On the existence of 2-fields in 8-dimensional vector bundles over 8-complexes

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Abstract. Necessary and sufficient conditions for the existence of two linearly independent sections in an 8-dimensional spin vector bundle over a CW-complex of the same dimension are given in terms of characteristic classes and a certain secondary cohomology operation. In some cases this operation is computed.

Keywords: span of the vector bundle, classifying spaces for spinor groups, characteristic classes, Postnikov tower, secondary cohomology operation

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1. Introduction

There are several papers devoted to the existence of tangent 2-fields on $4k$-dimensional manifolds. In [T1] E. Thomas used the method of the Postnikov tower to show that a spin vector bundle ξ over a $4k$-dimensional manifold M has two linearly independent sections if and only if the Euler class $e(\xi)=0$, the Stiefel-Whitney class $\delta w_{4k-2}=0$, and $\Phi(U)=0$, where U is the Thom class of ξ and Φ is a certain secondary operation. In the case of the tangent bundle of a compact spin manifold and under some additional assumptions on $H^*(M;\mathbb{Z}_2)$ he found that the last condition is equivalent to the fact that the Euler characteristic is divisible by 4.

For general $4k$-dimensional manifolds the problem of the existence of tangent 2-fields was solved by D. Frank in [F] using K-theory and by M. Atiyah and J. Dupont in [AD] using index theory. The necessary and sufficient conditions here are the vanishing of the Euler characteristic and divisibility of the signature by 4. In both papers the fact that the vector bundle is a tangent bundle is essential.

The aim of this note is to present results concerning the existence of two linearly independent sections in 8-dimensional spin vector bundles over a CW-complex X of the same dimension. The main results (Theorems 5.1 and 5.2) use a secondary cohomology operation $\Omega : H^4(X;\mathbb{Z}) \to H^8(X;\mathbb{Z}_2)$ applied on a cohomology class which can be computed from the Pontrjagin and Stiefel-Whitney classes. The computation of Ω is often possible also for non-tangent bundles. As a corollary we obtain the following theorem given in terms of the Euler and Pontrjagin classes.

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Theorem 1.1. *Let M be a compact smooth spin manifold of dimension 8 and let ξ be an 8-dimensional oriented vector bundle over M with $w_2(\xi) = 0$ and $w_4(\xi) = w_4(M)$. Suppose $H^4(M, \mathbb{Z})$ has no element of order 4. Then ξ has two linearly independent sections if and only if the Euler class of ξ vanishes and*

$$\{4p_2(\xi) - 2p_1^2(\xi) - p_1^2(M) + 2p_1(\xi)p_1(M)\}[M] \equiv 0 \quad mod\ 32.$$

The computation of Ω needed in the proof of Theorem 1.1 was carried out in [T2]. To prove our results we build the Postnikov tower for the fibrations $BSpin(6) \to BSpin(8)$ and $BSpin(6) \to BSpin$. In our considerations we use the fact that the groups $Spin(6)$ and $SU(4)$ are isomorphic.

Notation and preliminary results on the cohomology groups of the classifying spaces $BSpin(n)$ and $BSpin$ are introduced in Section 2. In Sections 3 and 4 we deal with spin characteristic classes and the secondary cohomology operation mentioned above. Section 5 contains the main results together with examples and proofs of their corollaries. There we also show that our results coincide with those of Atiyah, Dupont and Frank in the case of the tangent bundle of an 8-dimensional spin manifold (which is not quite obvious). In the last section the remaining proofs are given.

2. Notation and preliminaries

All vector bundles will be considered over a connected CW-complex X and will be oriented. The mapping $\delta : H^*(X;\mathbb{Z}_2) \to H^*(X;\mathbb{Z})$ is the Bockstein homomorphism associated with the exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 0$. The mappings $i_* : H^*(X;\mathbb{Z}_2) \to H^*(X;Z_4)$ and $\varrho_k : H^*(X;\mathbb{Z}) \to H^*(X;\mathbb{Z}_k)$ are induced from the inclusion $\mathbb{Z}_2 \to \mathbb{Z}_4$ and reduction mod k, respectively.

In our considerations we will explore the Steenrod squares Sq^i and the Pontrjagin square $\mathfrak{P}$, a cohomology operation from $H^{2k}(X;\mathbb{Z}_2)$ into $H^{4k}(X;\mathbb{Z}_4)$ satisfying the following relation

$$\mathfrak{P}\varrho_2 x = \varrho_4 x^2 \tag{1}$$

for $x \in H^{2k}(X;\mathbb{Z})$. See [MT, Chapter 2].

We will use $w_s(\xi)$ for the s-th Stiefel-Whitney class of the vector bundle ξ, $p_s(\xi)$ for the s-th Pontrjagin class, and $e(\xi)$ for the Euler class. For a complex vector bundle ξ the symbol $c_s(\xi)$ denotes the s-th Chern class. The classifying spaces for spinor groups $Spin(n)$ and $Spin$ will be denoted by $BSpin(n)$ and $BSpin$, respectively. The letters $w_s(n)$, $p_s(n)$, $e(n)$ and w_s, p_s will stand for the characteristic classes of the universal bundles over the classifying spaces $BSpin(n)$ and $BSpin$, respectively. The results on the cohomology groups of the classifying spaces given below are based on the following relations among the characteristic classes

$$\varrho_4 p_1(\xi) = \mathfrak{P}w_2(\xi) + i_* w_4(\xi) \tag{2}$$

$$\varrho_4 p_2(\xi) = \mathfrak{P} w_4(\xi) + i_*\{w_8(\xi) + w_2(\xi)w_6(\xi)\}. \tag{3}$$

See [M] and [T3].

We say that $x \in H^*(X;\mathbb{Z})$ is an element of order r ($r = 2, 3, 4, \dots$) if and only if $x \neq 0$ and r is the least positive integer such that $rx = 0$ (if it exists).

The Eilenberg-MacLane space with n-th homotopy group G will be denoted $K(G,n)$ and ι_n will stand for the fundamental class in $H^n(K(G,n);G)$. Writing the fundamental class it will be always clear which group G we have in mind.

The classifying space $BSpin(n)$ can be considered as the fibration

$$K(\mathbb{Z}_2, 1) \xrightarrow{l} BSpin(n) \to BSO(n)$$

induced by the map $w_2 : BSO(n) \to K(\mathbb{Z}_2, 2)$ from the fibration

$$\Omega K(\mathbb{Z}_2, 2) = K(\mathbb{Z}_2, 1) \to PK(\mathbb{Z}_2, 2) \to K(\mathbb{Z}_2, 2).$$

In this way the natural multiplication

$$m : BSpin(n) \times K(\mathbb{Z}_2, 1) \to BSpin(n)$$

can be defined. The letter l will stand for the inclusion of the fibre $K(\mathbb{Z}_2, 1)$ into $BSpin(n)$.

There are several papers concerning the cohomology groups of $BSpin(n)$ and $BSpin$. The ring $H^*(BSpin;\mathbb{Z}_2)$ has been completely computed and the generators of the ring $H^*(BSpin;\mathbb{Z})$ have been described in [T4]. The complete ring structure of $H^*(BSpin(n);\mathbb{Z}_2)$ is described in [Q], and in [K] the computation of the groups $H^s(BSpin(n);\mathbb{Z})$ has been carried out. As far as the authors know the ring structure of $H^*(BSpin(n);\mathbb{Z})$ has not been determined yet for general n. Here we summarize and complete some of these results in the case of $BSpin(6)$, $BSpin(8)$ and $BSpin$.

Lemma 2.1. *The cohomology rings of $BSpin(6)$ are*

$$H^*(BSpin(6);\mathbb{Z}_2) \cong \mathbb{Z}_2[w_4(6), w_6(6), \varepsilon(6)],$$
$$H^*(BSpin(6);\mathbb{Z}) \cong \mathbb{Z}[q_1(6), q_2(6), e(6)],$$

where $q_1(6)$, $q_2(6)$ and $\varepsilon(6)$ are uniquely determined by the relations

$$p_1(6) = 2q_1(6), \qquad p_2(6) = q_1^2(6) + 4q_2(6), \qquad \varepsilon(6) = \varrho_2 q_2(6). \tag{4}$$

Moreover,

$$\varrho_2 q_1(6) = w_4(6), \qquad \varrho_2 e(6) = w_6(6) \tag{5}$$

$$m^* q_1(6) = q_1(6) \otimes 1, \qquad m^* e(6) = e(6) \otimes 1 \tag{6}$$

$$m^* q_2(6) = q_2(6) \otimes 1 + e(6) \otimes \delta\iota_1 + q_1(6) \otimes \delta\iota_1^3 + 1 \otimes \delta\iota_1^7. \tag{7}$$

Proof: The group $SU(4)$ acts naturally on $\Lambda^2(\mathbb{C}^4)$. On this complex vector space there is an involutive antihomomorphism, which commutes with the action of $SU(4)$. It means that $\Lambda^2(\mathbb{C}^4)$ is the complexification of a 6-dimensional real vector space and this real space is a real representation of $SU(4)$. It yields a homomorphism $SU(4) \to SO(6)$ with kernel $\pm Id$. Hence $SU(4)$ is isomorphic to $Spin(6)$ and consequently

$$H^*(BSpin(6);\mathbb{Z}) \cong H^*(BSU(4);\mathbb{Z}) \cong \mathbb{Z}[c_2, c_3, c_4]$$

where c_2, c_3, c_4 are the Chern classes of the complex vector bundle η which is associated with the universal $SU(4)$-bundle. Let μ be the fibration $BSU(4) \cong BSpin(6) \to BSO(6)$ given by the double covering of $SO(6)$. Then $\Lambda^2\eta$ is a complexification of the real vector bundle $\mu^*\gamma$ where γ is the real vector bundle over $BSO(6)$ associated with the universal $SO(6)$-bundle. Then

$$p_r(6) = \mu^* p_r(\gamma) = (-1)^r c_{2r}(\Lambda^2\eta)$$
$$-e^2(6) = -\mu^* e^2(\gamma) = e((\Lambda^2\eta)_{\mathbb{R}})$$

for $r = 1, 2$. According to [H] we have

$$1 + \sum_{1\le t\le 6} c_t(\Lambda^2\eta)x^t = \prod_{1\le r<s\le 4} (1 + (\alpha_r + \alpha_s)x),$$

where

$$1 + \sum_{1\le t\le 4} c_t(\eta)x^t = \prod_{r=1}^{4} (1 + \alpha_r x).$$

That is why

$$c_2(\Lambda^2\eta) = 2c_2(\eta), \qquad c_4(\Lambda^2\eta) = c_2^2(\eta) - 4c_4(\eta), \qquad c_6(\Lambda^2\eta) = -c_3^2(\eta).$$

We put $q_1(6) = -c_2(\eta)$, $q_2(6) = -c_4(\eta)$. Moreover, we can arrange that $e(6) = c_3(\eta)$. Then $H^*(BSpin(6);\mathbb{Z}) \cong \mathbb{Z}[q_1(6), q_2(6), e(6)]$ and we get the first two relations in (4). The first relation in (5) follows from (2). Define $\varepsilon(6) = \varrho_2 q_2(6)$. Then $H^*(BSpin(6);\mathbb{Z}_2) \cong \mathbb{Z}_2[w_4(6), w_6(6), \varepsilon(6)]$. Comparing this result with [Q] we obtain that

$$l^*\varepsilon(6) = \iota_1^8, \qquad \varepsilon(6) = w_8(\Delta)$$

where Δ is the spin representation of the group $Spin(6)$ in $\mathbb{C}^4$. Since $l^* w_4(6) = 0$, we get $l^* q_1(6) = 0$ and $m^* q_1(6) = q_1(6) \otimes 1$.

Further $m^*\varepsilon(6) = \varepsilon(6) \otimes 1 + a w_6(6) \otimes \iota_1^2 + b w_4(6) \otimes \iota_1^4 + 1 \otimes \iota_1^8$ where $a, b \in \mathbb{Z}_2$. We have

$$Sq^2\varepsilon(6) = Sq^2 w_8(\Delta) = (w_2 w_8)(\Delta) = w_8(\Delta) w_2(\Delta) = 0$$

since $w_2(\Delta) = 0$. (See [Q].) Hence

$$0 = m^* Sq^2 \varepsilon(6) = Sq^2 m^* \varepsilon(6) = a w_6(6) \otimes \iota_1^4 + b w_6 \otimes \iota_1^4$$

and that is why $a = b$. According to [Q], $w_4(\Delta) = w_4(6)$ and therefore $Sq^4\varepsilon(6) = Sq^4 w_8(\Delta) = w_4(6)\varepsilon(6)$. It yields

$$w_4(6)\varepsilon(6) \otimes 1 + a w_4(6) w_6(6) \otimes \iota_1^2 + a w_4^2(6) \otimes \iota_1^4 + w_4(6) \otimes \iota_1^8 =$$
$$m^*(w_4(6)\varepsilon(6)) = m^* Sq^4 \varepsilon(6) = Sq^4 m^* \varepsilon(6) =$$
$$w_4(6)\varepsilon(6) \otimes 1 + a w_4(6) w_6(6) \otimes \iota_1^2 + a w_4^2(6) \otimes \iota_1^4 + a w_4(6) \otimes \iota_1^8$$

which implies $a = 1$. Now, since $\varrho_2 q_2(6) = \varepsilon(6)$ and

$$H^8(BSpin(6) \times K(\mathbb{Z}_2, 1); \mathbb{Z}) \cong \oplus_{r=0}^{8} \{H^r(BSpin(6); \mathbb{Z}) \otimes H^{8-r}(K(\mathbb{Z}_2, 1); \mathbb{Z})\},$$

we get (7) for $m^* q_2(6)$. In the similar way we can show that $m^* e(6) = e(6) \otimes 1$. $\square$

The fibrations $BSpin(6) \to BSpin(8)$ and $BSpin(6) \to BSpin$ will be denoted by π. It will be always clear from the context which case we consider.

Lemma 2.2. *The mod 2 cohomology ring of $BSpin(8)$ is*

$$H^*(BSpin(8); \mathbb{Z}_2) \cong \mathbb{Z}_2[w_4(8), w_6(8), w_7(8), w_8(8), \varepsilon(8)].$$

The only nonzero integer cohomology groups through dimension 8 are

$$\begin{aligned}
&H^0(BSpin(8); \mathbb{Z}) \cong \mathbb{Z} && \\
&H^4(BSpin(8); \mathbb{Z}) \cong \mathbb{Z} && \textit{with generator } q_1(8) \\
&H^7(BSpin(8); \mathbb{Z}) \cong \mathbb{Z}_2 && \textit{with generator } \delta w_6(8) \\
&H^8(BSpin(8); \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} && \textit{with generators } q_1^2(8), q_2(8), e(8)
\end{aligned}$$

where $q_1(8)$, $q_2(8)$ and $\varepsilon(8)$ are defined by the relations

$$(8) \quad p_1(8) = 2q_1(8), \qquad p_2(8) = q_1^2(8) + 2e(8) + 4q_2(8), \qquad \varrho_2 q_2(8) = \varepsilon(8).$$

Moreover

$$(9) \qquad \varrho_2 q_1(8) = w_4(8), \qquad \varrho_2 e(8) = w_8(8)$$

$$(10) \qquad m^* q_1(8) = q_1(8) \otimes 1, \qquad m^* e(8) = e(8) \otimes 1$$

$$(11) \qquad m^* q_2(8) = q_2(8) \otimes 1 + \delta(w_6(8) \otimes \iota_1) + q_1(8) \otimes \delta\iota_1^3 + 1 \otimes \delta\iota_1^7$$

$$(12) \qquad \pi^* q_1(8) = q_1(6), \qquad \pi^* q_2(8) = q_2(6), \qquad \pi^* e(8) = 0.$$

Remark. It can be shown that

$$H^*(BSpin(8);\mathbb{Z}) \cong \mathbb{Z}[q_1(8), q_2(8), p_3(8), e(8), \delta w_6(8)]/(2\delta w_6(8)).$$

The proof will be given elsewhere. □

Proof of Lemma 2.2: From (1), (2) and (3) we get the existence of $q_1(8)$ and $q_2(8)$ such that the first two formulas in (8) hold. Using the Serre exact sequences for the fibrations $S^6 \to BSpin(6) \to BSpin(7)$ and $S^7 \to BSpin(7) \to BSpin(8)$ we can compute $H^*(BSpin(8);\mathbb{Z})$ through dimension 8 from $H^*(BSpin(6);\mathbb{Z})$. Simultaneously, we get (9) and (12). Comparison with [Q] gives the formula for the mod 2 cohomology ring where $\varepsilon(8)$ is defined in (8) and satisfies $l^*\varepsilon(8) = \iota_1^8$. The first formula in (10) is a consequence of the fact that $l^*w_4(8) = 0$.

It remains to prove the second formula in (10) and (11), which is similar to the proof of (7) in Lemma 2.1. From [Q] it follows that there is $\varepsilon' = \varepsilon(8) + rw_4^2(8) + sw_8(8)$, $r, s \in \{0,1\}$ such that

$$\varepsilon' = w_8(\Delta) \tag{13}$$

where Δ is the real spin representation of $Spin(8)$ in $\mathbb{R}^8$. We look for $m^*\varepsilon'$ in the form

$$\varepsilon' \otimes 1 + aw_7(8) \otimes \iota_1 + bw_6(8) \otimes \iota_1^2 + cw_4(8) \otimes \iota_1^4 + 1 \otimes \iota_1^8. \tag{14}$$

Computing $Sq^2m^*\varepsilon'$, $Sq^4m^*\varepsilon'$ and $Sq^1m^*\varepsilon'$ from (13) and (14) and using the formula $w_4(8) = w_4(\Delta)$ from [Q], we obtain $a = b = c = 1$. Using $l^*w_8(8) = 0$ we can show that $m^*w_8(8) = w_8(8) \otimes 1$ in the similar way. Now we can easily find out that the formula for $m^*\varepsilon(8)$ has the same form as that for $m^*\varepsilon'$. It gives the only possibility for $m^*q_2(8)$, namely the formula (11). The same applies to $m^*e(8)$. This completes the proof. □

Lemma 2.3. *In the cohomology ring $H^*(BSpin;\mathbb{Z}_2)$ the Stiefel-Whitney classes w_{2^r+1} are equal to zero for $r \geq 0$ and*

$$H^*(BSpin;\mathbb{Z}_2) \cong \mathbb{Z}_2[w_4, w_6, w_7, w_8, w_{10}, \dots].$$

The only nonzero integer cohomology groups through dimension 8 are

$$\begin{aligned} &H^0(BSpin;\mathbb{Z}) \cong \mathbb{Z} && \\ &H^4(BSpin;\mathbb{Z}) \cong \mathbb{Z} && \text{with generator } q_1\\ &H^7(BSpin;\mathbb{Z}) \cong \mathbb{Z}_2 && \text{with generator } \delta w_6\\ &H^8(BSpin;\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z} && \text{with generators } q_1^2, q_2 \end{aligned}$$

where q_1 and q_2 are determined by the relations

$$p_1 = 2q_1, \qquad p_2 = q_1^2 + 2q_2.$$

Moreover,

$$\varrho_2 q_1 = w_4, \qquad \varrho_2 q_2 = w_8$$

$$m^* q_1 = q_1 \otimes 1, \qquad m^* q_2 = q_2 \otimes 1 \tag{15}$$

$$\pi^* q_1 = q_1(6), \qquad \pi^* q_2 = 2q_2(6). \tag{16}$$

PROOF: Much more on $H^*(BSpin)$ was proved in [T4]. (4) of Lemma 2.1 implies (16) and (15) follows from (7) using the fact that $\pi \circ m = m \circ (\pi \times id)$. □

3. Spin characteristic classes

Let ξ be an 8-dimensional oriented vector bundle over a CW-complex X with $w_2(\xi) = 0$. Then there is a mapping $\eta : X \to BSpin(8)$ such that the following diagram is commutative.

$$\begin{tikzcd}
 & K(\mathbb{Z}_2,1) \arrow[d, "i"] \\
 & BSpin(8) \arrow[d, "\mu"] \\
X \arrow[ur, dotted, "\eta"] \arrow[r, "\xi"] & BSO(8)
\end{tikzcd}$$

We define

$$q_1(\xi) = \eta^* q_1(8).$$

The definition is correct since for two liftings η_1, η_2 of ξ we have $\eta_2 = m(\eta_1, \zeta)$, where $\zeta : X \to K(\mathbb{Z}_2, 1) \cong \Omega K(\mathbb{Z}_2, 2)$ and

$$\eta_2^* q_1(8) = (\eta_1 \times \zeta)^* m^* q_1(8) = \eta_1^* q_1(8).$$

Further, we define

$$Q_2(\xi) = \{\eta^* q_2(8); \mu \circ \eta = \xi\}.$$

The indeterminacy of this class is given by $m^* q_2(8)$ (see Lemma 2.2) and is equal to

$$\text{Indet}\,(Q_2, \xi, X) = \{\delta(w_6(\xi)x) + q_1(\xi)\delta x^3 + \delta x^7; x \in H^1(X; \mathbb{Z}_2)\}.$$

Analogously,

$$E(\xi) = \{\eta^* \varepsilon(8); \mu \circ \eta = \xi\}.$$

and the indeterminacy of this class is equal to

$$\text{Indet}\,(E, \xi, X) = \{w_7(\xi)x + w_6(\xi)x^2 + w_4(\xi)x^4 + x^8; x \in H^1(X; \mathbb{Z}_2)\}.$$

In the same way we can define stable spinor classes $q_1^s(\xi)$ and $q_2^s(\xi)$ for every oriented stable vector bundle ξ with $w_2(\xi) = 0$. These classes are determined uniquely since $m^*q_r = q_r \otimes 1$ for $r = 1, 2$ (see Lemma 2.3). Moreover, for every 8-dimensional oriented vector bundle ξ with $w_2(\xi) = 0$ we get that

$$q_1^s(\xi) = q_1(\xi)$$
$$q_2^s(\xi) \in 2Q_2(\xi) + e(\xi).$$

So we will abandon the upper index in $q_1^s(\xi)$.

Lemma 3.1. *Let one of the following conditions be satisfied*

(i) $H^8(X;\mathbb{Z})$ *has no element of order* 2,

(ii) X *is simply connected.*

Then

$$\mathit{Indet}\,(Q_2, \xi, X) = \mathit{Indet}\,(E, \xi, X) = 0.$$

Proof: (i) Since $2\mathrm{Indet}\,(Q_2, \xi, X) = 0$ and $\mathrm{Indet}\,(E, \xi, X) = \varrho_2 \mathrm{Indet}\,(Q_2, \xi, X)$, we get the conclusion immediately.

(ii) is obvious since $H^1(X;\mathbb{Z}_2) = 0$. □

Notation: If the indeterminacy of $Q_2(\xi)$ or $E(\xi)$ is zero, we shall write $q_2(\xi)$ and $\varepsilon(\xi)$ instead of $Q_2(\xi)$ and $E(\xi)$, respectively, to emphasize this fact. Then $q_2^s(\xi) = 2q_2(\xi) + e(\xi)$. □

Lemma 3.2 (Computation of $q_1(\xi)$)**.** *If* $H^4(X;\mathbb{Z})$ *has no element of order* 4, *then the class* $q_1(\xi)$ *is uniquely determined by the relations*

$$2q_1(\xi) = p_1(\xi)$$
$$\varrho_2 q_1(\xi) = w_4(\xi).$$

Proof: Let two classes x_1 and x_2 satisfy the above relations. Then $x_2 = x_1 + 2y$ for some $y \in H^4(X;\mathbb{Z})$, and

$$p_1(\xi) = 2x_2 = 2x_1 + 4y = p_1(\xi) + 4y.$$

Hence $4y = 0$ implies $2y = 0$, and we get $x_1 = x_2$. □

Lemma 3.3 (Computation of $q_2(\xi)$ and $q_2^s(\xi)$)**.** *If* $H^8(X;\mathbb{Z})$ *has no element of order* 2, *then the classes* $q_2(\xi)$ *and* $q_2^s(\xi)$ *are uniquely determined by the relations*

$$16q_2(\xi) = 4p_2(\xi) - p_1^2(\xi) - 8e(\xi)$$
$$8q_2^s(\xi) = 4p_2(\xi) - p_1^2(\xi).$$

Proof: $q_2(\xi)$ and $q_2^s(\xi) \in H^8(X;\mathbb{Z})$ satisfy the formulas. □

4. Secondary operation

On integral classes of dimension 4 we have

$$Sq^2\varrho_2(\delta Sq^2\varrho_2 u) = Sq^2 Sq^1 Sq^2 \varrho_2 u = Sq^2 Sq^3 \varrho_2 u = Sq^1 Sq^4 \varrho_2 u + Sq^4 Sq^1 \varrho_2 u$$
$$= Sq^1 \varrho_2 u^2 = 0.$$

Let Ω denote a secondary operation associated with the relation

$$(Sq^2\varrho_2) \circ (\delta Sq^2 \varrho_2) = 0. \tag{17}$$

Its indeterminacy on the CW-complex X is

$$\mathrm{Indet}\,(\Omega, X) = Sq^2\varrho_2 H^6(X;\mathbb{Z}).$$

The operation is not uniquely specified by the above relation, for $\Omega' = \Omega + Sq^4$ is a second operation also associated with (17). We normalize the operation as follows. Let $\mathbb{H}P^2$ denote the quaternionic projective plane. We can regard $\mathbb{H}P^2$ as 8-skeleton of the classifying space for the special unitary group $SU(2)$. Let $x \in H^4(\mathbb{H}P^2;\mathbb{Z})$ denote the restriction of the universal Chern class c_2 to $\mathbb{H}P^2$. Then $H^*(\mathbb{H}P^2;\mathbb{Z}) \cong \mathbb{Z}[x]/x^3$. We will let Ω denote the unique operation associated with (17) such that

$$\varrho_2 x^2 \in \Omega(x).$$

According to [T2] this operation satisfies the following

Lemma 4.1. (i) *Let $u, v \in H^4(X;\mathbb{Z})$ be elements from the domain of Ω. Then*

$$\Omega(u+v) = \Omega(u) + \Omega(v) + \{u \cdot v\},$$

where $\{u\cdot v\}$ denotes the image of $\varrho_2(u\cdot v)$ in $H^8(X;\mathbb{Z}_2)/Sq^2\varrho_2 H^6(X;\mathbb{Z})$.

(ii) *Let w be any element in $H^4(X;\mathbb{Z})$. Then $2w$ belongs to the domain of Ω, and $\Omega(2w) = \{w^2\}$.*

Let M be a smooth 8-dimensional spin manifold, i.e. $w_1(M) = w_2(M) = 0$. We denote by τ_M the tangent bundle of M. The indeterminacy of Q_2 and E on the manifold M is zero and we write $q_1(M)$, $q_2(M)$, $\varepsilon(M)$ and $q_2^s(M)$ instead of $q_1(\tau_M)$, $Q_2(\tau_M)$, $E(\tau_M)$ and $q_2^s(\tau_M)$, respectively.

Lemma 4.2. *Let M be an 8-dimensional compact spin manifold, and let $H^4(M;\mathbb{Z})$ have no element of order 4. Then*

$$\Omega(q_1(M)) = 0,$$

where Ω is the secondary cohomology operation associated with the relation (17).

Proof: First, $\mathrm{Indet}\,(\Omega, M) = Sq^2\varrho_2 H^6(M;\mathbb{Z}) = w_2(M)\cdot\varrho_2 H^6(M;\mathbb{Z}) = 0$. Further, let M_6 denote the 6-skeleton of M. Since $\delta w_2(M) = 0$, τ_M restricted to M_6 has a stable complex structure ω. Let $c_i(\omega)$ denote the i-th Chern class of ω. E. Thomas in [T2] proved that

$$w_4^2(M) \in \Omega(c_2(\omega)).$$

Since $p_1(M) = c_1^2(\omega) - 2c_2(\omega)$ and $\varrho_2 c_1(\omega) = w_2(M) = 0$ we have

$$2q_1(M) = p_1(M) = 2(2x^2 - c_2(\omega))$$

for some $x \in H^2(M;\mathbb{Z})$. Further

$$\varrho_2(2x^2 - c_2(\omega)) = w_4(M) = \varrho_2(q_1(M)).$$

Due to Lemma 3.2 we get

$$q_1(M) = 2x^2 - c_2(\omega).$$

Consequently, Lemma 4.1 yields

$$\begin{aligned}\Omega(q_1(M)) &= \Omega(2x^2) + \Omega(-c_2(\omega)) = \varrho_2 x^4 + \Omega(c_2(\omega)) + \Omega(-2c_2(\omega))\\ &= \varrho_2 x^4 + w_4^2(M) + w_4^2(M) = \varrho_2 x^4.\end{aligned}$$

Since $\varrho_2 x^4 = Sq^2\varrho_2 x^3 = w_2(M)\cdot\varrho_2 x^3 = 0$, we obtain $\Omega(q_1(M)) = 0$. □

5. Existence of 2-fields

In this section ξ will denote either an 8-dimensional oriented vector bundle or a stable oriented vector bundle of geometric dimension 8 over an 8-dimensional CW-complex X with $w_2(\xi) = 0$. The maximal number of linearly independent sections in a vector bundle ξ is called span of ξ. If a stable vector bundle ξ (over an 8-dimensional complex) is stably equivalent to a 6-dimensional vector bundle, we say that stable span of ξ is ≥ 2. Now we are in position to state the main results.

Theorem 5.1. *Let ξ be an 8-dimensional oriented vector bundle over a CW-complex X of dimension 8 with $w_2(\xi) = 0$. Then span $(\xi) \geq 2$ if and only if*

(i) $e(\xi) = 0$, $\delta w_6(\xi) = 0$,

(ii) *There is $\varepsilon \in E(\xi)$ such that*

$$\varepsilon \in \Omega(q_1(\xi)),$$

where $q_1(\xi)$ and $E(\xi)$ are the spin characteristic classes defined in Section 3, and Ω is the secondary cohomology operation defined in Section 4.

Theorem 5.2. *Let ξ be a stable oriented vector bundle of geometric dimension 8 over a CW-complex X of dimension 8 with $w_2(\xi)=0$. Then stable span$(\xi)\geq 2$ if and only if*

(i) $w_8(\xi)=0$, $\delta w_6(\xi)=0$,
(ii) $\varrho_4 q_2^s(\xi)\in i_*\Omega(q_1^s(\xi))$,

where $q_1^s(\xi)$ and $q_2^s(\xi)$ are the spin characteristic classes defined in Section 3, and Ω is the secondary cohomology operation defined in Section 4.

Remark. The condition (ii) of Theorem 5.2 can be replaced by

(iii) $q_2^s(\xi)=2q$ and $\varrho_2 q\in\Omega(q_1^s(\xi))$.

□

Proof of Theorem 1.1: In [Ma] the author proved that $\delta w_{2n-2}(M)=0$ on $2n$-dimensional compact smooth manifolds. Hence

$$\delta w_6(\xi)=\delta Sq^2 w_4(\xi)=\delta Sq^2 w_4(M)=\delta w_6(M)=0.$$

Since $\varrho_2 q_1(\xi)=w_4(\xi)=w_4(M)=\varrho_2 q_1(M)$ there is $y\in H^4(M;\mathbb{Z})$ such that $2y=q_1(\xi)-q_1(M)$, and consequently

$$4y=p_1(\xi)-p_1(M).$$

Due to Lemma 4.1 and 4.2 we get

$$\Omega(q_1(\xi))=\Omega(q_1(M)+2y)=\Omega(q_1(M))+\Omega(2y)=\varrho_2 y^2.$$

Then (ii) of Theorem 5.1 is equivalent to

$$\varrho_2 q_2(\xi)=\varrho_2 y^2.$$

Since $H^8(M;\mathbb{Z})\cong\mathbb{Z}$, this is the same as

$$\begin{aligned}0&=\varrho_{32}(16q_2(\xi)-(p_1(\xi)-p_1(M))^2)=\\&=\varrho_{32}(4p_2(\xi)-p_1^2(\xi)-p_1^2(\xi)+2p_1(\xi)p_1(M)-p_1^2(M)),\end{aligned}$$

which yields the formula in Theorem 1.1. □

Remark. Using Theorem 5.2 and the remark following it, one can prove a similar result for the stable span replacing the condition $e(\xi)=0$ by $w_8(\xi)=0$. □

In the case of tangent bundle, Theorem 1.1 yields a necessary and sufficient condition for the existence of 2 linearly independent vector fields in the form $\chi(M)=0$ and $\varrho_2 q_2(M)=0$. The second condition is equivalent to $2\mid q_2(M)[M]$. In [AD] and [F] this condition is given in terms of the Euler characteristic and the signature: $\chi(M) = 0$ and $4 \mid \sigma(M)$. Using the Signature Theorem, the second condition reads for spin manifolds as

$$4 \mid q_1^2(M)[M].$$

Now, we shall show that the both conditions are equivalent. According to [H, Theorem 26.3.1], the $\hat{A}$-genus of the spin manifold M

$$\hat{A}_2[M] = \frac{1}{2^8} \cdot \frac{2}{45} \cdot (-4p_2(M) + 7p_1^2(M))[M]$$

is an integer. In terms of the spin characteristic classes this implies that

$$\left(\frac{3q_1^2(M)}{4} - \frac{q_2(M)}{2} \right) [M]$$

is an integer, which yields the equivalence of the above conditions.

Corollary 5.3. *Let ξ be an 8-dimensional oriented vector bundle over a CW-complex X of dimension 8 with $w_2(\xi) = w_4(\xi) = 0$. Then span$(\xi) \geq 2$ if and only if*

(i) $e(\xi) = 0$, $\delta w_6(\xi) = 0$,

(ii) *There is $\varepsilon \in E(\xi)$ such that*

$$\varepsilon + \varrho_2 y^2 \in Sq^2 \varrho_2 H^6(X;\mathbb{Z})$$

where $2y = q_1(\xi)$.

Remark. A similar corollary can be formulated for the stable span. □

PROOF: Since $\varrho_2 q_1(\xi) = w_4(\xi) = 0$, there is $y \in H^4(X;\mathbb{Z})$ such that $q_1(\xi) = 2y$. Lemma 4.1 implies that

$$\Omega(q_1(\xi)) = \Omega(2y) = \varrho_2 y^2 + Sq^2 \varrho_2 H^6(X;\mathbb{Z}).$$

After substituting this formula into (ii) of Theorem 5.1, we obtain (ii) of Corollary 5.3.

□

Next we show two examples where Theorem 5.1 can be directly applied.

Example 5.4. Let us consider an 8-dimensional oriented vector bundle ξ over $X = S^4 \times S^4$ with $e(\xi) = 0$. We take generators $g_1, g_2 \in H^4(S^4 \times S^4;\mathbb{Z})$ and $g \in H^8(S^4 \times S^4;\mathbb{Z})$ with $g_1 g_2 = g$. All characteristic classes in this example are the characteristic classes of ξ. There are $k_1, k_2 \in \mathbb{Z}$ such that $q_1 = k_1 g_1 + k_2 g_2$. Then

$$p_1 = 2(k_1 g_1 + k_2 g_2). \tag{18}$$

Now we get (the indeterminacy of Ω is zero)

$$\Omega(q_1) = \Omega(k_1g_1 + k_2g_2) = \Omega(k_1g_1) + \Omega(k_2g_2) + \varrho_2(k_1k_2g) = \varrho_2(k_1k_2g).$$

Let $q_2 = mg$. Because $p_2 = q_1^2 + 4q_2$, we get easily

$$p_2 = 2k_1k_2g + 4mg. \tag{19}$$

Thus, according to Theorem 5.1, ξ admits two linearly independent sections if and only if

$$\varrho_2(mg) = \varrho_2(k_1k_2g).$$

Now it suffices to change the form of this condition. We get easily

$$\varrho_8((4m - 4k_1k_2)g) = 0.$$

Using (19), we obtain

$$\varrho_{32}(4p_2 - 24k_1k_2g) = 0.$$

(18) implies

$$p_1^2 = (8k_1k_2)g.$$

Using this we have

$$\varrho_{32}(4p_2 - p_1^2 - 16k_1k_2g) = 0.$$

From (19) we have $\varrho_{32}(8p_2 - 16k_1k_2g) = 0$. Using this relation we get finally

$$\varrho_{32}(4p_2 + p_1^2) = 0.$$

Summarizing, we have proved that an oriented 8-dimensional vector bundle ξ over $S^4 \times S^4$ admits two linearly independent sections if and only if $e(\xi) = 0$ and $\varrho_{32}(4p_2(\xi) + p_1^2(\xi)) = 0$. $\square$

Example 5.5. Let us take the complex Grassmann manifold $G_{4,2}(\mathbb{C})$. It is a compact real manifold of dimension 8. We shall consider a spin vector bundle ξ over $G_{4,2}(\mathbb{C})$.

$H^*(G_{4,2}(\mathbb{C}); \mathbb{Z}) \cong \mathbb{Z}[x_1, x_2]/(x_1^3 - 2x_1x_2, x_2^2 - x_1^2x_2)$. The isomorphism is given by $x_1 \mapsto c_1$, $x_2 \mapsto c_2$, where c_1 and c_2 are Chern classes of the canonical complex vector bundle γ_2 over $G_{4,2}(\mathbb{C})$.

Let us write

$$p_1(\xi) = Ac_1^2 + Bc_2, \quad p_2(\xi) = Cc_1^2c_2, \quad e(\xi) = Dc_1^2c_2.$$

We have $p_1(\xi) = 2q_1(\xi)$, and consequently A and B are even.

We shall now investigate the relation $\varepsilon \in \Omega(q_1(\xi))$. An easy computation gives $\delta Sq^2\varrho_2(c_1^2) = \delta Sq^2\varrho_2(c_2) = 0$, which shows that the domain of Ω is the whole group $H^4(G_{4,2}(\mathbb{C}); \mathbb{Z})$. Furthermore, $Sq^2\varrho_2(c_1c_2) = 0$, which implies that

Indet $(\Omega, G_{4,2}(\mathbb{C})) = 0$. Let us compute now $\Omega(c_1^2), \Omega(c_2) \in H^8(G_{4,2}(\mathbb{C}); \mathbb{Z}_2)$. E. Thomas [T2] proved that the stable Chern classes $c_i(\infty) \in H^*(BU; \mathbb{Z})$ satisfy

$$\varrho_2(c_4(\infty) + c_2^2(\infty) + c_1^2(\infty)c_2(\infty)) \in \Omega(c_2(\infty)). \tag{20}$$

For the total Chern class of the complex vector bundle $\gamma_2 \oplus \gamma_2$ over $G_{4,2}(\mathbb{C})$ we find easily

$$c(\gamma_2 \oplus \gamma_2) = 1 + 2c_1 + (2c_2 + c_1^2) + 2c_1c_2 + c_2^2.$$

Using (20), we get

$$\varrho_2(c_2^2 + (2c_2 + c_1^2)^2 + 4c_1^2(2c_2 + c_1^2)) \in \Omega(2c_2 + c_1^2),$$

or equivalently

$$\Omega(2c_2 + c_1^2) = w_2^2 w_4.$$

Now, we have

$$\begin{aligned}\Omega(c_1^2) &= \Omega((2c_2 + c_1^2) + (-2c_2)) = \\ &= \Omega(2c_2 + c_1^2) + \Omega(-2c_2) = \\ &= w_2^2 w_4 + \varrho_2(c_2^2) = 0.\end{aligned}$$

An easy induction shows that

$$\Omega(nc_1^2) = 0 \quad \text{for every} \quad n \in \mathbb{Z}.$$

Using (20) for the vector bundle γ_2, we get

$$\varrho_2(c_2^2 + c_1^2 c_2) \in \Omega(c_2),$$

or equivalently

$$\Omega(c_2) = 0.$$

Here the induction gives

$$\Omega(nc_2) = \varrho_2\left(\frac{n(n-1)}{2} c_1^2 c_2\right) \quad \text{for every} \quad n \in \mathbb{Z}.$$

Using the above results, we can compute

$$\begin{aligned}\Omega(q_1(\xi)) &= \Omega\left(\frac{A}{2}c_1^2 + \frac{B}{2}c_2\right) = \\ &= \Omega\left(\frac{A}{2}c_1^2\right) + \Omega\left(\frac{B}{2}c_2\right) + \varrho_2\left(\frac{AB}{4}c_1^2 c_2\right) = \\ &= \varrho_2\left(\left(\frac{1}{2}\cdot\frac{B}{2}\left(\frac{B}{2} - 1\right) + \frac{AB}{4}\right) c_1^2 c_2\right) = \\ &= \varrho_2\left(\left(\frac{1}{8}B(B-2) + \frac{1}{4}AB\right) c_1^2 c_2\right).\end{aligned}$$

On the other hand, it is obvious that Indet $(E,\xi,G_{4,2}(\mathbb{C}))=0$. Consequently,

$$E(\xi)=\varrho_2(q_2(\xi)).$$

Obviously *span* $(\xi)\geq 2$ if and only if

$$\varrho_2(q_2(\xi))=\varrho_2\left(\left(\frac{1}{8}B(B-2)+\frac{1}{4}AB\right)c_1^2c_2\right),$$

or equivalently

$$\varrho_{32}(16q_2(\xi))=\varrho_{32}((2B(B-2)+4AB)c_1^2c_2).$$

Setting $D=0$, we get

$$\begin{aligned}16q_2(\xi)&=4p_2(\xi)-p_1^2(\xi)=4Cc_1^2c_2-(Ac_1^2+Bc_2)^2=\\&=(4C-2A^2-2AB-B^2)c_1^2c_2.\end{aligned}$$

The above condition can now be written in the form

$$4C-2A^2-2AB-B^2\equiv 2B(B-2)+4AB \mod 32,$$

or equivalently

$$4C\equiv 2A^2+6AB+3B^2-4B \mod 32. \tag{21}$$

We have proved that an 8-dimensional spin vector bundle ξ over $G_{4,2}(\mathbb{C})$ has two linearly independent sections if and only if $D=0$ and the condition (21) is satisfied. □

The results on the stable span make possible further applications; for instance to decide whether a given map $f:M^8\to M^{14}$ between two spin manifolds of dimension 8 and 14 is homotopic to an immersion. See [Ng].

6. Proof of Theorems 5.1 and 5.2

In this section we prove Theorem 5.1 in detail and we only sketch the proof of Theorem 5.2 since using Lemma 3.3 it proceeds in a very similar way.

We will build the Postnikov tower for the fibration

$$V_{8,2}\to BSpin(6)\xrightarrow{\pi}BSpin(8).$$

According to [P], $V_{8,2}$ is 5-connected, $\pi_6(V_{8,2})\cong\mathbb{Z}$ and $\pi_7(V_{8,2})\cong\mathbb{Z}\oplus\mathbb{Z}_2$. In [B] it is shown that $H^6(V_{8,2};\mathbb{Z})\cong\mathbb{Z}$ with a generator a_6 and $H^7(V_{8,2};\mathbb{Z})\cong\mathbb{Z}$ with a generator a_7. Moreover, their transgressions are $\delta w_6(8)\in H^7(BSpin(8);\mathbb{Z})$ and the Euler class $e(8) \in H^8(BSpin(8);\mathbb{Z})$, respectively. Denote by E the first stage of the Postnikov tower.

$$\begin{tikzcd}
\bar F \arrow[r] & V_{8,2} \arrow[r, "\bar q"] \arrow[d] & K(\mathbb{Z},6)\times K(\mathbb{Z},7) \arrow[d, "j"] & \\
F \arrow[r] & BSpin(6) \arrow[r, "q"] \arrow[d, "\pi"] & E \arrow[d, "p"] & \\
 & BSpin(8) \arrow[r, equal] & BSpin(8) \arrow[r, "(\delta w_6(8){,}e(8))"] & K(\mathbb{Z},7)\times K(\mathbb{Z},8)
\end{tikzcd}$$

Consider the situation described by the diagram. F and $\bar F$ are homotopy equivalent. Hence F is 6-connected and $\pi_7(F) \cong \mathbb{Z}_2$. That is why the next invariant k belongs to $H^8(E;\mathbb{Z}_2)$. Using the Serre exact sequence for the fibration

$$K(\mathbb{Z},6)\times K(\mathbb{Z},7) \xrightarrow{j} E \xrightarrow{p} BSpin(8)$$

we get that $H^8(E;\mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $p^*w_4^2(8)$, $p^*\varepsilon(8) \in H^8(E;\mathbb{Z}_2)$. Since $q^*p^*w_4^2(8) = \pi^*w_4^2(8) = w_4^2(6)$, $q^*p^*\varepsilon(8) = \varepsilon(6)$, we obtain that there is just one class k such that

$$j^*k = Sq^2\varrho_2\iota_6 \otimes 1, \qquad q^*k = 0. \tag{22}$$

For the secondary operation Ω associated with the relation (17) we will prove that

$$\Omega(\pi^*q_1(8)) = \Omega(q_1(6)) = \varepsilon(6) \tag{23}$$

in $H^8(BSpin(6);\mathbb{Z}_2)$. Using the identification $BSpin(6) \cong BSU(4)$, the inclusion $BSU(4) \overset{h}{\hookrightarrow} BU$ and the computations from [T2], we get

$$\begin{aligned}\Omega(q_1(6)) &= \Omega(-h^*c_2) \supseteq h^*(\Omega(-c_2)) = h^*(\Omega(c_2 + 2(-c_2))) = \\ &= h^*(\Omega(c_2) + \varrho_2 c_2^2) \ni h^*(\varrho_2(c_4 + c_2^2 + c_1^2 c_2) + \varrho_2 c_2^2) = \\ &= h^*\varrho_2 c_4 = \varepsilon(6).\end{aligned}$$

Since $\operatorname{Indet}(\Omega, BSpin(6)) = 0$, we get equality (23).

Now we are able to finish the proof of Theorem 5.1. Let $\xi : X \to BSpin(8)$ be a bundle such that $e(\xi) = \delta w_6(\xi) = 0$. Then there is a mapping $\zeta : X \to E$ such that $p\circ\zeta = \xi$. Define

$$k(\xi) = \{\zeta^*k, p\circ\zeta = \xi\}.$$

This class is the coset of $Sq^2\varrho_2 H^6(X,\mathbb{Z})$, which is the same as the indeterminacy of the secondary operation Ω. So Theorem 5.1 is proved when we show

$$k + p^*(\varepsilon(8)) \in \Omega(p^*q_1(8)), \tag{24}$$

since the application of ζ^* yields (ii) of Theorem 5.1.

Consider the following diagram

$$\begin{tikzcd}
 & K(\mathbb{Z},6)\times K(\mathbb{Z},7) \arrow[r, equal] \arrow[d, "\bar{j}"] & K(\mathbb{Z},6)\times K(\mathbb{Z},7) \arrow[d, "j"] & \\
 & Y\times K(\mathbb{Z},7) \arrow[d] & E \arrow[l, "f"'] \arrow[d, "p"] & BSpin(6) \arrow[l, "q"'] \arrow[d, "\pi"] \\
K(\mathbb{Z},7)\times K(\mathbb{Z},8) & K(\mathbb{Z},4) \arrow[l, "(\delta Sq^2\varrho_2\iota_4{,}0)"'] & BSpin(8) \arrow[l, "q_1(8)"'] \arrow[r, equal] & BSpin(8)
\end{tikzcd}$$

where Y is the universal example for the operation Ω and $\omega \in H^8(Y;\mathbb{Z}_2)$ defines Ω. We have

$$\begin{aligned} j^*(f^*(\omega\otimes 1)) &= \bar{j}^*(\omega\otimes 1) = Sq^2\varrho_2\iota_6\otimes 1 \\ f^*(\omega\otimes 1) &\in \Omega(p^*q_1(8)). \end{aligned}$$

Consequently

$$q^*f^*(\omega\otimes 1) \in \Omega(q^*p^*(q_1(8))) = \Omega(q_1(6)) = \varepsilon(6).$$

It means

$$\begin{aligned} j^*(f^*(\omega\otimes 1) + p^*(\varepsilon(8))) &= \bar{j}^*(\omega\otimes 1) = Sq^2\varrho_2\iota_6\otimes 1 \\ q^*(f^*(\omega\otimes 1) + p^*(\varepsilon(8))) &= 0 \end{aligned}$$

and consequently, (22) yields $k = f^*(\omega\otimes 1) + p^*(\varepsilon(8))$, which implies (24).

Remark. q_1 is a generating class for the invariant k in the sense of [T5]. $\square$

Sketch of the proof of Theorem 5.2: For similar objects as in the previous proof we will use the same letters (j, p, E). First we will build the Postnikov tower for the fibration

$$V \to BSpin(6) \xrightarrow{\pi} BSpin.$$

Since $\pi_6(V)\cong\mathbb{Z}$ and $\pi_7(V)\cong\mathbb{Z}_4$, the first obstruction is equal to δw_6. Let E be the first stage of the Postnikov tower. The next invariant is $i_*k \in H^8(E;\mathbb{Z}_4)$ where $i_* : H^8(E;\mathbb{Z}_2)\to H^8(E;\mathbb{Z}_4)$ is an isomorphism and k is uniquely determined by the relations

$$\begin{aligned} j^*k &= Sq^2\iota_6 \\ q^*k &= 0. \end{aligned}$$

As in the previous proof we define $i_*k(\xi)$ and get

$$i_*k + p^*\varrho_4 q_2^s \in i_*\Omega(p^*q_1) \quad \text{in } H^8(E,\mathbb{Z}_4)$$

using the facts that $\pi^*q_2^s = 2q_2(6)$ and $i\circ\varrho_2 = \varrho_4\circ 2$. It yields the condition (ii) in Theorem 5.2 and completes the proof. $\square$

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