Measures of compactness in approach spaces

R. Baekeland, R. Lowen

Abstract. We investigate whether in the setting of approach spaces there exist measures of relative compactness, (relative) sequential compactness and (relative) countable compactness in the same vein as Kuratowski's measure of compactness. The answer is yes. Not only can we prove that such measures exist, but we can give usable formulas for them and we can prove that they behave nicely with respect to each other in the same way as the classical notions.

Keywords: approach space, compactness
Classification: 54E99, 54A05

1. Preliminaries

1.1 Approach spaces and extended pseudo-quasi-metric spaces

We shall use the following symbols $\mathbf{R}_+ := [0,\infty[$, $\mathbf{R}_+^* :=]0,\infty[$ and $\bar{\mathbf{R}}_+ := [0,\infty]$. If $A \subset X$ then Θ_A stands for the function $X \longrightarrow \bar{\mathbf{R}}_+$ taking the value 0 in points of A and ∞ elsewhere. We put an $\overset{=}{\uparrow}$ (respectively $\uparrow$) for an increasing (respectively a strictly increasing) function, system, sequence or whatever. We shall also use the symbols $\downarrow$ and $\overset{=}{\downarrow}$ respectively for strict decreasing respectively decreasing functions, system, sequence or whatever.

We shall recall some definitions from [9] and [8]. An *extended pseudo-quasi-metric* (shortly extended p-q-metric space) is a pair (X,d) where $d : X \times X \longrightarrow \bar{\mathbf{R}}_+$ fulfils

(M1) $\{d = 0\} \supset \triangle_X := \{(x,x) \mid x \in X\}$.
(M2) d fulfils the triangle inequality.

The map d is then called an *extended pseudo-quasi-metric* (shortly extended p-q-metric). Other properties d may fulfil are:

(M3) d is symmetric.
(M4) $\{d = 0\} \subset \triangle_X$.
(M5) d is finite.

If d fulfils also (M3) we drop "quasi-" ("q-"), if it fulfils (M4) we drop "pseudo-" ("p-") and if it fulfils (M5) we drop "extended".

If $A \in X$ then $d(A) := \sup\{d(a,b) \mid a,b \in A\}$ stands for the *diameter* of A.

A map $\delta : X \times 2^X \longrightarrow \bar{\mathbf{R}}_+$ is called a *distance* if it fulfils

(D1) $\forall\, A \in 2^X, \forall\, x \in X : x \in A \Rightarrow \delta(x,A) = 0$.

(D2) $\forall\, x \in X : \delta(x,\emptyset) = \infty$.
(D3) $\forall\, A, B \in 2^X, \forall\, x \in X : \delta(x,A) \wedge \delta(x,B) = \delta(x, A \cup B)$.
(D4) $\forall\, A \in 2^X, \forall\, x \in X, \forall\, \varepsilon \in \bar{\mathbf{R}}_+ : \delta(x,A) \leq \delta(x, A^{(\varepsilon)}) + \varepsilon$ where $A^{(\varepsilon)} := \{x \mid \delta(x,A) \leq \varepsilon\}$.

A collection $(\Phi(x))_{x\in X}$ of ideals in $\bar{\mathbf{R}}_+^X$ is called an *approach system* if it fulfils

(A1) $\forall\, x \in X, \forall\, \phi \in \Phi(x) : \phi(x) = 0$.
(A2) $\forall\, x \in X, \forall\, \phi \in \bar{\mathbf{R}}_+^X : \forall\, \varepsilon, N \in \mathbf{R}_+^*, \exists\, \phi_\varepsilon^N \in \Phi(x)$:

$$\phi_\varepsilon^N + \varepsilon \geq \phi \wedge N \Rightarrow \phi \in \Phi(x).$$

(A3) $\forall\, x \in X, \forall\, \phi \in \Phi(x), \forall\, N \in \mathbf{R}_+^*, \exists\, \phi' \in \prod_{x\in X} \Phi(x), \forall\, z, y \in X$:

$$\phi'(x)(z) + \phi'(z)(y) \geq \phi(y) \wedge N.$$

We shall denote an approach system by $(\Phi(x))_{x\in X}$ or shortly Φ if no confusion is possible.

If Φ is an approach system then $\Lambda := (\Lambda(x))_{x\in X}$ is called a *basis* or *base* for Φ if it fulfils the properties:

(B1) $\forall\, x \in X : \Lambda(x)$ is a basis for an ideal.
(B2) $\forall\, x \in X : \Phi(x) = \hat{\Lambda}(x)$ where:

$$\hat{\Lambda}(x) := \{\phi \mid \forall\, \varepsilon, N \in \mathbf{R}_+^*, \exists\, \psi \in \Lambda(x) : \psi + \varepsilon \geq \phi \wedge N\}.$$

Further [8] if Φ is an approach system on X then the map

$$\delta_\Phi : X \times 2^X \longrightarrow \bar{\mathbf{R}}_+ : (x,A) \longrightarrow \sup_{\phi\in\Phi(x)} \inf_{a\in A} \phi(a)$$

is a distance on X. From a distance δ on X we can construct the approach system Φ_δ defined by:

$$\Phi_\delta(x) := \{\phi \mid \forall\, A \subset X : \inf_{a\in A} \phi(a) \leq \delta(a,A)\} \tag{1}$$

for all $x \in X$. Further we have $\Phi_{\delta_\Phi} = \Phi$ and $\delta_{\Phi_\delta} = \delta$. A space with an approach system or a distance is called an approach space.

Let X be an approach space with a distance δ and an approach system $(\Phi(x))_{x\in X}$. Then for each $A \subset X$ and for each $N > 0$ we consider the p-q-metric $d_A^N : X \times X : (x,y) \to (\delta(x,A) \wedge N - \delta(y,A) \wedge N) \vee 0$. Further let $\mathcal{D} := \{d_A^N \mid A \subset X, N > 0\}$ and $\mathcal{D}(x) := \{\sup_{j\in J} d_{A_j}^{N_j}(x,.) \mid A_j \subset X, N_j > 0, j \in J \text{ finite}\}$. Then this set $\mathcal{D}$ will determine our approach system. First we need a lemma:

Lemma 1.1. *For an approach space (X, δ) we have that for all $x \in X$ and $A, B \subset X$:*

$$\delta(x, A) \leq \delta(x, B) + \sup_{b \in B} \delta(b, A).$$

Proof: Consider $N > 0$. Then

$$\begin{aligned}
\delta(x, A) \wedge N &= \sup_{\phi \in \Phi(x)} \inf_{a \in A} \phi(a) \wedge N \\
&\leq \inf_{a \in A} (\phi_0(a) + \varepsilon/2) \wedge N \qquad \text{for a certain } \phi_0 \in \Phi(x) \\
&\leq \inf_{b \in B} \inf_{a \in A} \phi'(x)(b) + \phi'(b)(a) + \varepsilon \ \{\text{use (A3) from above}\} \\
&\leq \inf_{b \in B} \phi'(x)(b) + \sup_{b \in B} \inf_{a \in A} \phi'(b)(a) + \varepsilon \\
&\leq \delta(x, B) + \sup_{b \in B} \delta(b, A) + \varepsilon.
\end{aligned}$$

And this for every $N > 0$ and every $\varepsilon > 0$ proving the lemma. □

If Λ is a basis for the approach space (X, Φ) then:

$$\delta_\Phi(x, A) := \sup_{\Psi \in \Lambda(x)} \inf_{a \in A} \Psi(a).$$

Proposition 1.2. *For each $x \in X$, $\Phi(x)$ is generated by $\mathcal{D}(x)$.*

Proof: For all $A, B \subset X$, $x \in X$, $N > 0$ we have:

$$\inf_{b \in B} d_A^N(x, b) = (\inf_{b \in B} \delta(x, A) \wedge N - \delta(b, A) \wedge N) \vee 0 \tag{2}$$

$$\leq \inf_{b \in B} ((\sup_{b \in B} \delta(b, A) + \delta(x, B)) \wedge N - \delta(b, A) \wedge N) \vee 0 \tag{3}$$

$$\leq \delta(x, B). \tag{4}$$

So by expression (1) we know that $d_A^N(x, .) \in \Phi(x)$. It is easy to check that for all $A \subset X$, $x \in X$ we have $\delta(x, A) = \sup_{\phi \in \mathcal{D}(x)} \inf_{a \in A} \phi(a)$. Because δ determines completely the approach system, $\mathcal{D}(x)$ shall generate $\Phi(x)$. □

Proposition 1.3. *If we have a set $\mathcal{D}$ of p-q-metrics on X stable for finite suprema and we define:*

$$\Lambda_{\mathcal{D}}(x) := \{d(x, .) \mid d \in \mathcal{D}\},$$

then $(\Lambda_{\mathcal{D}}(x))_{x \in X}$ is a base for an approach system on X.

Proof: Verify conditions (A1), (A2) and (A3) for $\hat{\Lambda}_{\mathcal{D}}(x)$. □

If (X, Φ) and (X', Φ') are approach spaces than a function $f : X \longrightarrow X'$ is called a contraction if it fulfils any of the following equivalent conditions [8]:

(C1) $\forall x \in X, \forall \phi' \in \Phi'(f(x)) : \phi' \circ f \in \Phi(x)$.
(C2) For any basis Λ' for Φ': $\forall x \in X, \forall \psi' \in \Lambda'(f(x)) : \psi' \circ f \in \Phi(x)$
(C3) $\forall x \in X, \forall A \subset X : \delta'(f(x), f(A)) \leq \delta(x, A)$.

Approach spaces and contractions form a topological category [8] denoted AP. TOP is bireflectively and bicoreflectively embedded in AP by:

$$(X,\mathbf{T}) \xrightarrow{id} (X, A_t(\mathbf{T})),$$

where the approach system of $A_t(\mathbf{T})$ is $\Phi_{\mathbf{T}}(x) := \{\nu \mid \nu(x) = 0, \text{ u.s.c. at } x\}$ for all $x \in X$. The associated distance is given by $\delta_{\mathbf{T}}(x, A) = 0$ iff $x \in \bar{A}$ and $\delta_{\mathbf{T}}(x, A) = \infty$ iff $x \notin \bar{A}$ for all $x \in X$, $A \subset X$. Approach spaces for which $\delta(X \times 2^X) = \{0, \infty\}$ are topological [8]. Given $(X, \Phi) \in |AP|$ its TOP-coreflection is given by:

$$(X, \mathbf{T}^*(\Phi)) \xrightarrow{id} (X, \Phi),$$

where $\mathbf{T}^*(\Phi)$ is the topology determined by the neighborhoodsystem:

$$N^*(\Phi)(x) := \{\{\nu < \varepsilon\} \mid \nu \in \Phi(x),\ \varepsilon \in \mathbf{R}^*_+,\ x \in X\}.$$

$\mathbf{T}^*$ is left inverse, right adjoint to A_t. The TOP-reflection is given by:

$$(X, \Phi) \longrightarrow (X, \mathbf{T}_*(\Phi)),$$

where $\mathbf{T}_*(\Phi)$ is the topological modification of the pretopology determined by the neighborhoodsystem:

$$N_*(\Phi)(x) := \langle\{\{\nu < \infty\} \mid \nu \in \phi(x)\}\rangle$$

for all $x \in X$.
Analogously p-q-MET$^\infty$ is bicoreflectively embedded in AP by:

$$\begin{aligned} \text{p-q-MET}^\infty &\xrightarrow{A_m} AP \\ (X, d) &\longrightarrow (X, A_m(d)), \end{aligned}$$

where $A_m(d)$ is determined by the approach system $(\Phi_d(x))_{x\in X}$ with $\Phi_d(x) := \{\nu \mid \nu \le d(x, .)\}$ for all $x \in X$. In this case the associated distance is given by $\delta_d(x, A) = \inf_{a\in A} d(x, a)$ for all $x \in X$, $A \subset X$. Given the approach space X with approach system Φ its p-q-MET$^\infty$-coreflection is given by:

$$(X, M(\Phi)) \xrightarrow{id_X} (x, \Phi),$$

where $M(\Phi)$ is the ∞-p-q-metric defined by $M(\Phi)(x, y) := \delta_\Phi(x, \{y\})$. M is of course left inverse, right adjoint to A_m.

1.2 Filters and nets in approach spaces

If $X \in |SET|$ then we shall denote the set of all finite (respectively countable) subsets of X by $2^{(X)}$ (respectively $2^{((X))}$). If $\mathcal{F}$ is a filter on X then we put $F(\mathcal{F})$ (respectively $U(\mathcal{F})$) for the set of all filters (respectively ultrafilters) finer than $\mathcal{F}$.

If $\mathcal{F} = \{B \subset X \mid A \subset B\}$ then we put $F(A)$ and $U(A)$ rather than $F(\mathcal{F})$ and $U(\mathcal{F})$. So we can put $F(X)$ for the set of all filters on X. We remark that there is a one-to-one correspondence between (ultra)filters on A and (ultra)filters in $F(A)$.

We put $n(X)$ (respectively $r(X)$) for the set of all nets (respectively all sequences) on X. Let us further recall that a filter $\mathcal{F}$ on X determines a net and vice versa. Indeed if $\mathcal{F} \in F(X)$ then $\Gamma_{\mathcal{F}} := \{(x, F) \mid x \in F \in \mathcal{F}\}$ is a directed set [15] by the relation $(x_1, F_1) \leq (x_2, F_2)$ iff $F_2 \subset F_1$, so the map $P : \Gamma_{\mathcal{F}} \longrightarrow X$ defined by $P(x, F) = x$ is a net on X. Conversely if $P : \Gamma \longrightarrow X$ is a net then the sets $B_{\kappa_0} = \{x_\kappa \mid \kappa \geq \kappa_0\}$ with $\kappa_0 \in \Gamma$ generate a filter base $\mathcal{B}_\kappa$ which leads to a filter $\mathcal{F}_\kappa$. If $P : \Gamma \longrightarrow X$ is a net we shall denote this shortly as the net $(x_\kappa)_{\kappa\in\Gamma}$ in X where it is taken for granted that Γ is a directed set.

A net $(x_\kappa)_{\kappa\in\Gamma}$ is called an ultranet if for each $E \subset X$ there exists a $\kappa_0 \in \Gamma$ such that $\{x_\kappa \mid \kappa \geq \kappa_0\} \subset E$ or $\{x_\kappa \mid \kappa \geq \kappa_0\} \subset X \setminus E$. It is well-known (see e.g. [15]) that for each ultranet the corresponding filter is an ultrafilter and vice versa.

We shall denote the set of all ultranets on X as $u(X)$.
In topology we have: a net converges to a point x iff the corresponding filter converges to x. In approach spaces we can generalize this if we introduce the following definition:

Definition 1.4. *For $(X, \Phi) \in |AP|$, $(x_\kappa)_{\kappa\in\Gamma} \in n(X)$ we define:*

$$\lambda_{net}(x_\kappa \to x) = \sup_{\phi\in\Phi(x)} \limsup_{\kappa} \phi(x_\kappa)$$
$$\alpha_{net}(x_\kappa \to x) = \sup_{\phi\in\Phi(x)} \liminf_{\kappa} \phi(x_\kappa)$$

For $\mathcal{F} \in F(X)$ the limit [9] is defined as:

$$\lambda(\mathcal{F})(x) = \sup_{\phi\in\Phi(x)} \inf_{F\in\mathcal{F}} \sup_{f\in F} \phi(f)$$

and the adherence [9] as:

$$\alpha(\mathcal{F})(x) = \sup_{\phi\in\Phi(x)} \sup_{F\in\mathcal{F}} \inf_{f\in F} \phi(f).$$

The following proposition is easy to verify.

Proposition 1.5. *Let $\mathcal{F} \in F(X)$ and $x \in X$. If we note $\{x_{\kappa_\mathcal{F}} \mid x_{\kappa_\mathcal{F}} \in \Gamma_\mathcal{F}\}$ for the corresponding net then we have:*

$$\lambda_{net}(x_{\kappa_\mathcal{F}} \to x) = \lambda(\mathcal{F})(x)$$
$$\alpha_{net}(x_{\kappa_\mathcal{F}} \to x) = \alpha(\mathcal{F})(x)$$

Further if $(x_\kappa)_{\kappa \in \Gamma}$ is a net then for the corresponding filter $\mathcal{F}_\kappa$:

$$\lambda(\mathcal{F}_\kappa)(x) = \lambda_{net}(x_\kappa \to x)$$
$$\alpha(\mathcal{F}_\kappa)(x) = \alpha_{net}(x_\kappa \to x)$$

□

Because of the above relationships between limits and adherences for nets $(\lambda_{net}, \alpha_{net})$ and filters (λ, α) we shall make no difference between them.
A basic result concerning convergence in approach spaces is:

Proposition 1.6 ([9])**.** *For $\mathcal{F}, \mathcal{G} \in F(X)$ we have:*

$$\alpha\mathcal{F} \leq \lambda\mathcal{F}$$
$$\mathcal{F} \subset \mathcal{G} \Rightarrow \alpha\mathcal{F} \leq \alpha\mathcal{G}$$
$$\mathcal{F} \subset \mathcal{G} \Rightarrow \lambda\mathcal{G} \leq \lambda\mathcal{F}.$$

From this and Proposition 1.5 we immediately deduce:

Corollary 1.7. *For $(x_\kappa)_{\kappa \in \Delta}$, $(y_\gamma)_{\gamma \in \Gamma} \in n(X)$ and $z \in X$ we have:*

$$\alpha(x_\kappa \to z) \leq \lambda(x_\kappa \to z).$$

If $(x_\kappa)_{\kappa \in \Delta}$ is a subnet of $(y_\gamma)_{\gamma \in \Gamma}$:

$$\alpha(y_\gamma \to z) \leq \alpha(x_\kappa \to z)$$
$$\lambda(y_\gamma \to z) \geq \lambda(x_\kappa \to z).$$

2. Approach spaces and the first countability criterium

It is well-known that in a topological space sequential compactness and countably compactness coincide for first countable spaces. In this section we introduce the concept of first countability in approach spaces, which shall e.g. be used to study the measure of sequential compactness and countably compactness in approach spaces.

Definition 2.1. *An approach space $(X, (\Phi(x))_{x \in X})$ shall be called first countable if we can find a countable basis for each of the ideals of local distances.*

A good definition of first countability (A1) will give us for topological approach spaces the topological definition of first countability.

Theorem 2.2. *A topological space X is first countable in TOP iff X is first countable in AP. Further a ∞-p-q-metric space is always first countable.*

Proof: If $\mathcal{T}$ is a topology on X and $\Phi_{\mathcal{T}}$ has a countable basis $\Lambda(x)$ then $\{\{\psi < 1/n\} \mid \psi \in \Lambda(x),\, n \in \mathbf{N}\}$ is a countable base for the neighborhoodsystem in x. If $\{V_n \mid n \in \mathbf{N}\}$ a countable base in x then $\{\Theta_{V_n} \mid n \in \mathbf{N}\}$ is a countable base for $\Phi_{\mathcal{T}}$. For a p-q-MET$^\infty$ space it is clear that $\Lambda(x) := \{d(x,.)\}$ is a countable basis for $\Phi_d(x)$. $\square$

It is well-known that for first countable topological spaces a map is continuous iff each converging sequence has a converging image sequence [15]. For approach spaces we have the following analogue.

Proposition 2.3. *For $X, X' \in |AP|$ with X first countable and $f : X \longrightarrow X'$ the following are equivalent:*

(a) *f is a contraction.*
(b) $\forall\, (x_n)_{n\in\mathbf{N}} \in r(X) : \lambda'(f(x_n) \to f(x)) \leq \lambda(x_n \to x)$.
(c) $\forall\, (x_n)_{n\in\mathbf{N}} \in r(X) : \alpha'(f(x_n) \to f(x)) \leq \alpha(x_n \to x)$.

Proof: The implications (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c) follow easily from Proposition 6.1 in [9] and Proposition 1.5. We now prove (b) $\Rightarrow$ (a): Suppose that f is not a contraction then we will show that there exist $x \in X$, $A \subset X$ and $\varepsilon > 0$ such that:

$$\epsilon + \delta_X(x, A) < \delta_{X'}(f(x), f(A)).$$

Now consider a basis $\Lambda(x) := \{\phi_n \mid n \in \mathbf{N}\} \uparrow$ for $\Phi(x)$. From Proposition 2.13 in [8] we know that

$$\delta(x, A) = \sup_{\phi\in\Lambda(x)} \inf_{a\in A} \phi(a).$$

And thus for all $n \in \mathbf{N}$ there is an $a_n \in A$ such that:

$$\phi_n(a_n) < \delta(x, A) + 1/n.$$

Furthermore for some $l \in \mathbf{N}$:

$$\begin{aligned}
\lambda(a_n \to x) &\leq \inf_{n\in\mathbf{N}} \sup_{m\geq n} \phi_l(a_m) + \varepsilon/2 \\
&\leq \inf_{n\geq l} \sup_{m\geq n} \phi_m(a_m) + \varepsilon/2 \\
&\leq \inf_{n\geq l} \sup_{m\geq n} [\delta(x, A) + 1/n] + \varepsilon/2 \\
&\leq \delta(x, A) + \varepsilon/2 \\
&< \delta_{X'}(f(x), f(A)) - \varepsilon/2 \\
&= \sup_{\phi'\in\Phi'(f(x))} \inf_{a\in A} \phi'(f(a)) - \varepsilon/2 \\
&\leq \inf_{a\in A} \phi'_0(f(a)),
\end{aligned}$$

for a certain $\phi'_0 \in \Phi'(f(x))$. And thus:

$$\lambda(a_n \to x) < \inf_{n\in\mathbf{N}} \sup_{m\geq n} \phi'_0(f(a_m))$$

and

$$\lambda(a_n \to x) < \sup_{\phi'\in\Phi'(f(x))} \inf_{n\in\mathbf{N}} \sup_{m\geq n} \phi'(f(a_m)) = \lambda'(f(a_n) \to f(x)).$$

To prove (c) $\Rightarrow$ (a): We suppose that f is not a contraction and then we can show with similar arguments that condition (c) cannot be fulfilled. $\square$

From [6, Theorems 3.1 and 3.2] we know that the operators λ and δ determine each other completely. Because of the foregoing property it should not surprise us that the distance δ in an A1 space is also completely determined by sequences.

Proposition 2.4. *For a first countable approach space X we have*:

$$\delta(x, A) = \inf_{(y_n)_{n\in\mathbf{N}}\in r(A)} \lambda(y_n \to x).$$

Proof: One inequality follows from the theorems mentioned above in [6] and is true for all approach spaces. To prove the other one let $(\phi_n)_{n\in\mathbf{N}}\overset{=}{\uparrow}$ be an approach base in x, then it is easy to see that for every $n \in \mathbf{N}$ and for every $\varepsilon > 0$ there exists an $y_n \in A$ such that $\delta(x, A) \geq \phi_n(y_n) - \varepsilon$ for all $m \geq n$. Since then $\phi_m(y_m) \geq \phi_n(y_m)$ we have for all $l \in \mathbf{N}$:

$$\delta(x, A) \geq \inf_{n\in\mathbf{N}} \sup_{m\geq n} \phi_l(y_m) - \varepsilon.$$

And thus:

$$\begin{aligned}\delta(x, A) &\geq \sup_{l\in\mathbf{N}} \inf_{n\in\mathbf{N}} \sup_{m\geq n} \phi_l(y_m) - \varepsilon \\ &= \lambda(y_n \to x) - \varepsilon.\end{aligned}$$

$\square$

Corollary 2.5. *In X a first countable topological space, $x \in \bar{A}$ iff there exists a sequence in A converging to x.* $\square$

The following result is also a useful generalization of a well-known topological fact [15].

Theorem 2.6. *A product of approach spaces is first countable iff all factors are first countable and all but a countable number are indiscrete.*

Proof: Suppose $\mathcal{B} \subset 2^{((\mathcal{A}))}$ and suppose that for $X_\alpha \in \mathcal{A} \setminus \mathcal{B} : X_\alpha$ is indiscrete. For every $\alpha \in \mathcal{B}$ consider a countable base $\Lambda_\alpha(x_\alpha)$ in x_α of the approach system $\Phi\alpha$ of X_α and for $x = (x_\alpha)_{\alpha\in\mathcal{A}}$ define:

$$\Lambda(x) = \{\sup_{\alpha\in K} \phi_j(pr_\alpha(.)) \mid K \in 2^{(\mathcal{B})},\ \phi_j \in \Lambda_\alpha(x_\alpha)\}.$$

From the definition of product spaces in $|AP|$ and the definition of an indiscrete space it follows easily that $\Lambda(x)$ is a base for $\Phi(x)$ (the approach system of x in $\prod_{\alpha\in\mathcal{A}} X_\alpha$) and since each Λ_α is countable, $\Lambda(x)$ is also countable. This proves the first implication.

Suppose now that we have a product of a countable number of non-indiscrete spaces. This means that there exists a $\mathcal{B} \subset \mathcal{A}$ uncountable and for all $\alpha \in \mathcal{B}$ there exist $y_\alpha \in X_\alpha$, $\varepsilon_\alpha > 0$ and $\phi_\alpha \in \Phi_\alpha(x_\alpha)$ such that: $\phi_\alpha(y_\alpha) = \varepsilon_\alpha > 0$ for a certain $\phi_\alpha \in \Phi_\alpha(x_\alpha)$. If $(\psi_n)_{n\in\mathbf{N}}$ is a countable base in the product space then for each $m, n \in \mathbf{N}$ there exists $\varphi_{n,m} \in \Lambda(x)$ such that:

$$\psi_n \wedge 1 \le \varphi_{n,m} + 1/m \tag{5}$$

where $\varphi_{n,m} = \sup_{\alpha\in K_{(n,m)}} \varphi_\alpha(pr_\alpha(.))$ where $\varphi_\alpha \in \Phi(x_\alpha)$ and $K_{(n,m)}$ is finite. It is clear that $\varphi_{n,m}(z) = 0$ if $z_\alpha = x_\alpha$ for all $\alpha \in K_{(n,m)}$. For $K = \bigcup_{(n,m)\in\mathbf{N}\times\mathbf{N}} K_{(n,m)}$ we have $\mathcal{B} \setminus K \neq \emptyset$ because K is countable. Take now $z \in \prod_{\alpha\in\mathcal{A}} X_\alpha$ such that: $z_\alpha = x_\alpha$ for all $\alpha \in K \cup (\mathcal{A} \setminus \mathcal{B})$ and $z_\alpha = y_\alpha$ for all $\alpha \in \mathcal{B} \setminus K$ then out of equation (5) it follows that $\psi_n(z) = 0$ for all $n \in \mathbf{N}$. But $\varphi_\alpha(pr_\alpha(.)) \in \Lambda(x)$ for $\alpha \in \mathcal{B} \setminus K$ we have: $\varphi_\alpha(pr_\alpha(z)) = \varphi_\alpha(y_\alpha) = \varepsilon_\alpha > 0$. But because $(\psi_n)_{n\in\mathbf{N}}$ is a base we should have a ψ_n such that for $\varepsilon = \varepsilon_\alpha/2$: $\varphi_\alpha(pr_\alpha(.)) \wedge 1 \le \psi_n + \varepsilon$ but it is clear that this can never be satisfied in z. □

3. Measures of compactness and relative compactness in approach spaces

If we consider a subset A of a set X then it can be of interest to know if a sequence in A has a converging subsequence in X. If the limit point itself belongs to the set A is often of minor importance (e.g. probability theory). For this reason we shall discuss a measure of relative compactness, which includes the measure of compactness as a special case.

3.1 The measure of relative compactness

Definition 3.1. *Given an approach space X and a subset A of X, we call the following expression the measure of relative compactness of A with respect to X:*

$$\overline{C}(A, X) := \sup_{\phi\in\prod_{x\in X}\Phi(x)} \inf_{Y\in 2^{(X)}} \sup_{z\in A} \inf_{x\in Y} \phi(x)(z).$$

The measure of compactness $M(X)$ for an approach space X defined in [9] is precisely $\overline{C}(X,X)$ also noted here as $C(X)$. If no confusion is possible about the space X we will also write $\overline{C}(A)$ and call this the measure of relative compactness of A. In [9] five expressions were given for the measure of compactness, for the measure of relative compactness we can do the same and following the proof in [9] step by step with some minor changes we become:

Theorem 3.2. *For X an approach space the following expressions also express the measure of relative compactness of A in X:*

$$\overline{C}(A,X) := \sup_{\psi\in\prod_{x\in X}\Lambda(x)} \inf_{Y\in 2^{(X)}} \sup_{z\in A} \inf_{x\in Y} \psi(x)(z)$$

$$\overline{C}(A,X) := \sup_{\mathcal{F}\in F(A)} \inf_{x\in X} \alpha\mathcal{F}(x)$$

$$\overline{C}(A,X) := \sup_{\mathcal{U}\in U(A)} \inf_{x\in X} \alpha\mathcal{U}(x)$$

$$\overline{C}(A,X) := \sup_{\mathcal{U}\in U(A)} \inf_{x\in X} \lambda\mathcal{U}(x)$$

where $\Lambda(x)$ is a base for $\Phi(x)$.

Remark 3.3. *If we replace filters by nets and ultrafilters by ultranets in the expressions above then using the relationship between convergence and adherence of filters and nets* (*see* Theorem 1.5) *we obtain three more expressions which yield the same measure.*

For topological approach spaces the concept of measure of relative compactness 'almost' coincide with the concept of relative compactness in Hausdorff spaces i.e. the closure of a set is compact.

Theorem 3.4. *For a topological space X and $A\subset X$ we have*:

(a) *If A is relatively compact then $\overline{C}(A)=0$.*
(b) *If X is a regular space and $\overline{C}(A)=0$ then A is relatively compact.*
(c) *$\overline{C}(A)\in\{0,\infty\}$.*

In particular for regular topological spaces we have:
$\overline{C}(A)=0$ iff A is relatively compact.

Proof: (a) If A is a relatively compact space in X then $\overline{A}$ is compact then

$$\forall\,(V(x))_{x\in X},\exists\, Y\in 2^{(\overline{A})}: \bigcup_{y\in\overline{A}} V_y\supset\overline{A}\supset A.$$

Because $\Lambda(x):=\{\Theta_{V(x)}\mid V(x)\text{ is a neighborhood of }x\}$ is a base for $\Phi(x)$ we have:

$$\sup_{z\in A}\inf_{y\in Y}\Theta_{V(y)}(z)=0$$

which implies that $\overline{C}(A) = 0$.

(b) We choose $(V(x))_{x\in X}$ such that if $x \notin \overline{A} : V(x) \cap \overline{A} = \emptyset$ with $V(x)$ closed (X is regular). From $\overline{C}(A) = 0$ we deduce $Y \in 2^{(\overline{A})}$:

$$\sup_{z\in A} \inf_{y\in Y} \Theta_{V_y}(z) = 0. \tag{6}$$

Suppose now that:

$$\sup_{z\in \overline{A}} \inf_{y\in Y} \Theta_{V_y}(z) = 0. \tag{7}$$

It is now easy to see that $\overline{A}$ is compact and thus A is relatively compact. So we only have to show equation (7).
Suppose this is false. Then there exists $z \in \overline{A} \setminus A = \partial A$:

$$\inf_{y\in Y} \Theta_{V_y}(z) = \infty$$

i.e. $z \notin \cup_{y\in Y} V_y$. Because the V_y are closed we have a neighborhood W_z of z such that:

$$W_z \cap \cup_{y\in Y} V_y = \emptyset.$$

But because $z \in \partial A : W_z \cap A \neq \emptyset$. But if $a \in W_z \cap A$ we have $a \notin \cup_{y\in Y} V_y$ which is in contradiction with equation (6) and which proves equation (7). □

The regularity of X is necessary as the following examples show.

Counterexample 3.5. *Consider an uncountable set X and take an element $a \in X$. We shall say that a set G is open iff G contains a or $G = \emptyset$. And thus the closed sets are the sets which does not contain a and the whole set X. For $A := \{a, b\}$ where b can be any element, $\overline{A} = X$ and the cover $\{\{a, x\} \mid x \in X\}$ has no finite subcover which shows that A is not relatively compact. On the other hand A is finite and thus $\overline{C}(A) = 0$ (in fact A is compact).*

It is also possible to give an example of a non-regular Hausdorff space, which has $\overline{C}(A) = 0$ and is not relatively compact. In the following we will note $\Im$ for the irrational real numbers.

Counterexample 3.6. *For a point (x, y) in $\mathbf{R}^2$ consider $\{I_x \cap \Im\} \times \{I_y \cap \Im\} \cup \{(x, y)\}$ as basic neighborhoods, where I_x and I_y are open intervals containing respectively x and y. It is not difficult to see that these neighborhoods are a base for a non-regular Hausdorff topology. Take now the set $A :=]0,1[\times]0,1[\cap \Im \times \Im$ then the set $\overline{A} = [0,1] \times [0,1]$. Take now for every element $p \in \overline{A}$ a basic neighborhood $V_p := \{I_{p_x} \cap \Im\} \times \{I_{p_y} \cap \Im\} \cup \{(p_x, p_y)\}$ in this topology and consider $W_p := I_{p_x} \times I_{p_y}$. Then it is clear that $\bigcup_{p\in\overline{A}} W_p \supset \overline{A}$, but in the usual Euclidean topology the set $\overline{A}$ is compact, hence has a finite subcover. If we replace W_p by*

V_p we only leave out points outside $\Im \times \Im \supset A$. And thus we have a finite number of V_p with $p \in \overline{A}$ which covers A. This proves that $\overline{C}(A) = 0$. On the other hand $\overline{A}$ is not compact. Indeed take for each point (x, y) in $\overline{A} \setminus \Im \times \Im$ an open set V_p as above with $I_x \supset [0,1]$ and $I_y \supset [0,1]$. These sets form an open cover of $\overline{A}$ but each point with one or more rational coordinates will only be covered by precisely one set, hence no finite subcover exists.

We remark that for a topological space X it was shown in [9] that $C(X) = 0$ for a compact space and $C(X) = \infty$ for a non-compact space.

3.2 Measure of relative sequential compactness and relative countable compactness

We now introduce measures of relative sequential and relative countable compactness.

Definition 3.7. *Given an approach space X, we define its measure of sequential compactness (respectively countable compactness) by:*

$$\overline{SC}(A, X) = \sup_{(x_n)_{n\in\mathbf{N}}} \in r(A) \inf_{k\uparrow:\mathbf{N}\longrightarrow\mathbf{N}} \inf_{x\in X} \lambda(x_{k(n)} \to x)$$

$$(\textit{respectively by}: \overline{CC}(A, X) = \sup_{(x_n)_{n\in\mathbf{N}}} \in r(A) \inf_{x\in X} \alpha(x_n \to x)).$$

Again we shall also write $\overline{CC}(A)$ and $\overline{SC}(A)$ if no confusion about the space X is possible. Further we shall note $CC(X) = \overline{CC}(X, X)$ and $SC(X) = \overline{SC}(X, X)$ for respectively the measure of countable compactness and sequential compactness.

Theorem 3.8. *For a topological approach space X we have:*

$$\overline{SC}(A, X), \overline{CC}(A, X) \in \{0, \infty\}. \tag{8}$$

Further:

(a) *$\overline{SC}(A, X) = 0$ iff every sequence in A has a converging subsequence to a point in X.*
(b) *$\overline{CC}(A, X) = 0$ iff every sequence in A has an accumulation point in X.*

Proof: This is a straightforward verification. □

Corollary 3.9. *A topological space X is sequentially (countably) compact iff $SC(X) = 0$ $(CC(X) = 0)$.*

Remark 3.10. *The following question can be of interest: Is the topological coreflection of an approach space $A \subset X$ with $\overline{C}(A, X) = 0$ relatively compact? The answer is no. Indeed consider the approach space $X := \{1/n \mid n \in \mathbf{N}\}$ with $\Phi(x) := \{d(x, .) \mid d$ the Euclidean metric$\}$ the topological coreflection is the usual natural topology on this set, every infinite subset A of X is clearly not* *relatively compact. On the other hand it is not difficult to see that* $\overline{CC}(A,X) = \overline{SC}(A,X) = \overline{C}(A,X) = 0$.

We recall that a filter $\mathcal{F}$ on X is called countable if it has a filter base with a countable number of elements.

Example 3.11. *The filter associated with a sequence* $(x_n)_{n\in\mathbf{N}}$ *called an elementary filter is generated by* $\mathcal{B} = \{\{x_m \mid m \geq n\} \mid n \in \mathbf{N}\}$ *so it is a countable filter.*

The converse is not true but we have:

Proposition 3.12 ([3])**.** *Every countable filter* $\mathcal{F}$ *is the intersection of the elementary filters which contain* $\mathcal{F}$.

In [5] the following useful result is proven:

Proposition 3.13. *For a countable filter* $\mathcal{F}$ *there exists a base* $\mathcal{B} = \{B_n \mid n \in \mathbf{N}\}$ *such that* $B_n \subset B_m$ *if* $m \leq n$.

We shall note the countable (resp. elementary) filters on X by $F_c(X)$ (resp. $F_e(X)$).

With some minor changes to the proof of the corresponding theorem for the measure of countably compactness, we have:

Theorem 3.14. *For any subset* A *of an approach space* X*:*

$$\overline{CC}(A) = \sup_{\mathcal{F}\in F_e(A)} \inf_{x\in X} \alpha\mathcal{F}(x)$$

$$\overline{CC}(A) = \sup_{\mathcal{F}\in F_c(A)} \inf_{x\in X} \alpha\mathcal{F}(x)$$

$$\overline{CC}(A) = \sup_{\phi\in\prod_{x\in X}\Phi(x)} \sup_{(x_n)_n\in r(A)} \inf_{x\in X} \liminf_{n\to\infty} \phi(x)(x_n)$$

$\square$

Proof: From Theorem 1.5 we deduce that $\overline{CC}(A) := \overline{CC}_1(A)$. For a countable filter $\mathcal{F}$ on A we can always consider an elementary filter $\mathcal{G}$ on A such that: $\mathcal{F} \subset \mathcal{G}$. From 1.6 it now follows that for all x in X: $\alpha(\mathcal{F})(x) \leq \alpha(\mathcal{G})(x)$ so $\overline{CC}_2(A) \leq \overline{CC}_1(A)$. The other inequality follows from Example 3.11. For the last equality we proceed as follows:

$$\begin{aligned}
\overline{CC}(A) &= \sup_{(x_n)_{n\in\mathbf{N}}\in r(A)} \inf_{x\in X} \alpha(x_n \to x) \\
&= \sup_{(x_n)_{n\in\mathbf{N}}\in r(A)} \inf_{x\in X} \sup_{\phi\in\Phi(x)} \liminf_{n\to\infty} \phi(x_n) \\
&= \sup_{(x_n)_{n\in\mathbf{N}}\in r(A)} \sup_{\phi\in\prod_{x\in X}\Phi(x)} \inf_{x\in X} \liminf_{n\to\infty} \phi(x)(x_n) \\
&= \overline{CC}_3(A)
\end{aligned}$$

□

As in topology we can easily prove some stability properties for the measures of relative compactness, relative countable compactness and relative sequential compactness:

Theorem 3.15. *For the approach spaces X, X', $A \subset X$ and a contraction $f : X \longrightarrow X'$ we have:*

(a) $\overline{CC}(f(A)) \leq \overline{CC}(A)$.
(b) $\overline{SC}(f(A)) \leq \overline{SC}(A)$.
(c) $\overline{C}(f(A)) \leq \overline{C}(A)$.

□

3.3 Some relations between the different measures of compactness

In general we only have the following relation between the different measures of (relative) compactness:

Proposition 3.16. *For an approach space X and $A \subset X$ we have:*

$$\overline{CC}(A) \leq \overline{SC}(A)$$
$$\overline{CC}(A) \leq \overline{C}(A).$$

Proof: It is clear from the definitions that $\overline{CC}(X) \leq \overline{C}(X)$. Further if $(x_{k(n)})_{n\in\mathbf{N}}$ is a subsequence of $(x_n)_{n\in\mathbf{N}}$ then from Corollary 1.7 we deduce that for all $x \in X$: $\alpha(x_n \to x) \leq \alpha(x_{k(n)} \to x) \leq \lambda(x_{k(n)} \to x)$. It is now easy to see that $\overline{CC}(X) \leq \overline{SC}(X)$. □

In combination with 3.8 the foregoing states that countably compactness is implied by sequentially compactness and compactness. Further from the relationships between compactness, countable compactness and sequential compactness in topology it is clear that there are (topological) approach spaces which contradict any other inequality than the ones stated above. On the other hand countable compactness and sequential compactness coincide for first countable topological spaces. For first countable approach spaces we have:

Proposition 3.17. *For a first countable approach space X and $A \subset X$ the measures of (relative) countable compactness and (relative) sequential compactness coincide.*

Proof: By property 3.16 we only have to prove that $\overline{SC}(A) \leq \overline{CC}(A)$. Remark that after inspection of the definitions it is sufficient to prove that:

$$\inf_{k\uparrow:\mathbf{N}\to\mathbf{N}} \lambda(x_{k(n)} \to x) \leq \alpha(x_n \to x). \tag{9}$$

Take $\Lambda(x) = (\phi_n)_{n\in\mathbf{N}} \overset{=}{\uparrow}$ a basis for $\Phi(x)$. Then we have for $(x_n)_{n\in\mathbf{N}} \in r(A)$:

$$\alpha(x_n \to x) = \sup_{m\in\mathbf{N}} \sup_{l\in\mathbf{N}} \inf_{n\geq l} \phi_m(x_n).$$

For $\varepsilon > 0$ we can find a $k\uparrow : \mathbf{N} \longrightarrow \mathbf{N}$ such that:

$$\phi_n(x_{k(n)}) \leq \alpha(x_n \to x) + \varepsilon.$$

Indeed if $m \leq n$ then $\phi_m \leq \phi_n$ so $\phi_m(x_{k(n)}) \leq \phi_n(x_{k(n)}) \leq \alpha(x_n \to x) + \varepsilon$. Consider now:

$$\begin{aligned}
\lambda(x_{k(n)} \to x) &= \sup_{n\in\mathbf{N}} \inf_{l\in\mathbf{N}} \sup_{j\geq l} \phi_n(x_{k(j)}) \\
&\leq \inf_{l\in\mathbf{N}} \sup_{j\geq l} \phi_n(x_{k(j)}) + \varepsilon \text{ (for } n \text{ sufficiently large)} \\
&\leq \sup_{j\geq l} \ \phi_j(x_{k(j)}) + \varepsilon \\
&\leq \alpha(x_n \to x) + 2\varepsilon.
\end{aligned}$$

So that:

$$\inf_{k\uparrow} \lambda(x_{k(n)} \to x) \leq \alpha(x_n \to x) + 2\varepsilon.$$

And because this is true for all $\varepsilon > 0$ we have proved equation (9). $\square$

From [1] we recall the definition of the measure of Lindelöf for the approach space X:

$$L(X) = \sup_{\phi\in\prod_{x\in X}\Phi(x)} \inf_{Y\in 2^{((X))}} \sup_{x\in X} \inf_{y\in Y} \phi(y)(x).$$

Hereby $(\Phi(x))_{x\in X}$ shall be an approach system or basis. As in topology, where it is trivial, Lindelöf and countably compactness implies compactness.

Theorem 3.18. *For an approach space X and $A \subset X$ we have:*

$$\overline{C}(A) \leq \overline{CC}(A) + L(X).$$

In particular:

$$C(X) \leq CC(X) + L(X).$$

Proof: From [1] we know that for each approach space we can find a set of p-q-metrics $\mathcal{D}$ such that $(\mathcal{D}(x))_{x\in X}$ is a basis for the approach system. Choose $\varepsilon > 0$ and put $r = \overline{C}(A) - \varepsilon$ then there exists a filter $\mathcal{F}$ on A such that:

$$\inf_{x\in X} \alpha\mathcal{F}(x) > r.$$

For all $x \in X$, $\exists\, d_x \in \mathcal{D}$:

$$\sup_{F\in\mathcal{F}} \inf_{f\in F} d_x(x,f) > r$$

or i.e. $\forall\, x \in X, \exists\, d_x \in \mathcal{D}, \exists\, F_x \in \mathcal{F}, \forall\, f \in F_x : d_x(x,f) > r$.
Consider now this collection of $(d_x)_{x\in X}$, then from the definition of the Lindelöf measure we know that there exists a $Y_\varepsilon \in 2^{((X))}$:

$$\sup_{x\in X} \inf_{y\in Y} d_y(y,x) < L(X) + \varepsilon.$$

Consider now the filter $\mathcal{F}_\varepsilon := \langle\{F_y \mid y \in Y\}\rangle$ this is clearly a countable filter on A, and thus:

$$\inf_{x\in X} \alpha\mathcal{F}_\varepsilon(x) \leq \overline{CC}(X).$$

This means that we can find an element $x \in X$:

$$\sup_{d\in\mathcal{D}} \sup_{F_y\in\mathcal{B}_\varepsilon} \inf_{f\in F_y} d(x,f) \leq \overline{CC}(X) + \varepsilon.$$

Take now $y \in Y_\varepsilon$ such that: $d_y(y,x) < L(X) + \varepsilon$ then consider the corresponding d_y and F_y and thus there exists $f_y \in F_y : d_y(x,f_y) \leq \overline{CC}(X) + 2\varepsilon$. Finally we have:

$$r = \overline{C}(A) - \varepsilon < d_y(y,f_y) \leq d_y(y,x) + d_y(x,f_y) \leq L(X) + \overline{CC}(X) + 3\varepsilon.$$

Because this is true for every $\varepsilon > 0$ we have $\overline{C}(X) \leq \overline{CC}(X) + L(X)$. □

3.4 The measures of relative compactness for products of approach spaces

In this section we shall discuss the relations between the measures of compactness of a product space and their component spaces.

Remark 3.19. *Given that projections are contractions it is clear from Theorem* 3.15 *that the measures of countable compactness, sequential compactness of the components are always less than or equal to the corresponding measure for the product space. So we only have to prove one equality for each of the measures.*

For each measure we now look at the other inequality.

Measure of relative compactness.

In [9] it is shown that the Tychonoff theorem can be generalized for approach spaces in the following way:

Theorem 3.20. *For an arbitrary index set J and approach spaces X_j, $j \in J$ we have*:

$$C(\prod_{j\in J} X_j) = \sup_{j\in J} C(X_j)$$

But if we inspect carefully the proofs of Theorem 6.6 and 6.7 in [9] and make some minor changes then we also have:

Theorem 3.21. *For an arbitrary index set J and approach spaces X_j, with $A_j \subset X_j$ for all $j \in J$ we have:*

$$\overline{C}(\prod_{j\in J} A_j, \prod_{j\in J} X_j) = \sup_{j\in J} \overline{C}(A_j, X_j).$$

Measure of relative countable compactness.

In [12] an example is given (Example 112, Novak space) of a countably compact topological space such that the product of this space with itself is not a countably compact space. Consequently by Theorem 3.8 the product of approach spaces can have a measure of countable compactness equal to ∞ while the measure of components equals 0.

Measure of relative sequential compactness.

Theorem 3.22. *For approach spaces X_i and $A_i \subset X_i$ for all $i \in \mathbf{N}$:*

$$\text{(a)} \qquad \overline{SC}(\prod_{i\in \mathbf{N}} A_i, \prod_{i\in \mathbf{N}} X_i) = \sup_{i\in \mathbf{N}} \overline{SC}(A_i, X_i).$$

In particular we have:

$$\text{(b)} \qquad SC(\prod_{i\in \mathbf{N}} X_i) = \sup_{i\in \mathbf{N}} SC(X_i).$$

Proof: (a) Consider the sequence $(x_n)_{n\in\mathbf{N}} \in r(\prod_{i\in\mathbf{N}} A_i)$. For all $\varepsilon > 0, \exists\, k_1 \uparrow$: $\mathbf{N} \to \mathbf{N}, \exists\, x_1 \in X_1$:

$$\lambda_1(pr_1(x_{k_1(n)}) \to x_1) \leq \overline{SC}(A_1, X_1) + \varepsilon.$$

Consider now the sequence $(pr_2(x_{k_1(n)}))_{n\in\mathbf{N}}$ in A_2. There $\exists\, k_2 \uparrow$: $\mathbf{N} \to \mathbf{N}, \exists\, x_2 \in X_2$:

$$\begin{aligned} \lambda_2(pr_2(x_{k_2(k_1(n))}) \to x_2) &\leq \overline{SC}(A_2, X_2) + \varepsilon \\ &\leq \sup_{i=1,2} \overline{SC}(A_i, X_i) + \varepsilon. \end{aligned}$$

Since $(x_{k_2(k_1(n))})_{n\in\mathbf{N}}$ is a subsequence of $(x_{k_1(n)})_{n\in\mathbf{N}}$ we have:

$$\lambda_1(pr_1(x_{k_2(k_1(n))}) \to x_1) \leq \sup_{i=1,2} \overline{SC}(A_i, X_i) + \varepsilon.$$

We will put $k_2' = k_2 \circ k_1$ and in general we can continue the process above for every $n \in \mathbf{N}$ leading to $k_m' = k_m \circ k_{m-1}'$ and $x_1, x_2, \dots, x_m$ such that $\forall\, j = 1, \dots, m$:

$$\lambda_j(pr_j(x_{k_m'(n)}) \to x_j) \leq \sup_{i=1,2,\dots,m} \overline{SC}(A_i, X_i) + \varepsilon. \tag{10}$$

Now put $x = (x_1, x_2, \dots, x_n, \dots)$. We shall show that:

$$\lambda(x_{k'_n(n)} \to x) \le \sup_{i \in \mathbf{N}} \overline{SC}(A_i, X_i) + \varepsilon$$

where λ is the convergence operator in the product space. Indeed $\lambda(x_{k'_m(m)} \to x) = \sup_{j \in \mathbf{N}} \lambda_j(pr_j(x_{k'_m(m)}) \to x_j)$ and thus for every $\varepsilon > 0$ there exists a certain j$\in \mathbf{N}$:

$$\lambda(x_{k'_m(m)} \to x) < \lambda_j(pr_j(x_{k'_m(m)}) \to x_j) + \varepsilon.$$

But for every $p \in \mathbf{N}$ with $p > j$ we can find l_p big enough such that the sequence $\{k'_m(m) \mid m > l_p\}$ is a subsequence of $\{k'_p(n) \mid n \in \mathbf{N}\}$. Therefore we also have:

$$\lambda_j(pr_j(x_{k'_m(m)}) \to x_j) \le \lambda_j(pr_j(x_{k'_p(m)}) \to x_j).$$

Equation (10) leads to:

$$\lambda(x_{k'_m(m)} \to x) < \sup_{i=1,\dots,p} \overline{SC}(A_i, X_i) + \varepsilon$$

and it follows that:

$$\lambda(x_{k'_m(m)} \to x) < \sup_{i \in \mathbf{N}} \overline{SC}(A_i, X_i) + \varepsilon.$$

This last inequality implies:

$$\sup_{i \in \mathbf{N}} \overline{SC}(A_i, X_i) \le \overline{SC}(\prod_{i \in \mathbf{N}} A_i, \prod_{i \in \mathbf{N}} X_i).$$

□

It is clear from the results from topology [12] that this is not true for uncountable products of approach spaces.

4. Completeness and the measure of compactness

If we want to study products of p-MET$^\infty$ spaces we can restrict ourselves to consider the epireflective hull $\mathcal{M}$ of these spaces, which are exactly those approach spaces who have a generating set of extended pseudometrics. In [10] the notion of completeness was introduced for these approach spaces together with a notion of Cauchy filter. Let us recall [10] that if $\mathcal{F}$ is a filter on $X \in |\mathcal{M}|$ with $\inf_{x \in X} \lambda\mathcal{F}(x) = 0$ then we call $\mathcal{F}$ a *Cauchy filter*. Further we call X complete if every Cauchy filter on X has a limit point (i.e. there is an $x \in X : \lambda\mathcal{F}(x) = 0$). We will note in the following the topological bicoreflection as $\mathbf{T}(X)$. For each $(X, \delta) \in \mathcal{M}$ we can consider the set p-∞ metrics $\mathcal{D}_\delta := \{d \mid \forall\, A \subset X, \forall\, x \in X : \inf_{a \in A} d(x, a) \le \delta(x, A)\}$ so we can associate with each $\mathcal{D}_\delta$ the uniform space $\mathcal{U}(\mathcal{D}_\delta)$ generated by these p-∞ metrics. In [9] it was shown for p-MET$^\infty$ spaces that total boundedness is equivalent with $C(X) = 0$. The following result is an extension of it.

Theorem 4.1. *For $X \in |\mathcal{M}|$, $\mathbf{T}(X)$ is compact iff $C(X) = 0$ and X is complete.*

PROOF: Suppose $C(X) = 0$ and X is complete then consider an ultrafilter on X because $C(X) = 0$ we have: $\inf_{x \in X} \lambda\mathcal{U}(x) = 0$ and thus $\mathcal{U}$ is a Cauchy filter. But X is complete hence $\exists\, x \in X : \lambda\mathcal{U}(x) = 0$, which means that every ultrafilter in $\mathbf{T}(X)$ has a limit point and thus $\mathbf{T}(X)$ is compact. For the other implication it is clear that if $\mathbf{T}(X)$ is compact $C(X) = 0$. So we only have to prove the completeness of X. Suppose $\mathcal{C}$ is a Cauchy filter consider now an ultrafilter $\mathcal{U}$ containing this Cauchy filter because $\mathbf{T}(X)$ is compact it has a point x: $\lambda\mathcal{U}(x) = 0$ from Proposition 3.3 [10] it follows that $\lambda\mathcal{C}(x) = 0$ which proves the completeness. □

References

[1] Baekeland R., LowenR., *Measures of Lindelöf and separability in approach spaces*, to appear in Internat. J. Math & Math. Sci.

[2] Beer G., *On a measure of non-compactness*, Rend. Mat. Appl. **72** (1987), 193–205 (1988).

[3] Bourbaki N., *Elements de Mathématique : Livre III, Topologie générale'*, Hermann, Paris, 1962.

[4] Dugundji J., *Topology*, Allan and Bacon, Boston, 1966.

[5] Gaal S., *Point Set Topology*, Academic Press, New York, 1964.

[6] Lowen E., Lowen R., *A quasi-topos containing CONV and MET as full subcategories*, Internat. J. Math. & Math. Sci. **11:3** (1988), 417–438.

[7] ———, *Topological quasitopos hulls of categories containing topological and metric objects*, Cahiers de Topologie et Géométrie différentielle catégoriques **30:3** (1989), 213–228.

[8] Lowen R., *Approach spaces: a common supercategory of TOP and MET*, Math. Nachr. **141** (1989), 183–226.

[9] ———, *Kuratowski's measure of non-compactness revisited*, Quart. J. Math. Oxford **39:2** (1988), 235–254.

[10] Lowen R., Robeys K., *Completion of products of metric spaces*, Quart. J. Math. Oxford **43:2** (1992), 319–338.

[11] Kuratowski C., *Sur les espaces complets*, Fund. Math. **15** (1930), 301–309.

[12] Steen L.A., Seebach J.A., Jr, *Counterexamples in Topology*, Holt, Rinehart and Winston, Inc, 1970.

[13] Thron W.J., *Topological Structures*, Holt, Rinehart and Winston, 1966.

[14] Wilansky A., *Topology for Analysis*, Ginn, Waltham, Massachusetts, Toronto, London, 1970.

[15] Willard S., *General Topology*, Addison-Wesley Series in Mathematics, Reading, Massachusetts, 1970.

Department of Mathematics and Computer Science, University of Antwerp, RUCA, Groenenborgerlaan 171, 2020 Antwerp, Belgium

(*Received* November 29, 1992, *revised* February 1, 1994)