A generic theorem in the theory of cardinal invariants of topological spaces

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Abstract. Relative versions of many important theorems on cardinal invariants of topological spaces are formulated and proved on the basis of a general technical result, which provides an algorithm for such proofs. New relative cardinal invariants are defined, and open problems are discussed.

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§0. Introduction

In many topological arguments and constructions we have to deal with the following question: how a given subset A of a topological space X is located in X? Here we touch upon a systematic approach to this problem; though it is very general, we do not pretend that it embraces all other possible approaches to it. Besides, in this article we are mostly concerned with the case of cardinal invariants.

The key idea can be briefly described as follows. With each topological property $\mathcal{P}$ one can associate a relative version of it, formulated in terms of location of Y in X in such a natural way, that when Y coincides with X, then this relative property coincides with $\mathcal{P}$. Our basic conjecture is that the great majority of the results, involving "absolute" topological properties, can be transformed into "location" results, that is, into theorems on relative topological properties, though by no means we claim that it should be always easy to make such a transformation. Technically, there are no reasons to expect that. Nevertheless, the above conjecture, the author believes, can serve as a guideline in the work on relative properties.

Situations involving relative topological properties have been encountered in topology many times. For example, some very delicate results on relative countable compactness were obtained by A. Grothendieck: he has proved, in particular, that if X is a countably compact Tychonoff space, and Y is a subspace of the space $C_p(X)$ of all real-valued continuous functions on X in the topology of pointwise convergence such that Y is countably compact in X then the closure of Y

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in $C_p(X)$ is compact [16]. In [13] we find applications of the concepts of regular and normal location of a subspace in a space; such examples are numerous, and should be considered in a special survey. The first systematic exposition of relative topological properties along the lines adopted in this article was given in [5].

Below we prove relative versions of many important theorems on cardinal invariants of topological spaces (see [1], [18], [10], [2], [3], [20], [8], [22], [4]). To make the argument more compact and more transparent, we formulate a general technical theorem, after which the proofs of many original results on cardinal inequalities acquire almost algorithmic character—they turn into a rather easy (though still not quite routine) verification of certain natural (mostly, technical) conditions. It took some experimenting and time for the author to find the general formulation even for the classical case, though the general idea behind the technique of the proof is the same as in the original article [1] and in the later versions of the proof given in [23], [25] and [19]. This idea may be informally described in a few words: "push away" from closed sets, move into new closed sets, disjoint from those which were defined at previous steps, do that for a long enough time; then the structure you get will approximate the space well and the desired result will be easily available. Maybe, for the first time this "push away" technique was introduced by M.E. Rudin (see a proof of Mischenko's theorem on metrizability of compact Hausdorff spaces with a point-countable base in [11]).

Following the inner logic of the approach, we introduce some new "absolute" notions, and prove some new theorems which, possibly, are interesting also in the absolute case; in particular, see Section 4, Theorems 5, 6 and 7. Many new open problems are formulated.

§1. Some definitions and preliminary results

In what follows, Y is always a subspace of X. All spaces considered are assumed to be T_1-spaces. Recall that Y is *compact in* X (*Lindelöf in* X), if from each open covering of X one can choose a finite (a countable) subfamily covering Y (see [24], [5]). For a cardinal number τ, we shall say that Y is *initially* τ*-Lindelöf* (*initially* τ*-compact*) *in* X, if for each open covering γ of X of cardinality $\leq \tau$ there is a finite (a countable) subfamily of γ covering Y. Further, we denote by $l(Y,X)$ the smallest infinite cardinal number τ such that from each open covering of X one can choose a subfamily η of cardinality $\leq \tau$ covering Y. This cardinal number obviously should be called *the Lindelöf degree of* Y *in* X.

We consider cardinal invariants of topological spaces (see [2], [20]); in particular, $d(X)$ is the density of X, $s(X)$ is the spread of X, $l(X)$ is the Lindelöf number of X, $\psi(X)$ is the pseudocharacter of X, $\chi(X)$ is the character of X, $|X|$ is the cardinality of X, $c(X)$ is the Souslin number of X, and $nw(X)$ is the networkweight of X. We write $hl(X) \leq \tau$ if $l(Y) \leq \tau$ for each subspace Y of X.

We define *the Souslin number of* Y *in* X (notation: $c(Y,X)$), as the smallest cardinal number τ such that the cardinality of every disjoint family of non-empty open subsets of X, each of which intersects Y, does not exceed τ.

The following assertion is a reformulation of a result of Shapirovskij, which is easy to prove (see [25]):

Lemma 1. *If the spread of X is countable, then for every family γ of open subsets of X and for each subset P of $\cup\gamma$, one can find a countable discrete subset A of P and a countable subfamily η of γ such that $P \subset \bar{A} \cup (\cup\eta)$.*

Our next three results are relative versions of elementary results, which are well known in the absolute case (see [13], [20], [19]).

Lemma 2. *If $c(Y, X) \leq \omega$, then for every family γ of open subsets of X there is a countable subfamily η of γ such that $(\cup\gamma) \cap Y \subset \overline{\cup\eta}$.*

Proof: Let $\mathcal{E}$ be the family of all open subsets V of X such that $V \cap Y$ is not empty and there is $U \in \gamma$ such that $V \subset U$. Take any maximal disjoint family ξ of members of $\mathcal{E}$. Then ξ is countable, since $c(Y, X) \leq \omega$. For each $V \in \xi$, fix $U_V \in \gamma$ such that $V \subset U_V$. Then $\eta = \{U_V : V \in \xi\}$ is what we want. Indeed, if there are $y \in Y \setminus \overline{\cup\eta}$ and $W \in \gamma$ such that $y \in W$, then we can find $V \in \mathcal{E}$ such that $y \in V \subset W$ and $(\cup\eta) \cap V = \emptyset$. Thus, $\xi \cup \{V\}$ is a disjoint subfamily of $\mathcal{E}$ strictly larger than ξ, in contradiction with maximality of ξ. $\square$

Let us define *the extent $e(Y, X)$ of Y in X* as the smallest cardinal number τ such that the cardinality of every closed in X discrete subspace of Y is not greater than τ. We recall that, for a point $p \in X$ (for a subset A of X) and a family γ of subsets of X, $st(p, \gamma) = \cup\{U \in \gamma : p \in U\}$ (respectively, $st(A, \gamma) = \cup\{U \in \gamma : U \cap A \neq \emptyset\}$).

Lemma 3. *If $P \subset Y$, and γ is a family of open subsets of X such that $\bar{P} \subset \cup\gamma$, then there is a closed in X discrete subspace A of P such that $P \subset st(A, \gamma)$.*

Proof: Take the family of all subsets B of P such that no two distinct points of B belong to the same element of γ. Let A be a maximal element of this family. Clearly, $P \subset st(A, \gamma)$. Assume that A is not closed in X, and fix $x \in \bar{A} \setminus A$. Then $x \in U$, for some $U \in \gamma$. Since x is a limit point for A, there are at least two distinct points of A in U—a contradiction. $\square$

The next assertion is an obvious corollary of Lemma 3.

Lemma 4. *If $e(Y, X) \leq \omega$, $P \subset Y$, and γ is a family of open subsets of X covering $\bar{P}$, then there is a countable subset A of P such that $P \subset st(A, \gamma)$.*

If X is a set, $x \in X$, and $<$ is a linear ordering on X, we put $X_x = \{y \in X : y < x\}$. The sets X_x are called *proper initial intervals* of $X, <$. For a topological space X, a linear ordering $<$ on X is said to be *left*, if each proper initial interval with respect to $<$ is closed in X. A space X is called *a left space*, if there is a left well ordering on X (see [2]). The following fact is well known [2]; for the sake of completeness, we present a proof of it—short and elegant.

Proposition 1. *Every topological space X contains a dense left subspace.*

Proof: Take any well ordering $<$ on X. For each non-empty open set V let m_V be the first element of V with respect to $<$. Then the restriction of $<$ to the subspace Z consisting of all such points m_V is a left well ordering on the space Z. Therefore Z is a left space. It is clear that Z is dense in X. □

The next assertion and its proof are taken from [2].

Proposition 2. *Every left space X of the countable spread is hereditarily Lindelöf.*

Proof: Since every subspace of a left space is a left space, it is enough to prove that X is Lindelöf. Assume the contrary, and fix a left well ordering $<$ on X. We may also assume that each proper initial interval of X is Lindelöf—otherwise we replace X with the smallest proper interval of it which is not Lindelöf. Then, clearly, $X, <$ is not countably cofinal. Since $<$ is left, it follows that for each countable subset A of X, the closure of A is contained in a proper initial interval of $X, <$ and, therefore, is Lindelöf. It remains to apply the following proposition, which is an easy corollary of Lemma 1. □

Proposition 3. *If X is a space of the countable spread such that the closure of every countable subset in X is Lindelöf, then X is Lindelöf.*

This implies, of course, that every monolithic space of countable spread is hereditarily Lindelöf. From Propositions 1 and 2 we immediately get the following result of Shapirovskij [25]:

Proposition 4. *Every space of the countable spread contains a dense hereditarily Lindelöf subspace.*

§2. Special notions and the main theorem

We fix the notation which will be used throughout the article.

Let X be a topological space and let Y be a subspace of X. Here and in what follows $\bar{A}$ is the closure of A in X. The closure of a subset A of Y in Y is denoted by $cl_Y(A)$.

Let τ be an infinite cardinal number, and let λ be a cardinal number not greater than the cofinality of τ. Then $\mu = |\{A \subset Z : |A| < \lambda\}|$, where Z is a set of cardinality τ. We also treat τ as the smallest ordinal of cardinality τ.

Let $\mathcal{L}$ be a family of subsets of Y of cardinality not greater than μ such that every subset of Y of cardinality not greater than μ is contained in an element of $\mathcal{L}$.

A τ-long increasing sequence in $\mathcal{L}$ is a transfinite sequence $\{F_\alpha : \alpha < \tau\}$ of elements of $\mathcal{L}$ such that $F_\alpha \subset F_\beta$ if $\alpha < \beta < \tau$.

A sensor is a pair $(\mathcal{A}, \mathcal{F})$, where $\mathcal{A}$ is a family of subsets of Y and $\mathcal{F}$ is a family of families of subsets of X.

We assume that with each sensor $s = (\mathcal{A}, \mathcal{F})$ a subset $\Theta(s)$ of X is associated, called *the* Θ*-closure of* s.

A sensor $s = (\mathcal{A}, \mathcal{F})$ will be called *small*, if the cardinalities of $\mathcal{A}$ and $\mathcal{F}$ are less than λ, the cardinalities of every γ in $\mathcal{F}$ and of every $A \in \mathcal{A}$ are also less than λ, and $Y \setminus \Theta(s) \neq \emptyset$.

Let H be a subset of Y and γ a family of subsets of X. A sensor $(\mathcal{A}, \mathcal{F})$ is said to be *generated* by the pair (H, γ), if $A \subset H$ for each $A \in \mathcal{A}$, and $\eta \subset \gamma$, for each $\eta \in \mathcal{F}$.

Let $\mathcal{Q}$ be the set of all families γ of subsets of X such that $|\gamma| \leq \mu$. If g is a mapping of $\mathcal{L}$ into $\mathcal{Q}$, $\xi \subset \mathcal{L}$, then $\mathcal{U}_g(\xi) = \cup\{g(F) : F \in \xi\}$.

Let g be a mapping of $\mathcal{L}$ into $\mathcal{Q}$, and let ξ be a subfamily of $\mathcal{L}$. A sensor s will be called *good for* ξ, if it is generated by the pair $(\cup\xi, \mathcal{U}_g(\xi))$ and $\cup\xi \subset \Theta(s)$.

A fast growing (*with respect to* (g, Θ)) *long sequence in* $\mathcal{L}$ is a τ-long increasing sequence ξ in $\mathcal{L}$ such that no small sensor s is good for ξ. For the sake of brevity, we call a fast growing long sequence in $\mathcal{L}$ *a propeller in* $\mathcal{L}$.

Theorem 1. *For each mapping g of $\mathcal{L}$ into $\mathcal{Q}$, there is a fast growing long sequence in $\mathcal{L}$, that is, there is a propeller in $\mathcal{L}$.*

Proof: We construct a propeller in $\mathcal{L}$ by transfinite recursion. Let F_0 be any member of $\mathcal{L}$; fix $\alpha < \tau$, and assume that $F_\beta \in \mathcal{L}$ are already defined for each $\beta < \alpha$. Put $H_\alpha = \cup\{F_\beta : \beta < \alpha\}$ and $\mathcal{U}_\alpha = \cup\{g(F_\beta) : \beta < \alpha\}$. Clearly, $|H_\alpha| \leq \mu$ and $|\mathcal{U}_\alpha| \leq \mu$.

For each small sensor s generated by $(H_\alpha, \mathcal{U}_\alpha)$, we fix a point $m(s) \in Y \setminus \Theta(s)$. Adding all these points to H_α we get a set B_α. Clearly, $|B_\alpha| \leq \mu$. Therefore, $F_\alpha \in \mathcal{L}$ can be chosen so that $B_\alpha \subset F_\alpha$.

Let us show that the τ-long sequence $\xi = \{F_\alpha : \alpha < \tau\}$ is fast growing. Put $P = \cup\xi$, and assume the contrary. Then there is a small sensor $s = (\mathcal{A}, \mathcal{F})$ generated by the pair $(P, \mathcal{U}_g(\xi))$ such that $P \subset \Theta(s)$. Since λ is less than the cofinality of τ, there is $\alpha < \tau$ such that $A \subset H_\alpha$, for each $A \in \mathcal{A}$, and $\eta \subset \mathcal{U}_\alpha$, for each η in $\mathcal{F}$. Then $m(s) \in F_\alpha \subset P \subset \Theta(s)$—a contradiction with the choice of $m(s)$. $\square$

§3. Corollaries of the main theorem: relative versions

In this section we derive some new results from Theorem 1. These results are relative versions of well known theorems on cardinal invariants of topological spaces, they involve relative cardinal topological invariants. Since the topic of relative topological properties is comparatively a new one, we recall some definitions.

Since this fits well all classical cases and almost all cases considered by us, we assume that $\mathcal{L}$ is the family of all subsets of Y of cardinality not greater than μ, if nothing to the contrary is explicitly stated.

Let us also agree (again, if nothing to the contrary is specified), that for each $x \in X$, $\mathcal{B}_x$ is a fixed base of X at x of the smallest possible cardinality, and $\mathcal{P}_x$ is a fixed family of open subsets of X of the smallest possible cardinality such that $\{x\} = \cap\mathcal{P}_x$. We follow the notation introduced in the previous section; in addition, if a fast growing sequence ξ in $\mathcal{L}$ is chosen, we put $P = P(\xi) = \cup\xi$. Finally, if in an argument below we define the Θ-closure only for sensors of a particular kind, that means that only such sensors are effectively involved in the argument, and for all other sensors the Θ-closure can be taken to be the empty set.

Corollary 1. *Let X be a first countable Hausdorff space, and let Y be a subspace of X which is dense in X and initially 2^ω-Lindelöf in X. Then the cardinality of X is not greater than 2^ω, and Y is Lindelöf in X.*

Proof: Put $\tau = \aleph_1 = \lambda$; then $\mu = 2^\omega$. For $F \in \mathcal{L}$ we put $g(F) = \cup\{\mathcal{B}_x : x \in \bar{F}\}$. Then $|\bar{F}| \leq 2^\omega$ and $|g(F)| \leq 2^\omega$. Further, $\Theta((\emptyset, \{\gamma\})) = \cup\gamma$ (and $\Theta(s) = \emptyset$ for all other sensors s, as it was agreed before). By Theorem 1, there is a propeller $\xi = \{F_\alpha : \alpha < \aleph_1\}$ in $\mathcal{L}$. The set $\Phi = \cup\{\bar{F}_\alpha : \alpha < \aleph_1\}$ is closed in X, since X is first countable. Let us show that $Y \subset \Phi$. Assume the contrary, and fix a point $y \in Y \setminus \Phi$. Obviously, the family $\gamma = \{V \in \mathcal{U}_g(\xi) : y \notin V\}$ covers Φ. Therefore, there is a countable subfamily η of γ such that $P \subset (\Phi \cap Y) \subset \cup\eta \subset Y \setminus \{y\}$. Then $s = (\emptyset, \{\eta\})$ is a small sensor good for ξ—a contradiction. Therefore, $Y \subset \Phi$ and $|Y| \leq |\Phi| \leq 2^\omega$. Since X is first countable, Hausdorff and $\bar{Y} = X$, $|X| \leq 2^\omega$. □

In particular, if X is a first countable Hausdorff space, and Y is a dense subspace of X which is compact in X and dense in X, then $|X| \leq 2^\omega$. From Corollary 1 we get the next result:

Corollary 2. *If X is a first countable Hausdorff space, and Y is dense in X and initially 2^ω-compact in X, then Y is compact in X.*

Compare the last result with an absolute theorem in [27].

A space X *is Hausdorff on its subspace* Y (or Y *is Hausdorff in* X) if every two distinct points of Y can be separated by disjoint neighbourhoods in X.

Corollary 3. *If X is Hausdorff on Y, the Souslin number of Y in X is countable, and X is first countable at all points of Y, then $|Y| \leq 2^\omega$.*

Proof: We put $\Theta((\emptyset, \mathcal{F})) = \cup\{\bar{\gamma} : \gamma \in \mathcal{F}\}$, $\tau = \aleph_1 = \lambda$, $\mu = 2^\omega$, and $g(F) = \cup\{\mathcal{B}_x : x \in F\}$, for $F \in \mathcal{L}$. By Theorem 1, there is a propeller ξ in $\mathcal{L}$. Let us show that $Y = P(\xi)$. Assume the contrary, and fix $y \in Y \setminus P$. For $V \in \mathcal{B}_y$, let $P_V = P \setminus \bar{V}$ and $\gamma_V = \{U \in \mathcal{U}_g(\xi) : U \cap \bar{V} = \emptyset\}$. Then γ_V covers P_V, and $\cup\{P_V : V \in \mathcal{B}_y\} = P$. Since $c(Y, X) \leq \omega$, there is a countable subfamily η_V of γ_V such that $P_V \subset \overline{\cup(\eta_V)}$, by Lemma 2. Since $\overline{(\cup\eta_V)} \cap V = \emptyset$, y is not in $\overline{\cup\eta_V}$. We also have:

$$P = \cup\{P_V : V \in \mathcal{B}_y\} \subset \cup\{\overline{\cup\eta_V} : V \in \mathcal{B}_y\}.$$

Therefore, $(\emptyset, \{\eta_V : V \in \mathcal{B}_y\})$ is a small sensor good for ξ—a contradiction. It follows that $P = Y$ and $|Y| = |P| \leq 2^\omega$. □

For a regular X, we get a related result of independent interest by a slightly simpler argument. Recall that Y is said to be *regular in* X, if for each closed subset B of X and each point $y \in Y \setminus B$, there is an open neighbourhood U of y in X such that $\bar{U} \cap B = \emptyset$. Further, we write $\pi\chi(y, X) \leq \mu$ for $y \in X$ if there is a family $\mathcal{E}_y$ of non-empty open subsets of X such that $|\mathcal{E}_y| \leq \mu$ and every neighbourhood V of y in X contains some $W \in \mathcal{E}_y$; if this is the case, we fix such $\mathcal{E}_y$.

Corollary 4. *If Y is regular in X, the Souslin number of X is countable, and $\pi\chi(y, X) \leq 2^\omega$, for each $y \in Y$, then $w(Y) \leq 2^\omega$.*

Proof: We put $\Theta(\emptyset, \{\gamma\}) = \overline{\cup\gamma}$, $\tau = \aleph_1 = \lambda$, $\mu = 2^\omega$ and $g(F) = \cup\{\mathcal{E}_x : x \in F\}$. Take a propeller ξ in $\mathcal{L}$. We claim that $P = \cup\xi$ is dense in Y. Assume the contrary, and fix $y \in Y \setminus \bar{P}$. Fix also an open neighbourhood V of y in X such that $\bar{V} \cap P = \emptyset$. Let $\gamma_V = \{U \in \mathcal{U}_g(\xi) : \bar{V} \cap U = \emptyset\}$. Since $c(X) \leq \omega$, there is a countable subfamily η_V of γ_V such that $\overline{\cup\eta_V} = \overline{\cup\gamma_V}$. Then $P \subset \overline{\cup\eta_V}$, and y is not in $\overline{\cup\eta_V}$; therefore, $s = (\emptyset, \{\eta_V\})$ is a small sensor good for ξ—a contradiction. Thus, P is dense in Y.

Put $\mathcal{E} = \cup\{\mathcal{E}_y : y \in P\}$. Then $|\mathcal{E}| \leq 2^\omega$. Let $\mathcal{S}$ be the family of all disjoint families of elements of $\mathcal{E}$. Since $c(X) \leq 2^\omega$, each $\eta \in \mathcal{E}$ is countable, therefore, $|\mathcal{S}| \leq 2^\omega$. For each $\eta \in \mathcal{S}$ let V_η be the interior in the space Y of the intersection of Y with the closure in X of the set $\cup\eta$. Take any open subset W of X, and let η be a maximal disjoint subfamily of the family $\{U \in \mathcal{E} : U \subset W\}$. Then $|\eta| \leq \omega$, since $c(X) \leq \omega$. From the definition of $\mathcal{E}$ it is easy to see that $Y \cap W \subset V_\eta$. Since Y is regular in X, it follows that $\{V_\eta : \eta \in \mathcal{S}\}$ is a base of the space Y. Therefore, $w(Y) \leq |\mathcal{S}| \leq 2^\omega$. □

Compare Corollary 4 with an absolute result in [26]; see also [20]. There is another version of Corollary 4. To formulate it, we have to introduce a relative version of π-character. We shall write $\pi\chi(y, Y, X) \leq \mu$ for $y \in Y$, if there is a family $\mathcal{E}_y$ of open subsets of X such that $|\mathcal{E}_y| \leq \mu$, every neighbourhood of y in X contains some $W \in \mathcal{E}_y$, and $U \cap Y$ is not empty for each $U \in \mathcal{E}_y$; if this is the case, we fix such $\mathcal{E}_y$ for each $y \in Y$, and call it *a π-base of X at y with respect to Y*. The infinite cardinal number $\pi\chi(y, Y, X)$, defined in an obvious way, is called *the π-character of X at y with respect to Y*. In these notation, we have:

Corollary 5. *If Y is regular in X, the Souslin number of Y in X is countable, and $\pi\chi(y, Y, X) \leq 2^\omega$, for each $y \in Y$, then $w(Y) \leq 2^\omega$.*

Proof: We take the same Θ, τ, λ, μ, g, ξ, P, y, V and γ_V as in the proof of Corollary 4. Since $c(Y, X) \leq \omega$, there is a countable subfamily η_V of γ_V such that $Y \cap \overline{\cup\gamma_V} \subset \overline{\cup\eta_V} \subset \overline{\cup\gamma_V}$, by Lemma 2. Since $P \subset Y$, it follows that $P \subset \overline{\cup\eta_V}$. Also y is not in $\overline{\cup\eta_V}$. Thus, $s = (\emptyset, \{\eta_V\})$ is a small sensor good for ξ—a contradiction. Therefore, P is dense in Y. The rest of the proof is the same as in the proof of Corollary 4. □

Let us say that Y *is quasi-τ-Lindelöf in* X if for every subset A of Y closed in Y, and for every family γ of open subsets of X such that $A \subset \cup\gamma$ and $|\gamma| \leq \tau$, there is a countable subfamily η of γ such that $A \subset \overline{\cup\eta}$. We say that Y *is quasi-Lindelöf* *in* X, if the above condition holds without any restriction on the cardinality of γ. Now we can unify Corollaries 1 and 3 under a slightly stronger assumption on separation properties of Y in X as follows:

Corollary 6. *If Y is regular in X and X is first countable at all points of Y, and Y is quasi 2^ω-Lindelöf in X, then $|Y| \leq 2^\omega$.*

Proof: We put $\Theta((\emptyset, \{\gamma\})) = \overline{\cup\gamma}$, $\tau = \aleph_1 = \lambda$, $\mu = 2^\omega$, and $g(F) = \cup\{\mathcal{B}_y : y \in cl_Y(F)\}$. There is a propeller $\xi = \{F_\alpha : \alpha < \aleph_1\}$ in $\mathcal{L}$. Let us show that $Y \subset \bar{P}$, where $P = \cup\xi$. Assume the contrary, and fix $y \in Y \subset \bar{P}$. Since Y is regular in X, we can also fix a neighbourhood W of y in X such that $\bar{W} \cap \bar{P} = \emptyset$. Put $\gamma = \{U \in \mathcal{U}_g(\xi) : U \cap W = \emptyset\}$. Since Y is first countable, the closure of P in Y is the set $\cup\{cl_Y(F_\alpha) : \alpha < \aleph_1\}$. Therefore, $cl_Y(P) \subset \cup\gamma$. Since $|\gamma| \leq 2^\omega$, there is a countable subfamily η of γ such that $P \subset \overline{\cup\eta}$. From $W \cap (\cup\eta) = \emptyset$ it follows that $W \cap (\overline{\cup\eta}) = \emptyset$. Therefore, y is not in $\overline{\cup\eta}$. Hence, $s = (\emptyset, \{\eta\})$ is a small sensor good for ξ,—a contradiction. It follows that $Y \subset \bar{P}$. Since $|P| \leq 2^\omega$, and Y is first countable, we have: $|Y| \leq 2^\omega$. □

The next result easily follows from Corollary 6.

Corollary 7. *If X is a sequential Hausdorff space first countable on Y, and Y is regular in X, dense in X, and quasi 2^ω-Lindelöf in X, then $|X| \leq 2^\omega$.*

If X itself is first countable, we can slightly weaken another assumption in Corollary 7. To do that, we introduce another relative invariant. Let us say that Y is 2^ω*-slim in* X, if for each subset A of Y and every family γ of open sets in X such that $\bar{A} \subset \cup\gamma$ and $|\gamma| \leq 2^\omega$, there is a countable subfamily η of γ such that $A \subset \overline{\cup\eta}$.

Corollary 8. *If X is a first countable Hausdorff space, and Y is regular in X, dense in X, and 2^ω-slim in X, then $|X| \leq 2^\omega$.*

Proof: The proof is virtually the same as the proof of Corollary 6; the only essential difference is in the definition of the correspondence g. We put: $g(F) = \cup\{\mathcal{B}_x : x \in \bar{F}\}$, for each $F \in \mathcal{L}$. We also observe that if ξ is a propeller in $\mathcal{L}$, then the set $\cup\{\bar{F} : F \in \xi\}$ is closed in X, since X is first countable. This suffices. □

For a normal space X, Corollary 8 comes very close to a relative version of a result of M. Bell, D. Grant and G. Woods in [8]. We say that Y is *weakly* 2^ω*-Lindelöf in* X, if for every open covering γ of X such that $|\gamma| \leq 2^\omega$, one can find a countable subfamily η of γ such that $Y \subset \overline{(\cup\eta)}$. A subset A of X is said to be *concentrated on* Y, if $A \subset \overline{A \cap Y}$. We shall say that X is *strongly normal on* Y, if for every closed subset F of X concentrated on Y and every open neighbourhood U of F in X there is an open neighbourhood V of F in X such that $\bar{V} \subset U$.

Corollary 9. *If X is a first countable Hausdorff space strongly normal on Y, and Y is dense in X and weakly 2^ω-Lindelöf in X, then $|X| \leq 2^\omega$.*

Proof: Since X is strongly normal on Y and Y is weakly 2^ω-Lindelöf in X, Y is 2^ω-slim in X and Y is regular in X. It remains to apply Corollary 8. □

Corollary 10 [18]**.** *If all points of X are G_δ's in X, and the spread of X does not exceed ω, then $|X| \leq 2^\omega$.*

Proof: Put $Y = X$ and $\Theta((\mathcal{A}, \mathcal{F})) = (\cup\{\cup\gamma : \gamma \in \mathcal{F}\}) \cup (\cup\{\bar{A} : A \in \mathcal{A}\})$. Let $\tau = \aleph_1 = \lambda$. Then $\mu = 2^\omega$. Let $g(F) = \cup\{\mathcal{P}_x : x \in F\}$, for $F \in \mathcal{L}$. By Theorem 1, we can fix a propeller ξ in $\mathcal{L}$. Let us show that $X = P$.

Assume the contrary, and fix a point x in X such that $x \in X \setminus P$. For $V \in \mathcal{P}_x$ we put $P_V = P \setminus V$ and $\gamma_V = \cup\{\{U \in \mathcal{P}_y : x \notin U\} : y \in P_V\}$. Clearly, $P_V \subset \cup\gamma_V$. By Lemma 1, there are a countable discrete subspace A_V of P_V and a countable subfamily η_V of γ_V such that $P_V \subset (\cup\eta_V) \cup \bar{A}_V$. Since $A_V \subset P_V$, $x \notin \bar{A}_V$. Thus, x is not in the Θ-closure of the sensor $s = (\{A_V : V \in \mathcal{P}_x\}, \{\eta_V : V \in \mathcal{P}_x\})$, while $P \subset \Theta(s)$. We see that s is a small sensor good for ξ—a contradiction. Therefore, $X = P$ and $|X| = |P| \leq 2^\omega$. □

For a subset A of X we put: $[A]_\omega = \cup\{\bar{B} : B \subset A, |B| \leq \omega\}$. We say that the *lower ω-density $d_\omega(X)$ of* X is not greater than μ, if there is a subset A of X such that $|A| \leq \mu$ and $[A]_\omega = X$.

Corollary 11 [26]**.** *If $s(X) \leq \omega$ and $\psi(X) \leq 2^\omega$, then $d_\omega(X) \leq 2^\omega$.*

Proof: We put $Y = X$, $\tau = \aleph_1 = \lambda$, and $\Theta((\{A\}, \{\gamma\})) = (\cup\gamma) \cup \bar{A}$. Let $g(F) = \cup\{\mathcal{P}_x : x \in F\}$ for $F \in \mathcal{L}$. By Theorem 1, we can fix a propeller ξ in $\mathcal{L}$. For $P = \cup\xi$ we have: $|P| \leq 2^\omega$. Let us check that $[P]_\omega = X$. Assume the contrary, and fix $x \in X \setminus [P]_\omega$. The family $\gamma = \{V \in \mathcal{U}_g(\xi) : x \notin V\}$ covers P. By Lemma 1, since the spread of X is countable, there are a countable subset A of P and a countable subfamily η of γ such that $(\cup\eta) \cup \bar{A}$ contains P. That is, P is contained in the Θ-closure $\Theta(s)$ of the sensor $s = (\{A\}, \{\eta\})$. On the other hand, x is not in $\Theta(s)$, since x is not in $\cup\eta$ and x is not in $\bar{A}$. Thus, s is small and good for ξ—a contradiction. Therefore, $[P]_\omega = X$ and $d_\omega(X) \leq |P| \leq 2^\omega$. □

Corollary 12 [26]**.** *If X is a Hausdorff space of countable spread, then $d_\omega(X) \leq 2^\omega$.*

Proof: It follows easily from Lemma 1 that the pseudocharacter of X does not exceed 2^ω (see [26], [20]). It remains to apply Corollary 11. □

Corollary 13. *Let X be a regular space of countable tightness, and let Y be a subspace of X which is initially ω_1-compact in X and satisfies the condition: $hl(Y) \leq \omega_1$. Then $d(Y) \leq \aleph_1$.*

Proof: We put $\Theta((\emptyset, \{\gamma\})) = \cup\gamma$, $\tau = \aleph_1$ and $\lambda = \omega$. Then $\mu = \aleph_1$, and $\mathcal{L}$ is the family of all subsets of Y of cardinality not greater than $\aleph_1$. For $F \in \mathcal{L}$ we fix a family $g(F)$ of open subsets of X such that $|g(F)| \leq \aleph_1$, $\bar{F} \subset \cap g(F)$, and $Y \cap \bar{F} = (\cap g(F)) \cap Y$; since X is regular and $l(Y \setminus \bar{F}) \leq \aleph_1$, we can do it. By Theorem 1, there is a propeller ξ in $\mathcal{L}$. Let us show that $P = P(\xi)$ is dense in Y. Assume the contrary. The set $\Phi = \cup\{\bar{F}_\alpha : \alpha < \aleph_1\}$ is closed in X, since $t(X) \leq \omega$; also $\Phi \subset \bar{P}$. Therefore, there is a point $y \in Y \setminus \Phi$. The family

$\gamma = \{V \in \mathcal{U}_g(\xi) : y \notin V\}$ covers Φ. Since Y is initially ω_1-compact in X, there is a finite subfamily η of γ such that $\Phi \cap Y \subset \cup\eta$. Clearly, $\Phi \cap Y = P$. Also $y \notin \cup\eta$. Thus, $s = (\emptyset, \{\eta\})$ is a small sensor good for ξ—a contradiction. Therefore P is dense in Y, and $d(Y) \leq |P| \leq \aleph_1$. $\square$

Corollary 14. *If X is an initially ω_1-compact regular space of countable tightness, and the spread of X is not greater than $\aleph_1$, then the hereditary density of X also does not exceed $\aleph_1$.*

PROOF: Since the tightness of X is countable, it is enough to show that the density of any closed subspace Z of X is not greater than $\aleph_1$. Now it is clear that we only have to show that $d(X) \leq \aleph_1$. By the $\aleph_1$-version of Proposition 4, there is a dense subspace Y in X such that $hl(Y) \leq \aleph_1$. Now we can apply Corollary 13. $\square$

Corollary 15. *If X is an initially ω_1-compact regular space of countable spread, then $d(X) \leq \aleph_1$.*

PROOF: The tightness of any initially ω_1-compact regular space of countable spread is countable [2]. It remains to apply Corollary 14. $\square$

We say that a subset A of X is *a G_τ-subset in X with respect to a subspace Y of X*, if there is a family γ of open subsets of X such that $|\gamma| \leq \tau$, $A \subset \cap\gamma$ and $A \cap Y = (\cap\gamma) \cap Y$. A space X is said to be *weakly τ-perfect on* Y, if for every infinite subset A of Y there is a subset B of Y such that $|B| = |A|$, $A \subset B$, and the closure of A in X is a G_τ-set in X with respect to Y. If the above condition holds with $B = A$, we say that X is *τ-perfect on* Y. Naturally, *'weakly perfect'* means 'weakly ω-perfect', and 'perfect' means 'ω-perfect'.

Corollary 16. *Let X be a space of countable tightness, and let Y be a subspace of X which is weakly ω_1-perfect in X and initially ω_1-compact in X. Then the density of Y is not greater than ω_1.*

PROOF: Again, $\Theta((\emptyset, \{\gamma\})) = \cup\gamma$, $\tau = \aleph_1$ and $\lambda = \aleph_0$. Then $\mu = \aleph_1$. Let $\mathcal{L}$ be the family of all subsets F of Y such that $|F| \leq \aleph_1$ and $\bar{F}$ is a G_τ-subset of X. For each $F \in \mathcal{L}$ fix a family $g(F)$ of open subsets of X such that $|g(F)| \leq \aleph_1$, $\bar{F} \subset \cap g(F)$ and $Y \cap \bar{F} = \cap g(F) \cap Y$. By Theorem 1, there is a propeller $\xi = \{F_\alpha : \alpha < \aleph_1\}$ in $\mathcal{L}$. It suffices to show that $P = \cup\xi$ is dense in Y.

Assume the contrary, and fix $y \in Y \setminus \bar{P}$. Then $\Phi = \cup\{\bar{F}_\alpha : \alpha < \tau\}$ is closed in X, since $t(X) \leq \omega$. Clearly, there is $\gamma \subset \mathcal{U}_g(\xi)$ such that $\Phi \subset \cup\gamma$ and $y \notin \cup\gamma$. Then $|\gamma| \leq \omega_1$, and there is a countable subfamily η of γ covering P. Since $y \notin \cup\eta$, $s = (\emptyset, \{\eta\})$ is a small sensor good for ξ—a contradiction. $\square$

A pseudobase of a space X with respect to a subspace $Y \subset X$ is an open covering $\mathcal{P}$ of X such that for each $x \in X$, $(\cap\{U \in \mathcal{P} : x \in U\}) \cap Y \subset \{x\}$.

Corollary 17. *f Y is countably compact in X and dense in X, and there is a pseudobase $\mathcal{P}$ of X with respect to Y which is point-countable at all points of Y, then $\mathcal{P}$ is countable.*

Proof: Put $\tau = \omega = \lambda$ and, again, let $\Theta((\emptyset, \{\gamma\})) = \cup\gamma$. Then $\mu = \omega$, and $\mathcal{L}$ is the family of all countable subsets of Y. Let $g(F) = \{U \in \mathcal{P} : U \cap F \neq \emptyset\}$, for $F \in \mathcal{L}$. Then $|g(F)| \leq \omega$, since $\mathcal{P}$ is point-countable at all points of Y. By Theorem 1, there is a propeller $\xi = \{F_n : n < \omega\}$ in $\mathcal{L}$.

Then $P = \cup\xi$ is countable, and it suffices to show that P is dense in Y. Assume the contrary, and fix $y \in Y \setminus \bar{P}$. The family $\mathcal{U}_g(\xi) = \{U \in \mathcal{P} : U \cap \bar{P} \neq \emptyset\}$ contains a subfamily γ such that $y \notin \cup\gamma$ and $\bar{P} \subset \cup\gamma$. Since γ is countable, there is a finite subfamily η of γ such that $P \subset \cup\eta$. Then $s = (\emptyset, \{\eta\})$ is a small sensor good for ξ—a contradiction. □

We say that *the diagonal number* $\Delta(Y, X)$ *of* Y *in* X is countable, if there is a countable sequence $\zeta = \{\mathcal{G}_i : i \in \omega\}$ of open covers of X such that if p and q are any two distinct points in Y, then for some $n \in \omega$, q is not in $st(p, \mathcal{G}_n)$. In this case we write: $\Delta(Y, X) \leq \omega$, and say that ζ *separates* Y *in* X. The definition of $\Delta(Y, X)$ in the general case is now obvious.

Corollary 18. *If $e(Y, X) \leq \omega$ and $\Delta(Y, X) \leq \omega$, then $|Y| \leq 2^\omega$.*

Proof: Let us fix a sequence $\zeta = \{\mathcal{G}_i : i \in \omega\}$ of open coverings of X, separating Y in X. Put $\tau = \aleph_1 = \lambda$, $\mu = 2^\omega$, and let $\mathcal{L} = \{F \subset Y : |Y| \leq 2^\omega\}$. For an indexed sensor $s = (\{A_i : i \in \omega\}, \emptyset)$ we define the Θ-closure $\Theta(s)$ as follows: $\Theta(s) = \cup\{st(A_i, \mathcal{G}_i) : i \in \omega\}$. The mapping g is trivial: $g(F)$ is empty, for each $F \in \mathcal{L}$. Now take a propeller ξ in $\mathcal{L}$. Let us show that $P = \cup\xi = Y$. Assume the contrary, and fix $y \in Y \setminus P$. Put $P_i = P \setminus st(y, \mathcal{G}_i)$, for $i \in \omega$. Since ζ is separating Y in X, $\cup\{P_i : i \in \omega\} = P$. By Lemma 4, there is a countable subset A_i of P_i such that $P_i \subset st(A_i, \mathcal{G}_i)$. Since $y \notin st(P_i, \mathcal{G}_i)$, $y \notin st(A_i, \mathcal{G}_i)$. Therefore, the Θ-closure of the sensor $s = (\{A_i : i \in \omega\}, \emptyset)$ contains P and does not contain y. Thus, s is a small sensor good for ξ—a contradiction. □

Corollary 19. *If the diagonal of X is a G_δ in $X \times X$, and $e(Y, X) \leq \omega$, then $|Y| \leq 2^\omega$.*

Generalizing R. Hodel's definition [19], let us denote by $psw(Y, X)$ the smallest cardinal number τ such that there is a pseudobase $\mathcal{P}$ of X with respect to Y such that each point of Y is in at most τ elements of $\mathcal{P}$.

Corollary 20. *If $psw(Y, X) \leq \omega$, and $e(Y, X) \leq \omega$, then $|Y| \leq 2^\omega$.*

Proof: We put $\tau = \aleph_1 = \lambda$, $\mu = 2^\omega$. Let us fix a pseudobase $\mathcal{P}$ of X with respect to Y, which is point-countable at each point of Y. For $F \in \mathcal{L}$, let $g(F) = \{U \in \mathcal{P} : U \cap F \neq \emptyset\}$. Then $|g(F)| \leq 2^\omega$. We put $\Theta((\emptyset, \{\gamma\})) = \cup\gamma$. By Theorem 1, there is a propeller ξ in $\mathcal{L}$. Let us show that $P = \cup\xi = Y$. Assume the contrary, and fix $y \in Y \setminus P$. Fix also $V \in \mathcal{P}$ such that $y \in V$, and put $P_V = P \setminus V$. The family $\gamma = \{U \in \mathcal{P} : U \cap P_V \neq \emptyset, y \notin U\}$ covers the closure of P_V in X. Since $e(Y, X) \leq \omega$, and y is contained only in countably many elements of $\mathcal{P}$, there is a countable subset A of P such that $P \subset st(A, \gamma)$. Since the family $\mathcal{P}$ is point countable at points of Y, the family $\eta = \{U \in \gamma : U \cap A \neq \emptyset\}$ is countable.

Clearly, $\eta \subset \gamma \subset \mathcal{U}_g(\xi)$, $P \subset \cup\eta$, and y is not in $\cup\eta$. Therefore, $s = (\emptyset, \{\eta\})$ is a small sensor good for ξ—a contradiction. □

§4. Some observations, examples, and problems

1. For Y to be Lindelöf in X, it is sufficient to have an intermediate Lindelöf subspace Z, lying in between Y and X. A natural question arises: is that the only reason for Y to be Lindelöf in X? In other words, is it true that if Y is Lindelöf in X then there is a subspace Z of X such that $Y \subset Z \subset X$ and Z is Lindelöf (in itself)? If that were the case, then Corollary 1 would be reduced to its well known absolute version. Unfortunately (or fortunately for the theory of relative properties) this does not happen: A. Dow and J. Vermeer [12] have constructed a counterexample in the class of Tychonoff spaces. It is much easier to find such a counterexample in the class of Hausdorff spaces, which should satisfy us since the classical result holds for Hausdorff spaces.

Example 1. Let $X = A \cup I$ be the Alexandroff double of the closed unit interval I, where A is the discrete copy of I (see [13]). Now take the weakest topology on X which contains the topology of the space X and turns I into a closed discrete subspace; let X_1 be the resulting space. Then X_1 is Hausdorff, and A is compact in X_1. Let us show that no subspace Z of X_1 such that $A \subset Z \subset X_1$ is Lindelöf in itself. Assume the contrary. Then the set $B = Z \cap I$, being closed and discrete in Z, must be countable. There is an open subset U of X (which is automatically open in X_1) such that $B \subset U$ and the set $I \setminus U$ is uncountable. Then the copy in A of the set $I \setminus U$ is an uncountable closed discrete subspace of the space Z—a contradiction. In fact, we have shown that the extent of Z is uncountable for any Z such that $A \subset Z \subset X_1$.

Problem 1. Find a "naive" example of a regular space X, in which all points are G_δ's, and of its subspace Y such that Y is Lindelöf in X and the cardinality of Y is greater than 2^ω.

2. Observe that for relative compactness the situation in the class of regular spaces is trivial, and one indeed can reduce some relative results to classical ones: if X is regular, and Y is compact in X, then the closure of Y in X is compact [24].

3. It follows from Corollary 10 that the cardinality of any hereditarily Lindelöf Hausdorff space is not greater than 2^ω. Is there a nice relative version of this result? Let us say that Y is *hereditarily Lindelöf in* X (*hereditarily compact in* X), if from every family γ of open subsets of X such that $X \setminus Y \subset \cup\gamma$ one can choose a countable (respectively, a finite) subfamily η such that $(\cup\gamma) \cap Y \subset \cup\eta$.

Proposition 5. *If Y is Lindelöf in X, then the extent of Y in X is countable, that is, every discrete subspace Z of Y closed in X is countable.*

Proof: There is a family ξ of open subsets of X such that each member of ξ contains not more than one point of Z and Z is covered by ξ. Then $\gamma = \xi \cup \{X \setminus Z\}$

is an open covering of X. A countable subfamily of γ, which covers Y, witnesses that Z is countable. □

Let us say that X is *strongly Hausdorff on* Y, if for every two distinct points of X at least one of which belongs to Y there are open disjoint neighbourhoods of these points in X.

Proposition 6. *If X is strongly Hausdorff on Y, and Y is hereditarily Lindelöf in X, then $\psi(Y) \leq \omega$.*

Proof: Fix $y \in Y$, and for each $x \in X \setminus \{y\}$ fix an open subset U_x of X such that $x \in U_x$ and y is not in $\overline{U_x}$. Then the family $\gamma = \{U_x : x \in X \setminus \{y\}\}$ covers $X \setminus \{y\}$. Therefore a countable subfamily η of γ covers $Y \setminus \{y\}$, and $\{y\} = \cap\{Y \setminus \bar{W} : W \in \eta\}$. Thus, $\psi(Y) \leq \omega$. □

Let us say that X is *weakly regular on* Y, if for each $x \in X$ and every subset P of Y closed in X such that x is not in P, there is a neighbourhood U of x in X such that $\bar{U} \cap P = \emptyset$. Clearly, if X is weakly regular on Y, then X is strongly Hausdorff on Y. We also say that Y is *weakly perfect in* X if every closed in X subset of Y is a G_δ in X.

Proposition 7. *If X is weakly regular on Y, and Y is hereditarily Lindelöf in X, then Y is weakly perfect in X.*

Proof: The proof is virtually the same as that of Proposition 6. □

Example 2. It is not true that if Y is hereditarily Lindelöf in a regular space X, then $|Y| \leq 2^\omega$, or that Y is separable. Indeed, every discrete space is hereditarily compact in every compactification of it.

A natural way to remedy the situation encountered in the case of relative hereditary Lindelöfness is to turn to the concept of relative cardinality of a set in a topological space. Let us say that *the cardinality of* Y *in* X is not greater than τ, if the cardinality of every closed in X subset of Y does not exceed τ. In this case we write: $|Y_X| \leq \tau$. We have:

Proposition 8. *If Y is hereditarily Lindelöf in a Hausdorff space X, then $|Y_X| \leq 2^\omega$.*

Proof: It is easy to see that every closed in X subspace Z of Y is a hereditarily Lindelöf space. Therefore $|Z| \leq 2^\omega$. □

Of course, Proposition 8 should be considered as a trivial relativization of the classical result on the cardinality of hereditarily Lindelöf Hausdorff spaces, since it reduces to it so easily. But now we can better appreciate the results in Section 3—Corollaries 1, 2, 3, 6, 7, and others, since the conclusion in all these assertions contains the information on the cardinality of Y, and not just on the relative cardinality of Y. Thus, the relative results in section 3 contain more information than the corresponding classical results, they are indeed stronger than the latter ones.

4. Actually, one can define relative versions of topological properties in a rather trivial and uniform way, which guarantees that almost all classical theorems of general topology remain true in the relative case. All we have to do is to follow the idea in the definition of the relative cardinality. Let $\mathcal{P}$ be a closed hereditary topological property. We will say that Y has $\mathcal{P}$ in X *from inside*, if every closed in X subspace of Y has the property $\mathcal{P}$ (in itself). Probably, this is the weakest among all natural definitions of the relative $\mathcal{P}$. Clearly, all theorems of General Topology, involving only closed hereditary properties, remain true in the relative case as well, provided, of course, we use only the 'inside' versions of relative properties. It is also clear that we must not use the 'inside' definition in the case of connectedness or in the case of the Souslin number. A question arises: can we get a non-trivial result, containing an information on the cardinality of Y itself (and not just on the relative cardinality of Y), if we accept this weakest definition of relative Lindelöfness, that is, if we assume that Y is Lindelöf in X from inside? Can we strengthen Corollary 1 in this way? The answer is 'no'.

Example 3. For any uncountable regular cardinal number τ there is a countably compact first countable Tychonoff space X such that the set Y of all isolated points of X is dense in X and has cardinality τ. Indeed, we get such an X when we remove from the ordinal space τ all limit ordinals which are not countably cofinal. Now, Y is obviously compact in X from inside: *a discrete subspace of a countably compact space X is always compact in X from inside.*

In any case, the 'inside approach' provides us with natural and interesting relative cardinal invariants. For example, we say that Y *is compact in* X (*Lindelöf in* X, *normal in* X) *from inside*, if every closed subspace of X contained in Y is compact (respectively, is Lindelöf, is normal). *The cardinality* $|Y|_X$ *of* Y *in* X is $\sup\{|A| : A \subset Y, A = \bar{A}\}$—the smallest cardinal number τ such that $|A| \leq \tau$, for each closed in X subset A of Y.

Here is a natural general question concerning the 'inside' version of relative properties.

Problem 2. Let $\mathcal{P}$ and $\mathcal{Q}$ be two topological properties. When a topological space Y can be embedded into a topological space Z with the property $\mathcal{Q}$ in such a way that Y has $\mathcal{P}$ in Z from inside?

5. Assume that Y is not closed in X. Then the family $\mathcal{F}(Y, X)$ of all closed in X subsets of Y satisfies all axioms for a family of all closed sets in a topological space, save one: if Y is not closed in X, then Y does not belong to this family. Adding Y to $\mathcal{F}(Y, X)$, we arrive at an interesting topological space Y_X: every two non-empty open subsets of Y_X have a non-empty intersection, while all proper closed subspaces might enjoy nice separation axioms—the same, as in the original space X.

Problem 3. Study $\mathcal{F}(Y, X)$ in the Vietoris topology and in the other natural topologies.

6. It is very natural to apply relative topological properties to the theory of extensions of topological spaces. Here are some typical general and concrete questions in this direction.

Problem 4. Let $\mathcal{Q}$ be a class of spaces (a topological property), and let $\mathcal{P}$ be a relative topological property. When for every Y in $\mathcal{Q}$ one can find X in $\mathcal{Q}$ such that Y is a subspace of X and Y has the property $\mathcal{P}$ in X?

For example, what happens if $\mathcal{Q}$ is a class of all first countable (Hausdorff, regular, Tychonoff) spaces, and $\mathcal{P}$ is the (relative) normality? Or if $\mathcal{P}$ is paracompactness? Or Lindelöfness?

Problem 5. Is it possible to embed every Tychonoff first countable space into a normal first countable space?

Problem 6. When a Tychonoff perfect space Y can be embedded into a compact Hausdorff space X such that X is perfect on Y? Is this always possible for a perfect Tychonoff space Y such that each compact subspace of Y has the countable character in Y?

Here are some interpretations of the relative theorems of the previous section in the subspace-extension spirit.

Theorem 2. *Let Y be a Hausdorff first countable space such that $|Y| > 2^\omega$. Then there is no Hausdorff first countable space X such that Y is a subspace of X and Y is 2^ω-Lindelöf in X.*

Theorem 3. *If X is Hausdorff and $c(X) \leq \omega$, then the cardinality of the set of all points of X at which the space satisfies the first axiom of countability, has the cardinality not greater than 2^ω.*

Observe that Theorem 3, which follows from Corollary 3, is a result on "absolute" topological properties, which cannot be obtained from the corresponding to Corollary 3 absolute assertion. The same holds true for the next theorem, which improves Theorem 3 for regular spaces. It follows from Corollary 4.

Theorem 4. *If X is regular and $c(X) \leq \omega$, then the subspace of X consisting of all points of X at which the π-character of X is countable, has the weight not greater than 2^ω.*

7. Proceeding along the lines of this article, it is natural to consider the following notion. Let us say that X is *a discretely Lindelöf space*, if every discrete subspace of X is Lindelöf in X. Now an invariant $dl(X)$ is introduced in a natural way: $dl(X)$ is the smallest cardinal number τ such that if A is any discrete subspace of X, and γ is any open covering of X, then there is a subfamily η of γ such that $A \subset \cup\eta$ and $|\eta| \leq \tau$. We might call $dl(X)$ *the discrete Lindelöf degree of* X. Clearly, $dl(X) \leq l(X)$. The next assertion is also obvious.

Proposition 9. *For every space X, $e(X) \leq dl(X) \leq s(X)$.*

Thus, if every discrete subspace of X is countable, then X is discretely Lindelöf in X, and if X is discretely Lindelöf in X, then every closed discrete subspace of X is countable. On the other hand, we have the next example:

Example 4. Let X be the space of all countable ordinals. Then X is countably compact, and therefore $e(X) = \omega$, while the subspace A of X consisting of all isolated ordinals is discrete and is not Lindelöf on X.

Results exhibited in this paper suggest the next question:

Problem 7. Is there in ZFC an example of a regular space X such that $dl(X) \leq \omega$, $\psi(X) \leq \omega$, and $2^\omega < |X|$?

In connection with the last question we mention the next simple result.

Proposition 10. *If X is perfect and discretely Lindelöf, then $s(X) \leq \omega$ and $|X| \leq 2^\omega$.*

Proof: Indeed, $e(X) \leq dl(X) \leq \omega$, and since X is perfect, $s(X) \leq \omega$ and $\psi(X) \leq \omega$. It remains to apply Corollary 11. □

At the moment the author does not know the answer to the following questions (which might turn out not to be too difficult, after all):

Problem 8. Is there in ZFC a regular discretely Lindelöf space which is not Lindelöf?

Problem 9. Is there in ZFC a Tychonoff discretely Lindelöf space which is not a Hewitt-Nachbin space?

Problem 10. Is there in ZFC a regular countably compact discretely Lindelöf space which is not compact?

Observe, that one easily gets consistent examples to Problems 8, 9, and 10 with the help of Fedorchuk's consistent example of a hereditarily separable compact Hausdorff space containing no non-trivial convergent sequences (see [14]).

Though the notion of discrete Lindelöfness is already based on the concept of relative Lindelöfness, it can also be relativized. Clearly, one should call Y *discretely Lindelöf in* X, if every discrete subspace of Y is Lindelöf in X. Now we can blend together Problems 1 and 7 in the following way:

Problem 11. Find in ZFC an example of a regular space X with a subspace Y such that all points of X are G_δ's in X, Y is discretely Lindelöf in X, and the cardinality of Y is greater than 2^ω.

8. The situation is much more transparent with discrete compactness, defined similarly to discrete Lindelöfness: a space X is *discretely compact*, if every discrete subspace of X is compact in X. If the closure of every discrete subspace of X is compact, we say that X is *strongly discretely compact.* Obviously, we can relativize both notions. In particular, Y is *discretely compact in* X, if every discrete subspace of Y is compact in X.

Theorem 5. *If X is regular and discretely compact, then X is compact.*

Proof: Since X is regular, the closure in X of every subspace of X which is compact in X, is compact (see [24]). Therefore, the closure in X of every discrete subspace of X is compact, that is, X is strongly discretely compact. It remains to prove the next assertion, in which separation properties of X are not important. □

Proposition 11. *If X is strongly discretely compact, then X is compact.*

Proof: Assume the contrary, and let $\gamma = \{U_\alpha : \alpha < \tau\}$ be an open covering of X of the smallest possible cardinality τ, from which one can not choose a finite subcovering of X. Fix $d \notin X$ and $x_0 \in U_0$. For $\alpha < \tau$ we put $\gamma_\alpha = \{U_\beta : \beta < \alpha\}$ and $G_\alpha = \cup\gamma_\alpha$. Let us assume that for each $\beta < \alpha$ a point $x_\beta \in X \cup \{d\}$ is already defined. To define x_α, we consider the set $V_\alpha = U_\alpha \setminus (G_\alpha \cup F_\alpha)$, where $F_\alpha = \overline{\{x_\beta : \beta < \alpha, x_\beta \neq d\}}$. If $V_\alpha = \emptyset$, we put $x_\alpha = d$; if V_α is not empty, then x_α is any point of V_α. Let $B = \{\beta : \beta < \tau, x_\beta \neq d\}$. The subspace $A = \{x_\beta : \beta \in B\}$ is discrete; indeed, for each $\beta \in B$, $W_\beta = U_\beta \setminus F_\beta$ is an open set containing only one point of A—the point x_β. Therefore, there is a finite subfamily $\xi \subset \gamma$ such that $\bar{A} \subset \cup\xi$. Since τ is infinite, there is $\alpha < \tau$ such that $\xi \subset \gamma_\alpha$. Then $\bar{A} \subset \cup\gamma_\alpha = G_\alpha$.

Take any $\beta > \alpha$. Then $x_\beta \notin G_\alpha$ and, hence, $x_\beta \notin A$. It follows that $x_\beta = d$, for each $\beta > \alpha$, which implies that $X = G_\alpha \cup F_\alpha$. Since, $F_\alpha \subset \bar{A} \subset G_\alpha$, we conclude that $X = G_\alpha = \cup\gamma_\alpha$. Clearly, $|\gamma_\alpha| < \tau$. Therefore, according to the choice of τ, there is a finite subcovering $\eta \subset \gamma_\alpha \subset \gamma$—a contradiction. □

Corollary 21. *If X is a Hausdorff space of point countable type, and x is a non-isolated point of X, then there is a discrete subspace A of X such that x is a limit point for A.*

Proof: Since X is of point countable type, there is a compact subspace $F \subset X$ such that $x \in F$ and the character of the set F in X is countable [13]. If x is isolated in the space F, then there is a countable base of X at x, and hence, there is a sequence in $X \setminus \{x\}$, converging to x. It remains to consider the case, when x is not isolated in F. Let us show that then there is a discrete subspace A of F such that x is a limit point for A. Assume the contrary. Then the space $Y = F \setminus \{x\}$ is strongly discretely compact and not compact—in contradiction with Proposition 11. □

Example 5. Take a non-empty Tychonoff space Z without isolated points such that each discrete subspace of Z is closed in Z. For example, any non-empty nodec space or submaximal space without isolated points would do (see [28]). Fix $a \in Z$, and let $X = \beta Z \setminus \{a\}$, $Y = Z \setminus \{a\}$, where βZ is the Stone-Čech compactification of Z. Then X is a Tychonoff non-compact space, and Y is dense in X. Therefore, Y is not compact in X [24]. The closure in X of any discrete subspace B of Y is compact, since the closure of B in βZ does not contain the point a. It follows that Y is strongly discretely compact in X. We see that the relative versions of Proposition 11 and Theorem 5 are not true.

9. To understand better discrete Lindelöfness, it could also be worthwhile to have a look at the following weaker condition:

(τ) For each discrete subspace A of X and each family γ of open subsets of X such that $|\gamma| \leq \tau$ and $\bar{A} \subset \cup\gamma$, there is a countable subfamily η of γ such that $A \subset \overline{\cup\eta}$.

In addition to discretely Lindelöf spaces, all quasi-Lindelöf spaces satisfy condition (τ). Therefore, it is natural to say that X *is discretely quasi-τ-Lindelöf* if X satisfies condition (τ). We omit τ if the condition is satisfied for all τ.

Problem 12. Is every discretely Lindelöf regular space quasi-Lindelöf?

Problem 13. Is it true that the cardinality of every regular, first countable, discretely quasi-Lindelöf space is not greater than 2^ω?

Let us show that under the continuum hypothesis (CH) the answer to the last question is positive.

Lemma 5. *Let P be a closed subset of X, and let $\gamma = \{U_\alpha : \alpha < \tau\}$ be a family of open subsets of X such that $P \subset \cup\gamma$, where τ is a not countably cofinal cardinal, and let X be discretely quasi-τ-Lindelöf. Then there is a subfamily ξ of γ such that $|\xi| < \tau$ and $P \subset \overline{\cup\xi}$.*

Proof: Put $W_\alpha = \overline{\cup\{U_\beta : \beta < \alpha\}}$, and $V_\alpha = P \cap U_\alpha \setminus W_\alpha$, for each $\alpha < \tau$, and let $B = \{\alpha : V_\alpha \neq \emptyset\}$. Fix $x_\alpha \in V_\alpha$ for each $\alpha \in B$. Then the set $A = \{x_\alpha : \alpha \in B\}$ is discrete, since the sets V_α are open in P and disjoint. There is a countable subfamily η of γ such that $A \subset \overline{\cup\eta}$. Since τ is not countably cofinal, there is $\alpha < \tau$ such that $\eta \subset \{U_\beta : \beta < \alpha\}$. Then $A \subset W_\alpha$. Since V_β and W_α are disjoint for $\alpha < \beta$, it follows that $V_\beta \cap A = \emptyset$, for $\alpha < \beta$, which implies that $V_\beta = \emptyset$, for $\alpha < \beta$. Therefore, $P \subset W_\alpha$. □

From Lemma 5 we immediately get the next result:

Proposition 12. *If X is a discretely quasi-ω_1-Lindelöf space, then X is quasi-ω_1-Lindelöf.*

From the absolute version of Corollary 6 and Proposition 12 we now have:

Theorem 6. *If (CH) holds, and X is a regular first countable discretely quasi-Lindelöf space, then $|X| \leq 2^\omega$.*

With the help of a result similar to Lemma 5, we can also prove a partial result in the direction of Problems 12 and 8. Let us say that X is *strongly discretely Lindelöf*, if the closure of every discrete subspace of X is a Lindelöf space. Observe that it is no longer true, that every space of the countable spread is strongly discretely Lindelöf.

Problem 14. Is every regular strongly discretely Lindelöf space Lindelöf?

To present a partial answer to this question, we need the next lemma.

Lemma 6. *Let X be a strongly discretely Lindelöf space, and let $\gamma = \{U_\alpha : \alpha < \tau\}$ be an open covering of X, where the cardinal τ is not countably cofinal. Then there is a subcovering $\eta \subset \gamma$ such that $|\eta| < \tau$.*

PROOF: Put $W_\alpha = \cup\{U_\beta : \beta < \alpha\}$, for $\alpha < \tau$. Let us define x_α for each $\alpha < \tau$ in the following way. First, fix an object d not in X. If $U_0 = \emptyset$, we put $x_0 = d$. If U_0 is not empty, take x_0 to be any point of U_0. If $\alpha < \tau$, and the points x_β are already defined for all $\beta < \alpha$, let x_α to be any point of the set $V_\alpha = U_\alpha \setminus (F_\alpha \cup (\cup\gamma_\alpha))$, where $F_\alpha = \overline{\{x_\beta : \beta < \alpha, x_\beta \neq d\}}$ and $\gamma_\alpha = \{U_\beta : \beta < \alpha\}$, if the set V_α is not empty, otherwise put $x_\alpha = d$. Clearly, $A = \{x_\alpha : \alpha < \tau, x_\alpha \neq d\}$ is a discrete subspace of X. Therefore, there is a countable subfamily η of γ such that $\bar{A} \subset \cup\eta$. Since τ is not of the countable cofinality, there is $\beta < \tau$ such that $\eta \subset \gamma_\beta$. Then x_α is not in A for $\beta < \alpha$; therefore, $x_\alpha = d$ whenever $\beta < \alpha$. This can happen only if $X = (\cup\gamma_\beta) \cup F_\beta$. Since $F_\beta \subset \bar{A} \subset \cup\eta \subset \cup\gamma_\beta$, it follows, that $\cup\gamma_\beta = X$. We have: $|\gamma_\beta| < \tau$ and $\gamma_\beta \subset \gamma$. The proof is complete. □

This lemma permits to prove easily the next result, providing a partial answer to Problem 14.

Theorem 7. *If X is countably paracompact and strongly discretely Lindelöf, then X is Lindelöf.*

PROOF: Assume the contrary. Let τ be the smallest cardinal number such that there are a countably paracompact, strongly discretely Lindelöf space X and an open covering γ of X of cardinality τ such that no countable subfamily of γ covers X. Since X is countably paracompact, the cofinality of τ is not countable (by an obvious standard argument in which we also take into account the choice of τ). Now, applying Lemma 6 and again referring to the choice of τ, we arrive at a contradiction. □

We could relativize the notions and theorems proved in this subsection, but since the absolute results are, it seems, not yet in the final form, we do not do that.

10. Going through the proof of Corollary 17, we come very close to the following notion. Let $\mathcal{P}$ be a family of subsets of X. Let us say that X is *Lindelöf with respect to* $\mathcal{P}$, if for every closed subset P of X, and for each subfamily γ of $\mathcal{P}$ covering P, there is a countable subfamily η of γ such that $P \subset \cup\eta$. We call a space X *pseudobase-Lindelöf*, if there is a pseudobase $\mathcal{P}$ of X such that X is Lindelöf with respect to $\mathcal{P}$. We say that X is *strongly pseudobase-Lindelöf*, if there is a pseudobase $\mathcal{P}$ in X such that for each $\gamma \subset \mathcal{P}$ there is a countable subfamily η of γ such that $\cup\eta = \cup\gamma$. The next assertion is obvious.

Proposition 13. *If there is a one-to-one continuous mapping of X onto a hereditarily Lindelöf space, then X is strongly pseudobase-Lindelöf.*

Problem 15. Is there a "naive" example of a regular pseudobase-Lindelöf space X of the countable pseudocharacter such that $2^\omega < |X|$?

11. Since there are so many formulations to grasp in this paper, we have restricted ourselves with the countable case, which is more open for the intuition. But the proofs presented are obviously valid in the general case as well. In this way, we arrive at the next theorems:

A. *For a Hausdorff space X and its subspace Y, $|Y| \leq 2^{\chi(X)l(Y,X)}$.*

B. *If Y is regular in X, then $w(Y) \leq (\pi\chi(Y,X))^{c(Y,X)}$, where $\pi\chi(Y,X)$ is the smallest cardinal number ν such that $\pi\chi(y,X) \leq \nu$, for each $y \in Y$.*

This is a relative version of a result of Shapirovskij [26] (see also [20]).

C. $|Y| \leq 2^{e(Y,X)\Delta(Y,X)}$.

We denote by $ql(Y,X)$ the smallest cardinal number τ such that for every subset A of Y closed in Y, and for every family γ of open subsets of X such that $A \subset \cup\gamma$, there is a subfamily η of γ such that $|\eta| \leq \tau$ and $A \subset \overline{\cup\eta}$. Now we can present the general version of Corollary 6:

D. *If Y is regular in X, $\chi(y,X) \leq \tau$ for each $y \in Y$, and $ql(Y,X) \leq \tau$, then $|Y| \leq 2^\tau$.*

To provide a general version of Corollary 8, we denote by $sl(Y,X)$ the smallest cardinal number τ such that for every subset A of Y and every family γ of open sets in X such that $\bar{A} \subset \cup\gamma$, there is a subfamily η of γ such that $A \subset \overline{\cup\eta}$ and $|\eta| \leq \tau$.

E. *If X is a Hausdorff space, and Y is regular in X and dense in X, then $|X| \leq 2^{\chi(X)sl(Y,X)}$.*

Corollary 20 generalizes as follows:

F. $|Y| \leq 2^{e(Y,X)psw(Y,X)}$.

12. Of course, it is not true that all important cardinal inequalities can be proved just following the algorithm described in Theorem 1. For example, the author does not know such a proof for R. Hodel's inequality $|X| \leq (e(X))^{psw(X)}$ (see [21]), or for Gryzlov's theorem (see [17]).

13. In conclusion, we would like to mention that a version of the addition theorem for weight in relatively compact Hausdorff spaces was proved in [7] by V.A. Arhangel'skii; as in the classical case, it is based on a relative version of the theorem on the equality of the networkweight to the weight for compact Hausdorff spaces (see [2]).

14. Corollary 6 does not fully unify Corollaries 1 and 3, since we had to assume regularity in Corollary 6. To remedy for this, we slightly modify the notion of a quasi-Lindelöf space, strengthening (or narrowing) it in such a way that it still embraces the class of Lindelöf spaces and the class of spaces with the countable Souslin number, which allows us to prove a Hausdorff version of Corollary 6. For the sake of simplicity, we treat only the absolutely countable case.

Let us call a space X *strictly quasi-Lindelöf*, if for every closed subset P of X and every countable family $\{\gamma_i : i \in \omega\}$ of families of open subsets of X such that $P \subset \cup\{\cup\gamma_i : i \in \omega\}$ one can choose a countable subfamily η_i of γ_i for each $i \in \omega$ so that $P \subset \cup\{\overline{\cup\eta_i} : i \in \omega\}$. The next assertion is obvious:

Proposition 14. *If X is Lindelöf then X is strictly quasi-Lindelöf.*

The next simple result is a Corollary of Lemma 2.

Proposition 15. *If the Souslin number of X is countable, then X is strictly quasi-Lindelöf.*

Now comes the main result of this section:

Corollary 22. *Let X be a first countable Hausdorff strictly quasi-Lindelöf space. Then $|X| \leq 2^\omega$.*

PROOF: We put $\tau = \aleph_1 = \lambda$, $\mu = 2^\omega$, and $g(F) = \cup\{\mathcal{B}_x : x \in \bar{F}\}$. There is a propeller $\xi = \{F_\alpha : \alpha < \aleph_1\}$ in $\mathcal{L}$. It is enough to show that the set $P = \cup\xi$ is dense in X, since $|P| \leq 2^\omega$ and X is first countable and Hausdorff.

Assume the contrary, and fix $z \in X \setminus \bar{P}$. For $V \in \mathcal{B}_z$ we put $\gamma_V = \{U \in \mathcal{U}_g(\xi) : U \cap V = \emptyset\}$. Since X is first countable, $\bar{P} = \cup\{\overline{F_\alpha} : \alpha < \aleph_1\}$. Therefore, $\mathcal{U}_g(\xi) = \cup\{\mathcal{B}_x : x \in \bar{P}\}$. Then $\bar{P} \subset \cup\{\cup\gamma_V : V \in \mathcal{B}_z\}$, since X is Hausdorff. Note that the family $\mathcal{B}_z$ is countable. Therefore, we can choose a countable subfamily η_V of γ_V for each V in $\mathcal{B}_z$ so that

$$\bar{P} \subset \cup\{\overline{\cup\eta_V} : V \in \mathcal{B}_z\}.$$

Put

$$s = (\emptyset, \{\eta_V : V \in \mathcal{B}_z\}).$$

Let us show that s is good for ξ. It is obvious that s is generated by $(\cup\xi, \mathcal{U}_g(\xi))$, and that all families entering s are countable. From $(\cup\gamma_V) \cap V = \emptyset$ it follows that $(\cup\eta_V) \cap V = \emptyset$; since V is open, this implies that $\overline{\cup\eta_V} \cap V = \emptyset$ for each $V \in \mathcal{B}_z$. Therefore, z does not belong to $\Theta(s)$, and s is a small sensor good for ξ,—a contradiction which completes the proof. □

15. After this paper was submitted to the journal, I have discovered that Proposition 11 in it is Lemma 4.13 of the following article: **V.V. Tkachuk**, *Spaces that are projective with respect to classes of mappings*, Trudy Moskov. Matem. Ob-va **50** (1987), 139–156.

Note that in the abstract to the paper above Tkachuk made a claim that it contains a proof that the answer to Problem 14 of this my paper is in positive. In fact, Tkachuk's paper does not contain such a proof, and he does not know such a proof at present—he acknowledged that in an e-mail letter to me.

16. A systematic study of relative topological properties began in 1989 (see A.V. Arhangel'skii and H.M.M. Genedi, "Beginnings of the Theory of Relative Topological Properties", p. 3–48, in: General Topology. Spaces and Mappings.– MGU, Moscow, 1989). At that time it was already clear that relative versions of many well known theorems on cardinal invariants remain true with practically the same proofs. I made several announcements about that at seminars in Moscow University and published a short notice to that end in Vestnik MGU, 1994 (ser.

mat. mech.). Meanwhile some relative versions of theorems on cardinal inequalities, similar or close to some of those which are discussed in the present article, were published by A.A. Gryzlov and D.N. Stavrova. See in particular A.A. Gryzlov, D.N. Stavrova, *Topological spaces with a selected subset-cardinal invariants and inequalities*, in Comptes rendus Acad. bulgare sci, 46:7, (1993), 17–19, and, under the same title, in Comment. Math. Univ. Carolinae, 35:3 (1994), 525–531. When this my article was already accepted, I received two preprints of further papers by Stavrova on relative cardinal invariants, yet to appear: "Upper bounds for cardinality of topological spaces with selected subset", and "Hausdorff pseudocharacter and cardinality of topological spaces with a selected subset".

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