An existence theorem for extended mildly nonlinear complementarity problem in semi-inner product spaces

M.S. Khan

Abstract. We prove a result for the existence and uniqueness of the solution for a class of mildly nonlinear complementarity problem in a uniformly convex and strongly smooth Banach space equipped with a semi-inner product. We also get an extension of a nonlinear complementarity problem over an infinite dimensional space. Our last results deal with the existence of a solution of mildly nonlinear complementarity problem in a reflexive Banach space.

Keywords: strongly smooth Banach space, mildly nonlinear complementarity problem
Classification: 49A10, 90C33, 49A29, 65K10

1. Introduction

Complementarity theory has become a rich source of inspiration in both mathematical and engineering sciences. This theory provides us with a natural and elegant framework for the study of many unrelated free boundary value and equilibrium problems. Much work in this field has been done either in inner product spaces or in Hilbert spaces, since it is generally felt that this is desirable, if not essential, for the results to hold. Further, it has been observed that these results depend upon properties, which are independent of any inner product or Hilbert space structure. Recently, an important and useful generalization of nonlinear complementarity problem was introduced by Noor [10] who studied the notion of mildly nonlinear complementarity problem. In this paper, we consider and study an extension of the mildly nonlinear complementarity problem in the framework of semi-inner product spaces.

2. Preliminaries

A real vector space E is said to be a semi-inner product space if there is a function $[,] : E \times E \to \mathbb{R}$ with the following properties:

(i) $[x, x] > 0$ for $x \neq 0$ $(x \in E)$,
(ii) $[x + y, z] = [x, z] + [y, z]$ for $x, y, z \in E$,
(iii) $[\lambda x, y] = \lambda[x, y]$ for $x, y \in E$, $\lambda \in \mathbb{R}$,
(iv) $|[x, y]|^2 \leq [x, x][y, y]$ for $x, y \in E$.

It was Lumer [6] who originally introduced and studied the concept of the semi-inner product spaces. We also note that a semi-inner product space is a normed space with the norm given by $\|x\| = [x,x]^{1/2}$. It has been shown by Lumer [6] that a normed space can be made a semi-inner product in a unique way if and only if it is smooth. However, in general, every normed space can be made a semi-inner product space in infinitely many different ways. Giles [5] has further observed that if a normed space X is a uniformly convex smooth Banach space, then the semi-inner product has the following properties:

(a) $[x, \lambda y] = \lambda[x,y]$ for all $x, y \in X$, $\lambda \in \mathbb{R}$,
(b) $[x,y] = 0$ if and only if y is orthogonal to x,
(c) (The generalized Riesz-Fischer Representation Theorem): Given $f \in X^*$, there is a unique $y \in X$ such that $f(x) = [x,y]$ for all $x \in X$. Here X^* denotes the dual of X.

Let X be a strongly smooth (see [9] for definition) and uniformly convex Banach space equipped with the semi-inner product $[,]$. Let K be a closed convex cone in X with a vertex at 0. The polar of K is the cone K^*, defined by $K^* = \{y \in K : [x,y] \geq 0 \text{ for all } x \in K\}$.

Definition 2.1. Let $A : X \to X$. Then A is said to be Lipschitz continuous if there exists a constant $b > 0$ such that

$$\|Ax - Ay\| \leq b\|x - y\|$$

for all $x, y \in X$.

Definition 2.2. Let $T : K \to X$. Then following Edelstein ([4]), we say that

(i) T is contractive if $\|Tx - Ty\| < \|x - y\|$ for all $x, y \in K$ with $x \neq y$,
(ii) T is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$.

Let the value of $u \in X^*$ at $x \in X$ be denoted by (u, x).

Definition 2.3. Let T be a mapping of subset $D(T)$ of X into X^*. Then T is said to be

(i) monotone if $(Tx - Ty, x - y) \geq 0$ for all $x, y \in D(T)$,
(ii) strongly monotone if there exists a constant $c > 0$ such that $(Tx - Ty, x - y) \geq c\|x - y\|^2$ for all $x, y \in D(T)$,
(iii) coercive if for $x \in D(T)$, $\frac{(Tx,x)}{\|x\|} \to \infty$ as $\|x\| \to \infty$,
(iv) hemicontinuous if for any $x, y \in K$ (where K is a closed cone in $D(T)$), the map

$$t \to (tx + (1-t)y)$$

of $[0,1]$ to X^* is continuous for the natural topology of $[0,1]$ and the weak* topology of X^*.

We also need the following result in order to prove some results in reflexive Banach spaces.

Lemma 2.4 (Browder [2]). *Let T be a monotone and hemicontinuous map of a closed convex set K of a reflexive Banach space X, with $0 \in K^*$, into X^*, and if K is not bounded, let T be coercive on K. Then there is an $x_0 \in K$ such that*

$$(Tx_0, y - x_0) \geq 0$$

for all $y \in K$.

3. Existence results

In this section, we study those conditions under which there does exist a unique solution of an extended type mildly nonlinear complementarity problem.

Theorem 3.1. *Let X be a uniformly convex and strongly smooth Banach space with semi-inner product $[,]$, and K be a closed convex cone in X. Let $T : K \to X$ be a contractive mapping, $S : K \to X$ be a nonexpansive mapping, and $A : X \to X$ be Lipschitz continuous with constant $b > 0$. Then there is a unique y_0 such that*

$$y_0 \in K, \; Ty_0 + Ay_0 \in K^*, \quad \text{and} \quad [Ty_0 - Sy_0 + (2+b)y_0, (2+b)y_0] = 0.$$

Proof: As K is a closed convex subset of a uniformly convex Banach space X, by a result of Edelstein [3], for every $y \in K$, there is a unique $x \in K$ such that

$$\|x - Sy + Ty + Ay\| \leq \|z - Sy + Ty + Ay\|$$

for every $z \in K$.

Let the correspondence $y \to x$ be denoted by Θ. Let $z \in K$ and let $0 \leq \lambda < 1$. As K is convex, $(1-\lambda)x + \lambda z \in K$. Define a function $h : [0,1] \to \mathbb{R}^+$ by setting

$$h(\lambda) = \|Sy - Ty - Ay - (1-\lambda)x - \lambda z\|^2 .$$

Since X is uniformly convex and strongly smooth, h is a continuously differentiable function of λ and

$$h'(\lambda) = 2[Sy - Ty - Ay - (1-\lambda)x - \lambda z, x - z] .$$

As x is a unique element closest to $Sy - Ty - Ay$, we must have $h'(0) \geq 0$. Therefore, it follows that

$$[Sy - Ty - Ay - x, \; x - z] \geq 0 \tag{1}$$

for every $z \in K$.

Let $y_1, y_2 \in K$ with $y_1 \neq y_2$. Let $\Theta(y_1) = x_1$ and $\Theta(y_2) = x_2$. Then it follows from (1) that

$$[Sy_1 - Ty_1 - Ay_1 - \Theta(y_1), \; - \Theta(y_2) + \Theta(y_1)] \geq 0 \tag{2}$$

and

$$[Sy_2 - Ty_2 - Ay_2 - \Theta(y_2),\ \Theta(y_2) - \Theta(y_1)] \geq 0. \tag{3}$$

From (2) and (3) we obtain

$$[Sy_1 - Ty_1 - Ay_1 - Sy_2 + Ty_2 + Ay_2 - \Theta(y_1) + \Theta(y_2),\ \Theta(y_1) - \Theta(y_2)] \geq 0\,.$$

This gives

$$[Sy_1 - Ty_1 - Ay_1 - Sy_2 + Ty_2 + Ay_2,\ \Theta(y_1) - \Theta(y_2)] \geq \|\Theta(y_1) - \Theta(y_2)\|^2\,.$$

Hence we get

$$\begin{aligned}\|\Theta(y_1) - \Theta(y_2)\| &\leq \|Sy_1 - Ty_1 - Ay_1 - Sy_2 + Ty_2 + Ay_2\| \\ &= \|(Sy_1 - Sy_2) - (Ty_1 - Ty_2) - (Ay_1 - Ay_2)\| \\ &\leq \|Sy_1 - Sy_2\| + \|Ty_1 - Ty_2\| + \|Ay_1 - Ay_2\| \\ &< (2+b)\|y_1 - y_2\|\,.\end{aligned}$$

Therefore, by a fixed point theorem of Edelstein [4], there exists a unique fixed point y_0 of the contractive mapping $(\Theta/(2+b))$ i.e. $\Theta y_0 = (2+b)y_0$.

So it follows from (1) that

$$[Sy_0 - Ty_0 - (2+b)y_0,\ (2+b)y_0 - z] \geq 0 \quad \text{for all} \quad z \in K\,. \tag{4}$$

As $0 \in K$, letting $z = 0$ in (4), we obtain

$$[Sy_0 - Ty_0 - (2+b)y_0,\ (2+b)y_0] \geq 0\,. \tag{5}$$

Further, K is a convex cone and $y_0 \in K$, so by taking $z = 4y_0 + 2by_0$ in (4), we get

$$[Sy_0 - Ty_0 - (2+b)y_0,\ (2+b)y_0] \leq 0\,. \tag{6}$$

Thus it follows that

$$[Sy_0 - Ty_0 - (2+b)y_0,\ (2+b)y_0] = 0\,.$$

This completes the proof. □

Remark 1. Let X, K, A, S and T be as in the statement of Theorem 3.1. Then the extended mildly nonlinear complementarity problem is to find $y \in K$ which satisfies the conclusion of Theorem 3.1.

Remark 2. Analogous to nonlinear complementarity problem, we can state the following property of T as a consequence of (4), (5) and (6):

$$(T - S + (2+b)I)\ y_0 \in K \quad \text{for} \quad y_0 \in K, \quad \text{where } I \text{ is the identity mapping.}$$

We can derive following corollaries from Theorem 3.1.

Corollary 3.2 (Nath. et al. [9]). *Let X be a uniformly convex and strongly smooth Banach space with semi-inner product $[,]$, and $T : K \to X$ be a contractive mapping, and $S : K \to X$ be nonexpansive, where K is a closed convex cone in X. Then there is a unique $y_0 \in K$ such that*

$$[Ty_0 - Sy_0 + 2y_0,\ 2y_0] = 0\,.$$

Proof: Follows from Theorem 3.1 when A is the zero nonlinear mapping. □

Corollary 3.3 (Nath. et al. [9]). *Let X be a uniformly convex and strongly smooth Banach space with semi-inner product $[,]$, and $T : K \to X$ be a contractive mapping, where K is a closed convex cone in X. Then there exists a unique $y_0 \in K$ such that*

$$[Ty_0 + y_0,\ y_0] = 0\,.$$

Proof: Follows from Corollary 3.2 and property (i) for semi-inner product on uniformly convex and smooth Banach space. □

The following result is an extension of the main result of Nanda [7].

Theorem 3.4. *Let X be a uniformly convex and strongly smooth Banach space with semi-inner product $[,]$, and let K be a closed convex cone in X. Let A and T be nonlinear mappings from K into X such that T is contractive and A is Lipschitz continuous with constant $b > 0$. Then there exists a unique $y_0 \in K$ satisfying*

$$Ty_0 + Ay_0 \in K^* \quad \textit{and} \quad [Ty_0 + Ay_0,\ y_0] = 0\,. \tag{7}$$

Proof: Proceeding exactly as in the proof of Theorem 3.1, we see that there exists a unique $y_0 \in K$ satisfying the conclusion of Theorem 3.1. Now, if we take $S = I$, the identity map, and $x = y_0$ in (1), then we get

$$[y_0 - Ty_0 - Ay_0 - y_0,\ y_0 - z] \geq 0 \quad \text{for all} \quad z \in K\,.$$

Thus

$$[Ty_0 + Ay_0,\ z - y_0] \geq 0 \quad \text{for all} \quad z \in K\,. \tag{8}$$

But as shown in Noor [10], (7) is equivalent to (8). Thus y_0 is the unique solution of (7). This completes the proof. □

Letting $A \equiv 0$, we get following

Corollary 3.5 (Nanda [7]). *Let X be a uniformly convex and strongly smooth Banach space with semi-inner product $[,]$, and let K be a closed convex cone in X with vertex at 0. Let $T : K \to X$ be Lipschitzian and strongly monotone with $b^2 < 2c < b^2 + 1$. Then there exists a unique y_0 such that*

$$y_0 \in K,\ Ty_0 \in K^* \ \text{ and } \ [Ty_0,\ y_0] = 0\,,$$

where c, b are respectively the strongly monotonicity and Lipschitzian constants.

Now, onwards we assume that X is reflexive real Banach space.

Theorem 3.6. *Let $A, T : K \to X^*$ be such that $A + T$ is hemicontinuous and monotone. Then there exists x_0 such that*

$$x_0 \in K,\ Ax_0 + Tx_0 \in K^* \ \text{ and } \ (Ax_0 + Tx_0,\ x_0) = 0 \tag{9}$$

provided one of the following conditions hold:

(a) *$A + T$ is coercive,*
(b) *$(A + T)(0) \in K^*$.*

PROOF:

(a) Let $S = T + A$. As S is hemicontinuous, monotone and coercive, by Lemma 2.4, it follows that there exists an $x_0 \in K$ such that

$$(Sx_0,\ y - x_0) \geq 0 \ \text{ for all } \ y \in K\,.$$

Since $0 \in K$, putting $y = 0$, we get

$$(Sx_0,\ x_0) \leq 0\,.$$

But K is a cone, so $2x_0 \in K$ giving thereby $(Sx_0, x_0) \geq 0$. Thus we have $(Sx_0, x_0) = 0$. Further, $Sx_0 \in K$, otherwise there will be a $y_0 \in K$ such that $(Sx_0, y_0) < 0$. Then we would have

$$0 > (Sx_0,\ y_0) \geq (Sx_0,\ x_0) = 0\,.$$

This contradiction completes the proof.

(b) Firstly, note that for each $r \geq 0$, the set $D_r = \{x \in K : \|x\| \leq r\}$ is a nonempty closed convex set in K with $0 \in D_r$. So it follows from Lemma 2.4 that for each $r \geq 0$, there exists a unique $x_r \in D_r$ such that

$$(Sx_r,\ y - x_r) \geq 0 \ \text{ for all } \ y \in D_r\,.$$

As $0 \in D_r$, it follows that $(Sx_r, x_r) \leq 0$.

Also, since S is monotone, we get

$$(Sx_r - S(0),\ x_r) \geq 0\,.$$

Further, if $S(0) \in K^*$, we obtain

$$(Sx_r,\ x_r) = 0\,.$$

Thus $(Sx_r, x_r) = 0$, i.e. $(Tx_r + Ax_r, x_r) = 0$.
Hence x_r is a unique solution of (9) for $r \geq 1$.

□

Remark 3. Let $A \equiv 0$, then Theorem 3.5 (a) is a result due to Bazaraa et al. [1]. Also, Theorem 3.6 (b) was obtained earlier by Nanda and Nanda [8].

Remark 4. Noor ([10], [11], [12]) has developed several iteration algorithms for different type complementarity problems. It will be interesting to develop an iteration scheme for complementarity problem discussed in this paper. It is also worth asking if the results of this paper can be extended to other types of nonlinear complementarity problems.

Acknowledgement. A part of this work was done while the author was a Visiting Fellow at the University of Wollongong, Australia. The author thanks Professor Sidney A. Morris, Dean, Faculty of Informatics, for providing the excellent research facilities and financial support.

References

[1] Bazaraa M.S., Goode J.J., Nashed M.Z., *A nonlinear complementarity problem in mathematical programming in Banach spaces*, Proc. Amer. Math. Soc. **35** (1972), 165–170.

[2] Browder F.E., *Nonlinear monotone operators and convex sets in Banach spaces*, Bull. Amer. Math. Soc. **71** (1965), 780–785.

[3] Edelstein M., *On nearest points of sets in uniformly convex Banach spaces*, J. London Math. Soc. **43** (1968), 375–377.

[3] ———, *On fixed and periodic points under contractive mappings*, J. London Math. Soc. **37** (1962), 74–79.

[5] Giles J.R., *Classes of semi-inner product spaces*, Trans. Amer. Math. Soc. **129** (1967), 436–446.

[6] Lumer G., *Semi-inner product spaces*, Trans. Amer. Math. Soc. **100** (1969), 29–43.

[7] Nanda S., *A non-linear complementarity problem in semi-inner product space*, Rendiconti di Matematica **1** (1982), 167–171.

[8] Nanda S., Nanda S., *On stationary points and the complementarity problem*, Bull. Austral. Math. Soc. **21** (1980), 351–356.

[9] Nath B., Lal S.N., Mukerjee R.N., *A generalized non-linear complementarity problem in semi-inner product space*, Indian J. Pure Appl. Math. **21:2** (1990), 140–143.

[10] Noor M.A., *On the non-linear complementarity problem*, J. Math. Anal. Appl. **123** (1987), 455–460.

[11] ———, *Mildly non-linear variational inequalities*, Math. Anal. Numer. Theory Approx. **24** (1982), 99–110.

[12] ———, *Generalized quasi complementarity problems*, J. Math. Anal. Appl. **120** (1986), 321–327.

Department of Mathematics and Computing, College of Science, Sultan Qaboos University, P.O.Box 36, Postal code 123, Al-Khod, Muscat, Sultanate of Oman

(*Received* November 22, 1993)