly</u> <u>$\mathscr{H}$-fine</u> (f <u>strongly refines</u> $\mathscr{H}$), if, for some finite open cover $\mathscr{V}$ of Y, $f^{-1}(\mathscr{V}) \prec \mathscr{H}$, (2) f is <u>weakly</u> <u>$\mathscr{H}$-fine</u> (f <u>weakly refines</u> $\mathscr{H}$), if, for some finite partition $\mathscr{B}$ of Y, $f^{-1}(\mathscr{B}) \prec \mathscr{H}$.

6.6 Theorem. *For every non-void space S, the partition dimension p-Dim$S$ of S is equal to the least $n \in N \cup \{\infty\}$ such that every rigged finite open cover of S is weakly refined by some continuous mapping of S onto a finite poset with height dimension not exceeding n.*

Proof: I. Assume p-Dim$S \leq n$. Let $\mathscr{F} = \langle \mathscr{U}, \mathscr{A} \rangle$ be a r.f.o.c. of S. There is a normal partition $\mathscr{B}$ of S of order $\leq n$ such that $\mathscr{B} \prec \mathscr{F}$. Put $f = f_{\mathscr{B}}$, $T = T(S,\mathscr{B}) = f(S)$. By 6.3, T is a poset and γ-Dim$T \leq n$. Since $f^{-1}(\{t\}$:

$t \in T) = \mathscr{B}$ and $\mathscr{B} \prec \mathscr{F}$, f is weakly $\mathscr{F}$-fine. — II. Assume that the condition in the theorem is fulfilled. Let $\mathscr{F} = \langle \mathscr{U}, \mathscr{A} \rangle$ be an arbitrary r.f.o.c. of S. There is a weakly $\mathscr{F}$-fine continuous $g : S \to T$, where T is a finite poset, p-Dim$(T) \leq n$. Since f is weakly $\mathscr{F}$-fine, we have $\mathscr{B} = f^{-1}(\{t\} : t \in T) \prec \mathscr{F}$. From 6.2, it follows that $\mathscr{B}$ is a normal partition of order $\leq n$. □

6.7 Theorem. *If S is a non-void normal space, then the Čech-Lebesgue dimension of S is equal to the least $n \in N$ such that every rigged finite open cover of S is strongly refined by some continuous $f : S \to T$, where T is a finite poset with height dimension not exceeding n.*

Proof: It follows from 6.6 that we have only to show that if $\dim S \leq n$, then for every r.f.o.c. $\mathscr{F} = \langle \mathscr{U}, \mathscr{A} \rangle$, there is a strongly $\mathscr{F}$-fine $g : S \to T$ with the properties stated in the theorem. Let $\mathscr{U} = (U_k : k \in K)$, $\mathscr{A} = (A_k : k \in K)$. Since S is normal, it is easy to prove that there is a f.o.c. $\mathscr{V} = (V_m : m \in M)$ such that $\mathscr{V} \prec \mathscr{F}$, $\dim \mathscr{V} \leq n$ and

($*$) if $m(i) \in M$, $i \leq q \leq n$, $V_{m(i)} \cap V_{m(i+1)} \neq \emptyset$ for $i < q$ and $V_{m(0)} \cap A_k \neq \emptyset$ for some $k \in K$, then $V_{m(q)} \subset U_k$. — Put $\mathscr{B} = \operatorname{at} \mathscr{V}$, $\mathscr{B}(B_t : t \in T)$. Put $g = f_{\mathscr{B}}$. Then, by 6.3, g is a continuous mapping onto $T = T(S, \mathscr{B})$, γ-Dim$T \leq n$. — We are going to show that g is strongly $\mathscr{F}$-fine. To this end it is sufficient to prove that if $k \in K$, $x \in T$, $B_x \cap A_k \neq \emptyset$, $x \leq y$ (in T), then $B_y \subset U_k$; indeed, this will imply $g^{-1}(\operatorname{op}\{x\}) \subset U_k$, hence $g^{-1}(\operatorname{op}\{t\} : t \in T) \prec \mathscr{F}$. Since $\mathscr{B}$ is a normal partition of order $\leq n$, there are disjoint $T_j \subset T$, $j \leq n$, such that γ-dim$(B_t : t \in T) \leq 0$ and $\bigcup(B_t : t \in \bigcup(T_j : j \leq i))$ is closed in S for all $i \leq n$. Since $x \leq y$, there are $t(i)$, $i \leq q$, such that $t(0) = x$, $t(q) = y$, $B_{t(i)} \cap \operatorname{cl} B_{t(i+1)} \neq \emptyset$ for $i < q$. If $t(i) \in T_{h(i)}$, this implies that $h(i) \subset h(i+1)$ for $i < q$, and therefore $q \leq n$. Due to $\mathscr{B} = \operatorname{at} \mathscr{V}$, there are $m(i) \in M$ such that $B_{t(i)} \subset V_{m(i)}$ for $i \leq q$. Clearly, $V_{m(i)}$ being open, we have $V_{m(i)} \cap V_{m(i+1)} \neq \emptyset$ for $i < q \leq n$. Since $V_{m(0)} \cap A_k \neq \emptyset$, we get, by ($*$), $V_{m(q)} \subset U_k$, hence $B_y = B_{t(q)} \subset V_{m(q)} \subset U_k$. □

6.8 Fact. Let φ stand for dim or Γ-dim or else γ-dim. Let S be a space and let $n \in N$. Assume that every finite open cover $\mathscr{U}$ of S is strongly refined by some continuous mapping f of S onto T, where T is a finite poset and $\varphi(T) \leq n$. Then $\varphi(S) \leq n$.

Proof: Put $M = \operatorname{Min}(T)$, $\mathscr{V} = (\operatorname{op}\{y\} : y \in M)$. Since $\mathscr{V}$ refines every f.o.c. of T, $f^{-1}(\mathscr{V})$ refines $\mathscr{U}$. If $\varphi = \dim$ or $\varphi = \Gamma$-dim, then, by 4.4, $\varphi(\mathscr{V}) \leq n$ and therefore $\varphi(f^{-1}\mathscr{V}) \leq n$. This proves $\varphi(S) \leq n$. — If $\varphi = \gamma$-dim, $\varphi(T) \leq n$, then there is a finite cover $\mathscr{B}$ of T refining $\mathscr{V}$ and satisfying γ-dim$\mathscr{B} \leq n$. Clearly, γ-dim$f^{-1}(\mathscr{B}) \leq n$ and $f^{-1}(\mathscr{B})$ refines $f^{-1}(\mathscr{V})$, hence $\mathscr{U}$. This proves γ-dim$S \leq n$. □

6.9 Proposition. *Let φ stand for* dim *or* Γ-dim. *For every non-void space S, $\varphi(S)$ is equal to the least $n \in N \cup \{\infty\}$ such that every finite open cover of S is strongly refined by some continuous mapping of S onto T, where T is a finite poset and $\varphi(T) \leq n$.*

Proof: Assume that $\varphi(S) \leq n$. Let $\mathscr{U}$ be a f.o.c. of S. There is a f.o.c. $\mathscr{V} \prec \mathscr{U}$ satisfying $\varphi(\mathscr{V}) \leq n$; clearly, we can assume that $\mathscr{V}$ is irreducible (see 6.4). Then $f_{\mathscr{U}}$ maps S onto $T = T(S, \mathscr{U})$, and, by 6.4, $\varphi(T) = \varphi(\mathscr{V}) \leq n$. Hence the condition in the theorem is necessary. By 6.8, it is sufficient. □

7.

In this concluding section, we introduce the Dushnik-Miller character $\mathrm{DM}\,\chi(S)$ and weight $\mathrm{DMw}(S)$ for arbitrary T_0-spaces S. However, in this paper, we will not examine their properties, and present only some simple facts and some open questions.

7.1. Recall that the Dushnik-Miller dimension of a poset $S = \langle S, \sigma \rangle$ is defined [DM 41] as the least cardinal of a set L of linear orders on S such that $\bigcap(\lambda : \lambda \in L) = \sigma$. We will denote this dimension by $\mathrm{DM}(S)$. — It is not difficult to show that, for a quasi-discrete T_0-space S, $\mathrm{DM}(S)$ is equal to the least κ such that S can be embedded into a box-product of κ linear posets. — Recall that the box-product of spaces X_a, $a \in A$, is defined as $\prod(X_a : a \in A)$ endowed with the topology an open base of which consists of all $\prod(G_a : a \in A)$, where every G_a is open in X_a.

7.2 Definition. For every T_0-space S, $\mathrm{DM}\,\chi(S)$, called the <u>Dushnik-Miller character</u> of S, and $\mathrm{DMw}(S)$, called the <u>Dushnik-Miller weight</u> of S, will denote, respectively, the least κ such that S can be embedded into a cartesian product of κ quasi-discrete T_0-spaces (κ linear quasi-discrete T_0-spaces). — Thus, $\mathrm{DM}\,\chi(S)$ and $\mathrm{DMw}(S)$ are introduced in a way, which reminds of $\mathrm{DM}(S)$; however, a substantial difference consists in using the cartesian product instead of the box product.

7.3. It is easy to prove that, for every T_0-space S, $\chi(S) \leq \mathrm{DM}\,\chi(S) \leq \mathrm{DMw}(S) \leq \mathrm{w}(S)$. It can be proved that $\mathrm{DM}(S) \leq \mathrm{DM}\,\chi(S)$ for every poset S, and $\mathrm{DMw}(S) \leq \omega$ whenever S is metrizable.

7.4. There is a number of open questions concerning $\mathrm{DM}\,\chi$ and DMw, e.g. the following ones: (1) to characterize, by means of intrinsic properties, spaces with $\mathrm{DM}\,\chi(S) = \omega$ and those with $\mathrm{DMw}(S) = \omega$, (2) to find a space S (with nice properties if possible) for which $\chi(S) < \mathrm{DM}\,\chi(S) < \mathrm{DMw}(S) < \mathrm{w}(S)$.

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