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A countably compact, separable space which is not absolutely countably compact

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Abstract. We construct a space havfng the properties in the title, and with the same technique, a countably compact T_2 topological group which is not absolutely countably compact.

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1. Introduction

A space X is called *countably compact* provided every countable open cover of X has a finite subcover. A characterization of countable compactness (see [4], [2] and [3, 3.12.22 (d)]) states that a T_2-space X is countably compact iff for every open cover $\mathcal{U}$ (countable or not) of X there exists a finite set $F \subset X$ such that

$$St(F,\mathcal{U}) = \cup\{U \in \mathcal{U} : U \cap F \neq \emptyset\} = X.$$

M.V. Matveev defined a space X to be *absolutely countably compact (acc)* [5] provided for every open cover $\mathcal{U}$ of X and every dense $Y \subset X$, there exists a finite set $F \subset Y$ such that $St(F,\mathcal{U}) = X$. We note that if D is a countable dense subset of X, then

$$\{St(x,\mathcal{U}) : x \in D\}$$

is a countable open cover of X; so obviously if X is countably compact, there exists a finite $F \subset D$ such that $St(F,\mathcal{U}) = X$. Hence, for a countably compact space X, if X is hereditarily separable (more generally, has countable tightness [5, 1.7]), or has a countable dense set of isolated points, then X is acc. Matveev raised the natural question [5, 1.11]: Is every countably compact, separable space acc? We answer this and another question of Matveev [5, 1.13] by constructing the following examples.

Example 1.1. *A separable, countably compact space which is not absolutely countably compact.*

Example 1.2. *A countably compact topological group which is not absolutely countably compact.*

The proofs that the spaces in 1.1 and 1.2 have the required properties follow easily from known results, and the following theorem:

Theorem 1.3. *If X is space having at least one nonempty closed, compact set K and X has an open cover $\mathcal{U}$ which does not have a finite subcover, then the product space X^κ, where $\kappa = |\mathcal{U}|$, is not acc.*

The hypothesis in 1.3 may be considered as a weak kind of separation property. For example if the space X has at least one point x such that $\{x\}$ is closed, then $K = \{x\}$ is compact and closed. Thus 1.3 holds in particular for T_1-spaces.

On the other hand, we have the following simple example:

Example 1.4. *A non-compact T_0-space X such that every power X^κ is acc.*

Proof: Let X be the set of natural numbers ω with the (well-known) T_0 topology whose only open sets are the empty set, ω, and $\{n : n < \omega\}$. Thus the only nontrivial open sets are initial intervals of integers. We show that X^κ is acc. In fact, X^κ satisfies the stronger property (†): for every open cover $\mathcal{U}$ and every point $f \in X^\kappa$, $St(\{f\},\mathcal{U}) = X^\kappa$. To see that (†) holds for this space, take any $f, g \in X^\kappa$, and define $h = max\{f, g\}$. Pick any $U \in \mathcal{U}$ such that $h \in U$. By definition of the product topology, there exists a finite $F \subset \kappa$ and integers n_α for $\alpha \in F$ such that $h \in B = \cap_{\alpha \in F} \pi^{-1}(n_\alpha) \subset U$. Thus for all $\alpha \in F$ we have

$$f(\alpha), g(\alpha) \leq h(\alpha) < n_\alpha;$$

so $f, g \in B \subset U$. □

The space X in 1.4 is not countably compact. A similar example which is countably compact can be given by using the analogous topology on ω_1 whose only non-trivial open sets are initial intervals of ω_1.

Theorem 1.3 improves the following similar theorem of Matveev, which we state here with an added "separation" condition that is implicitly used in his proof.

Theorem 1.5 [5, 2.15]**.** *For every non-compact space X [in which certain points are closed], X^τ is not acc for some τ.*

Matveev does not explicitly mention any separation axioms in the statement of 1.5, but implicitly his proofs of [5, Propositions 2.7] and [5, Propositions 2.14] (which are used in the proof of 1.5) each require that the a certain point, which arises during the proof, be closed. Thus the separation hypothesis of 1.3 is weaker that the implicit separation hypothesis of 1.5.

The main improvement of 1.3 over 1.5 is that we can take the cardinal κ in 1.3 to be much smaller than the cardinal τ in 1.5. We do not give here the definition of Matveev's cardinal $\tau = \tau(X)$ (it is defined in the proof of [5, Lemma 2.13] wherein it is not possible to use a cardinal smaller that τ). If in 1.3 we take κ

to be as small as possible (i.e., for a noncompact space X, let $\kappa(X)$ denote the smallest cardinal number of an open cover of X having no finite subcover), then it is not hard to show that $\kappa(X)$ is the cofinality of $\tau(X)$, and moreover

$$\kappa(X) < \beth_{\kappa(X)} \leq \tau(X).$$

Thus $\kappa(X)$ is much smaller than $\tau(X)$ (recall $\beth_0 = \omega$, $\beth_{\alpha+1} = 2^{\beth_\alpha}$, and for γ a limit ordinal, $\beth_\gamma = \sup\{\beth_\alpha : \alpha < \gamma\}$).

To show that the spaces in 1.1 and 1.2 are countably compact, we use the following version of a theorem of C.T. Scarborough and A.H. Stone [9] (or see [10, Theorem 3.3]). Recall that $\mathfrak{c}$ denotes the cardinality of the continuum, $\mathfrak{t}$ denotes the smallest cardinality of a mod-finite tower on the set ω of natural numbers, and that $\omega_1 \leq \mathfrak{t} \leq \mathfrak{c}$ (see [1], [11]).

Theorem 1.6. *Every product of no more than $\mathfrak{t}$ sequentially compact spaces is countably compact.*

To show that our Example 1.1 is separable we use the following special case of the Hewitt-Marczewski-Pondiczery theorem (see [3, 2.3.15]).

Theorem 1.7. *The product of no more than $\mathfrak{c}$ separable spaces is separable.*

A preliminary version of this paper was presented at the first joint meeting of the American Mathematical Society and the Sociedad Matematica Mexicana, Merida, Mexico December 1–5, 1993 [12].

2. Proofs

Proof of Theorem 1.3. Let $\mathcal{U}$ be an open cover of X such that $|\mathcal{U}| = \kappa$ and $\mathcal{U}$ has no finite subcover. We will assume that $\kappa = \kappa(X)$, i.e. that $|\mathcal{U}|$ is the smallest cardinality of an open cover of X having no finite subcover. This suffices to prove the theorem because acc is preserved by continuous open maps (or certain more general maps [5, Proposition 3.2]); so if X^κ is not acc, then neither is X^μ for all $\mu \geq \kappa$.

By hypothesis there is a non-empty closed set K and a finite family $\mathcal{U}_0 \subset \mathcal{U}$ covering K. Define

$$\mathcal{U}' = \mathcal{U}_0 \cup \{U \setminus K : U \in \mathcal{U} \setminus \mathcal{U}_0\}.$$

Then $\mathcal{U}'$ is an open cover of X, $\mathcal{U}'$ does not have a finite subcover, $|\mathcal{U}'| = \kappa$, and the elements in $\mathcal{U}_0$ are the only element of $\mathcal{U}'$ that intersect K. We may assume that the original cover $\mathcal{U}$ has these properties, and that $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$, with $\mathcal{U}_0 = \{U_0, U_1, \cdots, U_n\}$, i.e. the elements of $\mathcal{U}_0$ are listed first in the well-order on $\mathcal{U}$.

Let $\mathcal{W} = \{\pi_0^{-1}(U_\alpha) \cap \pi_\alpha^{-1}(U_\beta) : \alpha, \beta < \kappa\}$, where π_α is the projection map onto the α^{th} coordinate. Then $\mathcal{W}$ covers X^κ since given any $f \in X^\kappa$, there exist $\alpha < \kappa$ such that $f(0) \in U_\alpha$, and there exists $\beta < \kappa$ such that $f(\alpha) \in U_\beta$.

Next we define a dense subset of X^κ by

$$Y = \{f \in X^\kappa : \{\alpha < \kappa : f(\alpha) \notin K\} \text{ is finite}\}.$$

To show that X^κ is not acc, we show that if $F \subset Y$ is finite, then

$$\cup\{W \in \mathcal{W} : W \cap F \neq \emptyset\} \neq X^\kappa.$$

For each $f \in F$, let $R_f = \{\alpha < \kappa : f(\alpha) \notin K\}$, and let $R = \cup\{R_f : f \in F\}$. Thus R is a finite subset of κ; so we may pick a point

$$p \in X - \cup\{U_\alpha : \alpha \in R \cup \{0, 1, \cdots, n\}\}.$$

Define $g \in X^\kappa$ to be the constant function with constant value p. We need to show that $g \notin \cup\{W \in \mathcal{W} : W \cap F \neq \emptyset\}$ and we verify this by contradiction. Suppose we have $f \in F$, and $W \in \mathcal{W}$ such that both $f, g \in W$. There exists $\alpha, \beta < \kappa$ such that $W = \pi_0^{-1}(U_\alpha) \cap \pi_\alpha^{-1}(U_\beta)$. Since $g(0) = g(\alpha) = p \in U_\alpha \cap U_\beta$, we know that $\alpha, \beta \notin R \cup \{0, 1, \cdots, n\}$. Since $\alpha \notin R$, we know that $f(\alpha) \in K$. Since the elements of $\mathcal{U}_0$ are the only elements of $\mathcal{U}'$ that intersect K, and we have $f(\alpha) \in U_\beta \cap K$, we get $\beta \in \{0, 1, \cdots, n\}$, but this is a contradiction. □

The hypothesis in 1.3 on K can be formally weakened to the following technical property (*): There exists an open cover $\mathcal{U}$ of X having no finite subcover, and having a finite $\mathcal{U}_0 \subset \mathcal{U}$ such that

$$(\cup\mathcal{U}_0) \setminus \cup(\mathcal{U} \setminus \mathcal{U}_0) \neq \emptyset.$$

This set plays the role of K in the proof.

Proof of Example 1.1. Let $\mathcal{T}$ denote a mod-finite tower of infinite subsets of ω with $|\mathcal{T}| = \mathfrak{t}$ (see [1], [11]), and let $X(\mathcal{T})$ denote the associated Franklin-Rajagopalan space (see [8]). Then $X(\mathcal{T})$ is sequentially compact, separable, and has an open cover of cardinality $\mathfrak{t}$ which has no finite subcover. Thus $X(\mathcal{T})^{\mathfrak{t}}$ is countably compact (by 1.6), separable (by 1.7, since $\mathfrak{t} \leq \mathfrak{c}$), and not acc (by 1.3).

Proof of Example 1.2. Let

$$G = \{f \in 2^{\omega_1} : |\{\alpha < \omega_1 : f(\alpha) = 1\}| \leq \omega\}.$$

One can easily check that G is sequentially compact (indeed, G is a countably compact, Fréchet-Urysohn space [7]). Further,

$$\{\pi_\alpha^{-1}(0) : \alpha < \omega_1\}$$

is an open cover of G of cardinality ω_1 which has no finite subcover. Thus G^{ω_1} is a countably compact group that is not acc. For this space G, unlike the separable space $X(\mathcal{T})$ in the previous example, we do not need to use the Scarborough-Stone Theorem to know that the product is countably compact. It is well-known that every power of G is countably compact, i.e. G is ω-bounded (cf. [10, §4]).

Remark. There are several other constructions of countably compact, non-acc topological groups. After we had proved 1.2, we were informed that B. Bokalo and I. Guran noticed that the existence of such a group is implicit in Matveev's Theorem 1.5: take any ω-bounded noncompact group (such as the group G above). Then by 1.5 there exists τ such that G^τ is not absolutely countably compact (in this case $\tau \geq \beth_{\omega_1}$). Later, Matveev constructed a countably compact group $H \subset G$, in fact a ring, that is not acc [6]. We included 1.2 since it is somewhat different from these other two examples.

Questions. (1) Is there a countably compact, separable T_2 group which is not acc? A more basic question: (2) Is there a sequentially compact, separable T_2 group which is not compact? If there is a group that answers question (2), and it has an open cover of cardinality $\leq \mathfrak{t}$ with no finite subcover, then by the method we have given, the answer to question (1) is "yes".

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