Coincidence points and maximal elements of multifunctions on convex spaces

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Abstract. Generalized and unified versions of coincidence or maximal element theorems of Fan, Yannelis and Prabhakar, Ha, Sessa, Tarafdar, Rim and Kim, Mehta and Sessa, Kim and Tan are obtained. Our arguments are based on our recent works on a broad class of multifunctions containing composites of acyclic maps defined on convex subsets of Hausdorff topological vector spaces.

Keywords: convex space, polytope, multifunction (map), upper semicontinuous (u.s.c.), lower semicontinuous (l.s.c.), compact map, acyclic, Kakutani map, acyclic map, admissible class, almost p-affine, almost p-quasiconvex, maximal element

Classification: 47H10, 49A40, 54H25, 55M20

0. Introduction

Recently, in [P1]–[P4], the author established very general coincidence and fixed point theorems on multifunctions in a broad class containing composites of acyclic maps defined on convex subsets of Hausdorff topological vector spaces. We also showed that our new results subsume many of historically well-known theorems.

On the other hand, there have appeared a number of coincidence theorems and maximal element theorems by several authors [Se], [MS], [BDG], [Be], [YP], [M1], [M2], [KT], [F1]–[F3], [Ha], [Ki], [Ta]. Those results are of the same nature as our recent works and can be deduced from the Knaster-Kuratowski-Mazurkiewicz theory originated from [KKM].

In the present paper, improved versions of those results are obtained from our new coincidence and fixed point theorems in [P2], [P3]. Section 2 deals with coincidence theorems and Section 3 with maximal element theorems. In Section 4, we give some equivalent formulations of the Himmelberg fixed point theorem [Hi].

1. Preliminaries

All Hausdorff topological vector spaces are abbreviated as t.v.s. For a t.v.s. E, $S(E,w)$ denotes the set of all weakly continuous seminorms. Let co, Int, and $\overline{}$ denote the convex hull, interior, and closure, respectively.

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A *convex space* X is a nonempty convex set with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. Such convex hulls are called *polytopes*.

A *multifunction* (or *map*) $F : X \to 2^Y$ is a function from a set X into the power set 2^Y of a set Y.

For topological spaces X and Y, a multifunction $F : X \to 2^Y \setminus \{\emptyset\}$ is said to be *upper semicontinuous* (u.s.c.) if for each closed subset C of Y, $F^{-1}(C) = \{x \in X : Fx \cap C \neq \emptyset\}$ is closed; *lower semicontinuous* (l.s.c.) if for each open subset D of Y, $F^{-1}(D)$ is open; and *compact* if the range $F(X)$ is contained in a compact subset of Y.

A nonempty topological space is *acyclic* if all of its reduced Čech homology groups over rationals vanish.

Given a class $\mathbb{L}$ of multifunctions, $\mathbb{L}(X,Y)$ denotes the set of multifunctions $T : X \to 2^Y$ belonging to $\mathbb{L}$, and $\mathbb{L}_c$ the set of finite composites of multifunctions in $\mathbb{L}$. For topological spaces X and Y, we define

$f \in \mathbb{C}(X,Y) \Longleftrightarrow f$ is a (single-valued) continuous function.

$T \in \mathbb{K}(X,Y) \Longleftrightarrow T$ is a *Kakutani map*; that is, Y is a convex space and T is u.s.c. with compact convex values.

$T \in \mathbb{V}(X,Y) \Longleftrightarrow T$ is an *acyclic map*; that is, T is u.s.c. with compact acyclic values.

An abstract class $\mathfrak{A}$ of multifunctions is defined by

(i) $\mathfrak{A}$ contains $\mathbb{C}$;
(ii) each $T \in \mathfrak{A}_c$ is u.s.c. and compact-valued; and
(iii) for any polytope P, each $T \in \mathfrak{A}_c(P,P)$ has a fixed point.

Note that $\mathbb{C}, \mathbb{K}$, and $\mathbb{V}$ are examples of $\mathfrak{A}$. See Park [P2], [P3]. Moreover, the class $\mathbb{A}$ of approachable maps in topological vector spaces [BD1]–[BD3], the class of admissible maps in the sense of Górniewicz [Go], and the class of permissible maps of Dzedzej [D] also belong to $\mathfrak{A}$. Moreover, we define

$T \in \mathfrak{A}_c^\sigma(X,Y) \Longleftrightarrow$ for any σ-compact subset K of X, there is a $\Gamma \in \mathfrak{A}_c(K,Y)$ such that $\Gamma x \subset Tx$ for $x \in K$.

$T \in \mathfrak{A}_c^\kappa(X,Y) \Longleftrightarrow$ for any compact subset K of X, there is a $\Gamma \in \mathfrak{A}_c(K,Y)$ as above.

Note that $\mathfrak{A}_c^\kappa(X,Y) \supset \mathfrak{A}_c^\sigma(X,Y) \supset \mathfrak{A}_c(X,Y) \supset \mathfrak{A}(X,Y)$. The class $\mathbb{K}_c^\sigma$ is due to Lassonde [L] and $\mathbb{V}_c^\sigma$ to Park, Singh, and Watson [PSW]. Note that $\mathbb{K}_c^\sigma$ includes classes $\mathbb{K}$, $\mathbb{R}$, and $\mathbb{T}$ in [L].

The following coincidence theorem is a particular form of Park [P3, Theorem 5].

Theorem 1.1. *Let X be a convex space, Y a Hausdorff space, $S,T : X \to 2^Y$ multifunctions, and $F \in \mathfrak{A}_c^\kappa(X,Y)$. Suppose that*

(1) *for each $x \in X$, $Sx \subset Tx$ and Sx is open;*
(2) *for each $y \in F(X)$, $T^{-1}y$ is convex;*

(3) *there exists a nonempty compact subset K of Y such that $\overline{F(X)} \cap K \subset S(X)$; and*
(4) *for each $N \in \langle X \rangle$, there exists a compact convex subset L_N of X containing N such that $F(L_N) \backslash K \subset S(L_N)$.*

Then there exists an $x_0 \in X$ such that $Fx_0 \cap Tx_0 \neq \emptyset$.

In Theorem 1.1, $\langle X \rangle$ denotes the set of all nonempty finite subsets of X. Note that Theorem 1.1 generalizes a number of known results as shown in [P3]. Moreover, a recent work of Mehta and Sessa [MS, Theorem 2.3] is included in Theorem 1.1. Note that, if F is single-valued, the Hausdorffness of Y is not necessary. See [P3].

We also need the following:

Theorem 1.2 [P2, Theorem 5 (vi)]**.** *Let X be a compact convex subset of a t.v.s. E on which E^* separates points. Then any $F \in \mathfrak{A}_c^\kappa(X, X)$ has a fixed point.*

Theorem 1.3 [P3, Theorem 4]**.** *Let X be a convex subset of a locally convex t.v.s. E. Then any compact map $F \in \mathfrak{A}_c^\sigma(X, X)$ has a fixed point.*

Let p be a seminorm on a vector space E. For a convex set A, a function $g : A \to E$ is said to be

(i) *almost p-affine* if

$$p(g(rx + (1 - r)y) - u) \leq rp(gx - u) + (1 - r)p(gy - u);$$

(ii) *almost p-quasiconvex* if

$$p(g(rx + (1 - r)y) - u) \leq \max\{p(gx - u), p(gy - u)\}$$

for $x, y \in A$, $u \in E$, and $r \in (0, 1)$.

Note that (i) implies (ii), but not conversely.

2. Coincidence theorems

In this section, we deal mainly with coincidence theorems.

If X itself is not convex in Theorem 1.1, then we have the following variation of Theorem 1.1 from Theorem 1.3.

Theorem 2.1. *Let D be a nonempty subset of a convex space X, Y a nonempty convex subset of a locally convex t.v.s. E, and $F, G : D \to 2^Y$ multifunctions such that*

(1) *$F \in \mathfrak{A}_c^\sigma(D, Y)$ is compact;*
(2) *for each $y \in Y$, $G^{-1}y$ is a convex subset of D; and*
(3) *$\{\operatorname{Int} Gx : x \in D\}$ covers Y.*

Then F and G have a coincidence point $x_0 \in D$; that is, $Fx_0 \cap Gx_0 \neq \emptyset$.

PROOF: Since $\overline{F(D)}$ is compact, $Y' = \operatorname{co}\overline{F(D)}$ is σ-compact (see [L]). Since Y' is regular, it is paracompact. Therefore, $G^{-1}|_{Y'} : Y' \to 2^D$ has a continuous selection $f : Y' \to D$ (see [BDG], [YP]). Hence $Ff \in \mathfrak{A}_c^\sigma(Y', Y')$. Since F is compact, so is Ff. By Theorem 1.3, there exists a $y_0 \in (Ff)y_0$. Let $x_0 = fy_0 \in D$. Then $y_0 \in Fx_0$ and $y_0 \in f^{-1}x_0 \subset Gx_0$, whence we have $Fx_0 \cap Gx_0 \neq \emptyset$. □

Remark. If D itself is convex, Theorem 2.1 is a simple consequence of Theorem 1.1 without assuming the local convexity of E.

Particular forms.

1. Sessa [Se, Theorem 8]: $X = E$, Y is paracompact, and $F \in \mathbb{K}(D, Y)$.
2. Mehta and Sessa [MS, Theorem 2.1]: Y is paracompact and $F \in \mathbb{K}(D, Y)$.

Corollary 2.2. *Let X be a nonempty convex subset of a locally convex t.v.s. E, D a nonempty compact subset of X, and $T : X \to 2^D$ a multifunction such that*

(1) *for each $x \in X$, Tx is convex; and*

(2) *$\{\operatorname{Int} T^{-1}y\}_{y \in D}$ covers X.*

Then T has a fixed point $x_0 \in D$; that is, $x_0 \in Tx_0$.

PROOF: Let $X = Y$, $F = 1_D : D \to D$ the identity map on D, and $G = T^{-1} : D \to 2^X$ in Theorem 2.1. Then there exists an $x_0 \in D$ such that $x_0 \in Gx_0$; that is, $x_0 \in Tx_0$. □

Remarks. 1. Corollary 2.2 is due to Ben-El-Mechaiekh *et al.* [BDG, Theorem 3.2]. Ben-El-Mechaiekh [Be] raised a question whether the local convexity in Corollary 2.2 can be eliminated.

2. It is well known that if $X = D$ is a compact convex space, then we do not need to assume the local convexity in Corollary 2.2. This case is known as the Fan-Browder fixed point theorem.

Particular forms.

1. Yannelis and Prabhakar [YP, Theorem 3.2]: X is paracompact.
2. Mehta [M1, Theorem 2.2]: X is paracompact. He also assumed the nonemptiness of each Tx in (1), which is a trivial consequence of (2).
3. Kim and Tan [KT, Lemma]: An equivalent form of Corollary 2.2.

The following are two Fan type coincidence theorems:

Theorem 2.3. *Let X be a Hausdorff compact convex space and E a t.v.s. on which E^* separates points, $F : X \to 2^E$ a u.s.c. map with nonempty closed convex values, and $g \in \mathbb{C}(X, E)$ such that*

(1) *$Fx \cap g(X) \neq \emptyset$ for all $x \in X$; and*

(2) *$g(X)$ is convex and $g^{-1}(u)$ is acyclic for all $u \in g(X)$.*

Then there exists an $x_0 \in X$ *such that* $gx_0 \in Fx_0$.

Proof: Note that $g(X)$ is a compact convex subset of E. Consider the map $T \in \mathbb{K}(X, g(X))$ defined by $Tx = Fx \cap g(X)$ for $x \in X$. In fact, Tx is nonempty, compact and convex for each $x \in X$. Moreover, if C is a closed subset of $g(X)$, then we have $T^{-1}(C) = \{x \in X : Fx \cap g(X) \cap C \neq \emptyset\} = \{x \in X : Fx \cap C \neq \emptyset\} = F^{-1}(C)$ is closed in X since F is u.s.c. Note that $g^{-1} : g(X) \to X$ has closed acyclic values and closed graph in $g(X) \times X$, where X is Hausdorff and compact. Therefore, $g^{-1} \in \mathbb{V}(g(X), X)$ and $Tg^{-1} \in \mathbb{V}_c(g(X), g(X))$. Hence, by Theorem 1.2, Tg^{-1} has a fixed point $y_0 \in (Tg^{-1})y_0$; that is, $y_0 \in Tx_0$ for some $x_0 \in g^{-1}y_0$. Note that $y_0 \in Tx_0 \subset Fx_0$ and $y_0 = gx_0$. This completes our proof. □

Theorem 2.4. *Let* X *be a compact convex space,* E *a t.v.s. on which* E^* *separates points,* $F : X \to 2^E$ *an u.s.c. map with nonempty closed convex values, and* $g \in \mathbb{C}(X, E)$ *such that*

(1) $Fx \cap g(X) \neq \emptyset$ *for all* $x \in X$*; and*

(2) $g(X)$ *is convex and* g *is almost* p*-quasiconvex for each* $p \in S(E, w)$.

Then there exists an $x_0 \in X$ *such that* $gx_0 \in Fx_0$.

Proof: As in the proof of Theorem 2.3, we have the map $T \in \mathbb{K}(X, g(X))$ defined by $Tx = Fx \cap g(X)$ for $x \in X$. Suppose that $gx \notin Fx$ for each $x \in X$. Then the origin of E does not belong to the compact set $K := gx - Tx$. For each $z \in K$, there exists a linear functional $\ell_z \in E^*$ such that $\ell_z(z) \neq 0$. Since ℓ_z is continuous, there exists an open neighborhood V_z of z such that $\ell_z(y) \neq 0$ for every $y \in V_z$. Let $\{V_{z_1}, \dots, V_{z_n}\}$ be a finite subcover of the cover $\{V_z\}_{z \in K}$ of K and

$$p_x(y) := \sum_{i=1}^{n} |\ell_{z_i}(y)| \quad \text{for each} \quad y \in E.$$

Then $p_x \in S(E, w)$ such that $p_x(z) > 2\delta_x$ for all $z \in K$ for some $\delta_x > 0$. Since g is continuous and T is u.s.c., there exists an open neighborhood U_x of $x \in X$ such that $p_x(gu - v) > \delta_x$ for all $u \in U_x$ and $v \in Tu$. Since $\{U_x : x \in X\}$ covers X and X is compact, there exists a finite subcover $\{U_{x_1}, \dots, U_{x_n}\}$ of X. Let $p := \max\{p_{x_i} : 1 \leq i \leq k\}$ and $\delta = \min\{\delta_{x_i} : 1 \leq i \leq k\} > 0$. Then $p \in S(E, w)$ and $p(gx - y) > \delta$ for all $(x, y) \in T$. Now define a map $G : X \to 2^{g(X)}$ by $Gx = \{y \in g(X) : p(gx - y) < \delta\}$ for each $x \in X$. Since g is almost p-quasiconvex, $G^{-1}y = \{x \in X : p(gx - y) < \delta\}$ is convex for each $y \in g(X)$. Moreover, since p and g are continuous, Gx is open in $g(X)$ for each $x \in X$. Further, for each $y \in g(X) = \overline{g(X)}$, there exists $x \in X$ such that $y = gx$ and hence $y \in Gx$. Therefore, by Theorem 1.1 with $X = K$, $Y = g(X)$ and $F = T \in \mathbb{K}(X, g(X))$, T and G have a coincidence point $x_0 \in X$; that is, there exists a $y_0 \in Tx_0 \cap Gx_0$. Since $y_0 \in Tx_0$, we have $p(gx_0 - y_0) > \delta$; and, since $y_0 \in Gx_0$, we have $p(gx_0 - y_0) < \delta$. This contradiction completes our proof. □

Particular forms of Theorems 2.3 and 2.4.

1. If X is a subset of E, $g = 1_X$, and $F : X \to 2^X$, then Theorems 2.3 and 2.4 include earlier fixed point theorems due to Brouwer [Br], Schauder [S1], [S2], Tychonoff [Ty], Kakutani [K], Bohnenblust and Karlin [BK], Glicksberg [G], Fan [F1], [F3], Granas and Liu [GL], and others. See Park [P1].

2. Fan [F2, Theorem 2]: E is a locally convex t.v.s., $F : X \to 2^E$ is a continuous (u.s.c. and l.s.c.) map with nonempty compact convex values, and g satisfies the following instead of condition (2) of Theorem 2.3:

(2)′ For every closed convex set C in E, $g^{-1}(C)$ is convex or empty.

3. Ha [Ha, Theorem 2]: The map g satisfies (2)′ instead of condition (2) of Theorem 2.3. Note that (2)′ implies (2) of Theorems 2.3 and 2.4, but not conversely. It is known that condition (2) of Theorem 2.4 implies condition (2) of Theorem 2.3 whenever p is a norm.

4. Mehta and Sessa [MS, Theorem 2.2]: E is a locally convex t.v.s. and g is almost affine.

Corollary 2.5. *Let X be a compact convex subset of a t.v.s. E on which E^* separates points and $g : X \to E$ a continuous almost p-quasiconvex map such that $X \subset g(X)$ and $g(X)$ is convex. Then g has a fixed point.*

Particular forms. In particular, if g is almost p-affine for each $p \in S(E, w)$, then g has a fixed point. In case g is affine, Corollary 2.5 reduces to Park [P2, Corollary 5.1], [P4, Theorem 7].

We have another coincidence theorem:

Theorem 2.6. *Let X be a nonempty convex subset of a locally convex t.v.s. E_1, D a metrizable subset of a complete locally convex t.v.s. E_2, $S \in \mathfrak{A}_c^\sigma(D, X)$ a compact map, and $T : X \to 2^D$ an l.s.c. map such that, for some metric on D, Tx is nonempty complete convex for each $x \in X$. Then there exist points $x' \in X$, $y' \in D$ such that $x' \in Sy'$ and $y' \in Tx'$.*

Proof: Note that $X' = \operatorname{co} \overline{S(D)}$ is σ-compact (see [L]) and hence paracompact. Since $T' = T|_{X'} : X' \to 2^D$ is l.s.c. and has complete convex values, by the Michael selection theorem [Mi, Theorem 1.2], it has a continuous selection $f : X' \to D$. Hence $Sf \in \mathfrak{A}_c^\sigma(X', X')$. Since S is compact, so is Sf. Therefore, by Theorem 1.3, there exists an $x' \in X'$ such that $x' \in Sfx'$. Let $y' = fx' \subset T'x' \subset Tx'$. Then $x' \in Sy'$ and $y' \in D$. This completes our proof. □

Remark. As Michael [Mi] observed, the completeness of E_2 can be replaced by the compactness of $\overline{\operatorname{co}}\, K$ for every compact $K \subset D$.

Particular form. Mehta and Sessa [MS, Theorem 2.4]: X is paracompact, D is closed, E_2 is a Banach space, and $S \in \mathbb{K}(D, X)$.

3. Existence of maximal elements

Any binary relation R in a set X can be regarded as a multifunction $T : X \to 2^X$ and conversely by the following obvious way:

$$y \in Tx \quad \text{if and only if} \quad (x, y) \in R.$$

Therefore, a point $x_0 \in X$ is called a *maximal element* of a multifunction $T : X \to 2^X$ if $Tx_0 = \emptyset$.

In this section, we deal with the existence of such maximal elements.

From Theorem 1.1, we have the following:

Theorem 3.1. *Let X be a convex space, Y a Hausdorff space, $S, T : X \to 2^Y$ multifunctions, and $F \in \mathfrak{A}_c^\kappa(X, Y)$. Suppose that*

(1) *for each $x \in X$, $Sx \subset Tx$ and Sx is open;*
(2) *for each $y \in F(X)$, $T^{-1}y$ is convex; and*
(3) *there exists a nonempty compact subset K of Y such that, for each $N \in \langle X \rangle$, there exists a compact convex subset L_N of X containing N such that $F(L_N)\backslash K \subset S(L_N)$.*

If $Fx \cap Tx = \emptyset$ for all $x \in X$, then there exists a $y \in \overline{F(X)} \cap K$ such that $S^{-1}y = \emptyset$.

Proof: Suppose that for each $y \in \overline{F(X)} \cap K$, there exists an $x \in S^{-1}y$. Then $\overline{F(X)} \cap K \subset S(X)$. This and (1)–(3) implies the existence of an $x_0 \in X$ such that $Fx_0 \cap Tx_0 \neq \emptyset$, by Theorem 1.1. This completes our proof. □

Particular forms.

1. Yannelis and Prabhakar [YP, Theorem 5.1]: $X = Y = K$ and $F = 1_X$.
2. Mehta [M2, Theorem 4]: $X = Y = K$ and $F = 1_X$.
3. Kim [Ki, Lemma]: $X = Y$, $F = 1_X$, with a stronger coercivity condition than (3). Similarly, other results in [Ki] can be improved.
4. Mehta and Sessa [MS, Theorem 3.3]: $Y = K$ and $F \in \mathbb{K}(X, Y)$.

From Theorem 2.1, we have the following:

Theorem 3.2. *Let D be a nonempty subset of a convex space X, Y a nonempty convex subset of a locally convex t.v.s. E, and $F, G : D \to 2^Y$ multifunctions such that*

(1) *$F \in \mathfrak{A}_c^\sigma(D, Y)$ is compact;*
(2) *for each $y \in Y$, $G^{-1}y$ is a convex subset of D;*
(3) *$Fx \cap Gx = \emptyset$ for all $x \in D$; and*
(4) *for each $y \in Y$ with $G^{-1}y \neq \emptyset$, there exists an $x \in D$ such that $y \in \operatorname{Int} Gx$.*

Then there exists a $y \in Y$ such that $G^{-1}y = \emptyset$.

Proof: Suppose that $G^{-1}y \neq \emptyset$ for each $y \in Y$. Then by (4), $\{\operatorname{Int} Gx : x \in D\}$ covers Y. This and (1), (2) implies the existence of an $x_0 \in D$ such that $Fx_0 \cap Gx_0 \neq \emptyset$, by Theorem 2.1. This contradicts (3). □

Particular forms.

1. Yannelis and Prabhakar [YP, Theorem 5.3]: Y is paracompact, D is a compact subset of Y, and $F = 1_D$.

2. Mehta and Sessa [MS, Theorem 3.1]: Y is paracompact and $F \in \mathbb{K}(D, Y)$.

From Theorem 2.6, we have the following:

Theorem 3.3. *Let X be a nonempty convex subset of a locally convex t.v.s. E_1, D a metrizable subset of a complete locally convex t.v.s. E_2, $S \in \mathfrak{A}_c^\sigma(D, X)$ a compact map, and $T : X \to 2^D$ a l.s.c. map such that, for some metric on D, Tx is complete convex for each $x \in X$. If*

$(*)$ for each $(x, y) \in X \times D$, $x \in Sy$ implies $y \notin Tx$,

then T has a maximal element.

Proof: Suppose that $Tx \neq \emptyset$ for each $x \in X$. Then there exists an $(x', y') \in X \times D$ such that $x' \in Sy'$ and $y' \in Tx'$, which violates $(*)$. □

Particular forms.

1. Yannelis and Prabhakar [YP, Theorem 5.2]: $X = D$ is compact, $S = 1_X$, and $E_1 = E_2 = \mathbb{R}^n$.

2. Mehta [M2, Theorem 5]: $X = D$ is compact, $E_1 = E_2$ is a Banach space, and $S = 1_X$.

3. Mehta and Sessa [MS, Theorem 3.5]: X is paracompact, D is closed, E_2 is a Banach space, and $S \in \mathbb{K}(D, X)$.

4. Remarks on Himmelberg's theorem

Theorem 1.3 with $\mathbb{K}$ instead of $\mathfrak{A}_c^\sigma$ was due to Himmelberg [Hi, Theorem 2], which generalizes Fan's well-known fixed point theorem [F1].

In [Ta], Tarafdar defined that a closed-valued map $T : X \to 2^Y$ is *almost upper semicontinuous* (a.u.s.c.) if for each $x \in X$ and each open set V containing Tx, there exists an open set U containing x such that $T(U) \subset \overline{V}$. However, as we noted in [PB], this concept is the same as the upper semicontinuity if Y is normal. Using this concept, Tarafdar [Ta] claimed a generalization of Fan's fixed point theorem. On the other hand, Rim and Kim [RK] claimed a generalization of Himmelberg's theorem for a.u.s.c. maps. Note that, for a single-valued map, the concept of a.u.s.c. reduces to that of weak continuity due to Levine [Le].

In this section, we show that those generalizations are actually equivalent to Himmelberg's theorem.

Lemma 4.1. *Let X be a topological space, Y a normal space, and $T : X \to 2^Y$ an a.u.s.c. map with nonempty closed values. Then T is u.s.c.*

The proof is elementary. See [PB] or [RK, Lemma 1].

Lemma 4.2. *Let X be a nonempty convex subset of a locally convex t.v.s. E, K a nonempty compact subset of X, and $T : X \to 2^K$ an a.u.s.c. map such that* $\operatorname{co} Tx \subset K$ *for each $x \in X$. Then $\overline{\operatorname{co}}\, T$ is u.s.c.*

Proof: As in the proof of [RK, Lemma 2], $\overline{\operatorname{co}}\, T$ is a.u.s.c. with closed values. Therefore, by Lemma 4.1, $\overline{\operatorname{co}}\, T$ is u.s.c. □

The following is the main result of this section:

Theorem 4.3. *Let X be a nonempty convex subset of a locally convex t.v.s. E, and K a nonempty compact subset of X. Then the following equivalent statements hold:*

(i) *Every u.s.c. map $T : X \to 2^K$ with nonempty closed convex values has a fixed point.*

(ii) *For any a.u.s.c. map $S : X \to 2^K$ with nonempty values such that* $\operatorname{co} Sx \subset$ *K for each $x \in X$, there exists an $\hat{x} \in K$ such that $\hat{x} \in \overline{\operatorname{co}}\, S\hat{x}$.*

(iii) *Any map $T : X \to 2^K$ such that there exists an a.u.s.c. map $S : X \to 2^K$ with $\emptyset \neq \overline{\operatorname{co}}\, Sx \subset Tx$ for each $x \in X$ has a fixed point.*

Proof: Note that (i) is due to Himmelberg [Hi] and a particular form of Theorem 1.3 with $\mathbb{K}$ replacing $\mathfrak{A}_c^\sigma$.

(i) $\Longrightarrow$ (ii) By Lemma 4.2, $\overline{\operatorname{co}}\, S$ is u.s.c. and hence, by (i), $\overline{\operatorname{co}}\, S$ has a fixed point.

(ii) $\Longrightarrow$ (iii) By (ii), $\overline{\operatorname{co}}\, S$ has a fixed point $\hat{x} \in \overline{\operatorname{co}}\, S\hat{x} \subset T\hat{x}$.

(iii) $\Longrightarrow$ (i) Put $S = T$ in (iii). Note that u.s.c. implies a.u.s.c. □

Remarks. 1. Rim and Kim [RK, Theorem 1] and Tarafdar [Ta, Theorem 2.1] proved (iii) under some additional restrictions. Moreover, (ii) is due to [RK, Corollary 1].

2. Note that closed convex values in Theorem 4.3 can be replaced by closed acyclic values (containing Sx) in view of Theorem 1.3 with $\mathbb{V}$ instead of $\mathfrak{A}_c^\sigma$.

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