A note on intersection dimensions of graph classes

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Abstract. The intersection dimension of a graph G with respect to a class $\mathcal{A}$ of graphs is the minimum k such that G is the intersection of some k graphs on the vertex set $V(G)$ belonging to $\mathcal{A}$. In this paper we follow [Kratochvíl J., Tuza Z.: *Intersection dimensions of graph classes*, Graphs and Combinatorics 10 (1994), 159–168] and show that for some pairs of graph classes $\mathcal{A}$, $\mathcal{B}$ the intersection dimension of graphs from $\mathcal{B}$ with respect to $\mathcal{A}$ is unbounded.

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1. Introduction

In this paper we consider finite undirected graphs without loops or multiple edges. Classes of graphs are understood to be closed under isomorphism. The vertex set (edge set) of a graph G is denoted $V(G)$ ($E(G)$), and we write $G = (V(G), E(G))$. Given a set V, we denote by K_V the complete graph and by D_V the discrete graph (graph with no edges) on the vertex set V. For some set V, $|V| = n$, we also denote by K_n (D_n) the graph K_V (D_V). The complement $-G$ of a graph G is the graph $(V(G), \binom{V}{2} - E(G))$. As the intersection (union) of two graphs G, H on the same vertex set we understand the graph $(V(G) = V(H), E(G) \cap E(H))$ $((V(G) = V(H), E(G) \cup E(H)))$. The Zykov sum $G \oplus H$ of two graphs G, H is defined as their disjoint union plus all edges between the graphs.

We call a set of vertices $U \subseteq V(G)$ in a graph G independent if there is no edge between the vertices of U. Similarly we call $U \subseteq V(G)$ a clique if there are all edges between them in G. A chromatic number of a graph G (denoted by $\chi(G)$) is the minimum number of colours needed to colour the vertices of G such that no edge has end vertices of the same colour. It is easy to see that for every two graphs G, H, $\chi(G \cup H) \leq \chi(G) \cdot \chi(H)$ (we colour vertices of $G \cup H$ by pairs of colours composed from colours of proper colourings of G and H). A graph of chromatic number 2 is called bipartite.

Several types of graph dimensions that could be seen as intersection dimensions with respect to some special graph classes (boxicity as intersection dimension with respect to interval graphs, circular dimension – circular-arc graphs, overlap dimension – circle graphs, see [1], [3]) have been previously studied. The notion of intersection dimension (with respect to graphs having some property) was actually introduced by Cozzens and Roberts in [2]. We define it in a slightly different way as in [8]:

Definition 1.1. *Given a class $\mathcal{A}$ of graphs and a graph $G = (V, E)$, the* intersection dimension of G with respect to $\mathcal{A}$ (*otherwise called the* $\mathcal{A}$-dimension of G) *is*

$$\dim_{\mathcal{A}} G = \min\{\ k \mid \exists E_1, \ldots, E_k \subseteq \binom{V(G)}{2}$$
$$\text{s.t. } (V(G), E_i) \in \mathcal{A} \ \text{ for each } i \text{ and } E = \bigcap_{i=1}^{k} E_i\}\,.$$

Definition 1.2. *Given classes $\mathcal{A}$, $\mathcal{B}$ of graphs, the* intersection dimension of $\mathcal{B}$ with respect to $\mathcal{A}$ (*the* $\mathcal{A}$-dimension of $\mathcal{B}$) *is*

$$\dim_{\mathcal{A}} \mathcal{B} = \sup_{G \in \mathcal{B}} \dim_{\mathcal{A}} G$$

(*we write ∞ if the dimension is unbounded*).

The intersection graph of some set family $\mathcal{M}$ is a graph isomorphic to the graph, whose vertices are sets from $\mathcal{M}$ and two vertices are adjacent iff these two sets have nonempty intersection. We define following special classes of graphs:

Definition 1.3. Intervals graphs (*denoted by INT*) *are intersection graphs of intervals on a line,* circular-arc graphs (*CA*) *are intersection graphs of intervals* (*arcs*) *on a circle,* circle graphs (*CI*) *are intersection graphs of chords of a circle,* permutation graphs (*PER*) *are intersection graphs of straight line segments with end points on two parallel lines,* line segment graphs (*SEG*) *are intersection graphs of line segments in the plane,* function graphs (*FUN*) *are intersection graphs of continuous functions on a closed interval,* chordal graphs (*CHOR*) *are graphs in which each cycle of length greater than* 3 *has a chord,* split graphs (*SP*) *are graphs whose vertices can be divided into a clique and an independent set.*

Note that for each above defined class $\mathcal{A}$ and any graph G the $\mathcal{A}$-dimension of G is well defined, because each of these classes contains all complete graphs and all complete graphs minus an edge. There are some clear inclusions between them, e.g. *INT* $\subseteq$ *CA*, *PER* $\subseteq$ *CI*, *SP* $\subseteq$ *CHOR*, *INT* $\subseteq$ *CHOR*, *INT* $\subseteq$ *SEG*, *CI* $\subseteq$ *SEG*. For characterizations of the classes *INT*, *PER* and *FUN* see [4]. We will use here the fact that function graphs are exactly the complements of comparability graphs. As was proved in [7], for each *SEG* graph we have line segment representation where no two segments are parallel.

We include the table of dimensions computed in [8], the number $\dim_{\mathcal{A}} \mathcal{B}$ is placed in the row $\mathcal{A}$ and the column $\mathcal{B}$. We add the class *SEG* to the table and fill its known dimensions — the numbers marked by $'$ follow from *PER* $\subseteq$ *CI* $\subseteq$ *SEG*, the one marked by * follow from *INT* $\subseteq$ *SEG* and the ones marked by $^\circ$ are consequences of *. The items marked by $^{(1)}$ are proved here in Section 2, the ones marked by $^{(2)}$ are proved in Section 3 and for the ones marked by $^{(3)}$ this paper gives a different proof than [8]. The number $\dim_{SEG} FUN = \infty$ answers a question whether $FUN \subseteq SEG$ asked in [6]. There are still two remaining unknown items marked by ? in the table.

	INT	*CA*	*CI*	*PER*	*SEG*	*SP*	*CHOR*	*FUN*
INT	1	∞	∞	∞	∞'	$\infty^{(3)}$	$\infty^{(3)}$	$\infty^{(3)}$
CA	1	1	$\infty^{(2)}$	$\infty^{(2)}$	$\infty^{(2)}$	$\infty^{(1)}$	$\infty^{(1)}$	$\infty^{(3)}$
CI	∞	∞	1	1	∞°	$\infty^{(1)}$	$\infty^{(1)}$	$\infty^{(3)}$
PER	∞	∞	∞	1	∞°	$\infty^{(1)}$	$\infty^{(1)}$	$\infty^{(3)}$
SEG	1^*	?	$1'$	$1'$	1	$\infty^{(1)}$	$\infty^{(1)}$	$\infty^{(1)}$
SP	∞	∞	∞	∞	∞'	1	∞	∞
CHOR	1	∞	∞	∞	∞'	1	1	∞
FUN	1	2	2	1	?	2	2	1

2. Small and large classes

The main idea of this section is to compute how "large" are graph classes (how many graphs on a fixed n-element set they contain). We prove that if a class $\mathcal{B}$ is "much larger" than a class $\mathcal{A}$, then there exists a graph $G \in \mathcal{B}$ with arbitrary high $\mathcal{A}$-dimension.

Definition 2.1. *For any class $\mathcal{A}$ of graphs and any integer $n > 0$ we denote the set*

$$\mathcal{A}_n = \{G \in \mathcal{A} \mid V(G) = \{1, 2, \ldots, n\}\}\,.$$

Lemma 2.2. *Let $\mathcal{A}$, $\mathcal{B}$ be classes of graphs such that $\mathcal{A}$ contains all complete graphs and all complete graphs minus an edge (i.e. the $\mathcal{A}$-dimension of any graph is well defined). Then*

$$\dim_{\mathcal{A}} \mathcal{B} \;\geq\; \sup_{n \in N, n > 1} \frac{\log |\mathcal{B}_n|}{\log |\mathcal{A}_n|}\,.$$

Proof: If $\dim_{\mathcal{A}} \mathcal{B} = \infty$, there is nothing to prove. Let us suppose that there exist $d, n_0 \in N$ for which $d = \dim_{\mathcal{A}} \mathcal{B} < \frac{\log |\mathcal{B}_{n_0}|}{\log |\mathcal{A}_{n_0}|}$, that means $|\mathcal{B}_{n_0}| > |\mathcal{A}_{n_0}|^d$. Due to the definition of the $\mathcal{A}$-dimension of $\mathcal{B}$, each graph $G \in \mathcal{B}_{n_0}$ should be an intersection of at most d graphs from $\mathcal{A}$. We may add complete graphs K_{n_0} so that G is the intersection of exactly d graphs (not necesarily distinct) $H_1, \ldots, H_d \in \mathcal{A}$. But from Definition 2.1 we have that $H_1, \ldots, H_d \in \mathcal{A}_{n_0}$ and we can choose d graphs from $\mathcal{A}_{n_0}$ in at most $|A_{n_0}|^d$ ways, so $|B_{n_0}| \leq |A_{n_0}|^d$, a contradiction. $\square$

To use the previous Lemma 2.2, we need to compute the aproximate number of graphs in the considered classes. For this purpose we use the notation $f(n) = \Theta(g(n))$, which means that there exist constants $b > a > 0$ such that $a \cdot g(n) < f(n) < b \cdot g(n)$ for almost all n. We divide the graph classes into two groups *INT*, *CA*, *CI*, *PER*, *SEG* of aproximate size $e^{\Theta(n \log n)}$ and *SP*, *CHOR*, *FUN* of aproximate size $e^{\Theta(n^2)}$. Although it is enough to prove here the upper bound for the first group and the lower bound for the second group, the proof of tight aproximate size is not difficult.

Lemma 2.3. *For every integer n, $|CA_n| \leq (2n)^{2n}$.*

Proof: For n arcs on a circle there are at most $2n$ distinct end points of these arcs, and the intersection graph of these arcs if fully determined by their circular order. So we can construct all CA_n graphs by taking $2n$ distinct points on a circle and for each of n arcs we choose from these $2n$ possibilities the start point and the end point. Therefore $|CA_n| \leq (2n)^{2n}$. □

Lemma 2.4. *For every integer n, $|SEG_n| \leq n^{6n} \cdot \binom{n}{2}^n$.*

Proof: Firstly we determine how many possibilities there are of placing n lines denoted by $p_1, p_2, \ldots, p_n$ into the plane such that no two of them are parallel and two configurations are distinct if some line crosses the other lines in different orders. This is exactly the dual problem to the problem to compute the number of distinct order types of simple numbered configurations of n points in the plane. The upper bound of at most n^{6n} distinct configurations for that problem is proved in [5].

Each graph from SEG_n we can represent (see [7]) as an intersection graph of line segments, where no two segments are parallel. Such representation is fully determined by the configuration of n lines that the segments lie on, and by the position of each segment on its line with respect to the crossing points with other lines. The number of possibly distinct configurations of lines we determined above, and for each line that has at most $n-1$ crossings with other lines, we have at most $\binom{n}{2}$ distinct possibilities to choose the line segment. Thus the number of distinct line segment graphs with n labelled vertices is

$$|SEG_n| \leq n^{6n} \cdot \binom{n}{2}^n .$$

□

Lemma 2.5. *For every integer n, $|SP_n| \geq 2^{\frac{(n-1)^2}{4}}$.*

PROOF:

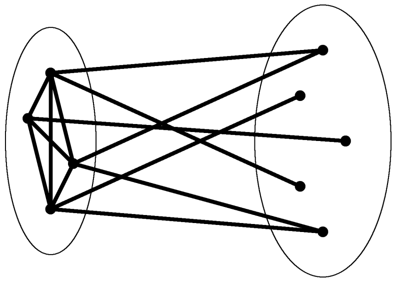

Figure 1

We may divide the vertices $\{1, 2, \ldots, n\}$ into two parts of size $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$. Then we add all edges inside the first part (see Figure 1), no edge inside the second part and an arbitrary choice of edges between them, and we get $2^{\lfloor n/2 \rfloor \cdot \lceil n/2 \rceil} \geq 2^{\frac{(n-1)^2}{4}}$ distinct graphs from SP_n. $\square$

Lemma 2.6. *For every integer n, $|FUN_n| \geq 2^{\frac{(n-1)^2}{4}}$.*

PROOF: We use nearly the same construction as in Lemma 2.5, the only difference is that we add all edges inside both parts (and arbitrary choice of edges between them). Constructed graphs are in *FUN* because their complements are bipartite, hence comparability graphs. $\square$

Theorem 2.7. *For any arbitrary pair $\mathcal{A}$ from INT, CA, CI, PER, SEG and $\mathcal{B}$ from SP, CHOR, FUN*

$$\dim_{\mathcal{A}} \mathcal{B} = \infty .$$

PROOF: From Lemma 2.3 and $INT \subseteq CA$ we have $\log |INT_n| \leq \log |CA_n| \leq \Theta(n \log n)$. Lemma 2.4 implies (we know $PER \subseteq CI \subseteq SEG$) $\log |PER_n| \leq \log |CI_n| \leq \log |SEG_n| \leq \Theta(n \log n)$. On the other hand, we know from Lemmas 2.5, 2.6 that $\log |CHOR_n| \geq \log |SP_n| \geq \Theta(n^2)$ and also $\log |FUN_n| \geq \Theta(n^2)$. Therefore for arbitrary $\mathcal{A}$ from *INT*, *CA*, *CI*, *PER*, *SEG* and $\mathcal{B}$ from *SP*, *CHOR*, *FUN* we have

$$\dim_{\mathcal{A}} \mathcal{B} = \sup_{n \in N, n > 1} \frac{\Theta(n^2)}{\Theta(n \log n)} = \infty .$$

$\square$

3. The CA-dimension

In this section we give arguments that the *CA*-dimension of permutation graphs and hence also circle and line segment graphs is unbounded, although these classes have aproximately the same number of graphs.

Lemma 3.1. *Suppose that graphs G, H are such that $-G$ is not bipartite and $G \oplus H \in CA$. Then H is a complete graph.*

Proof:

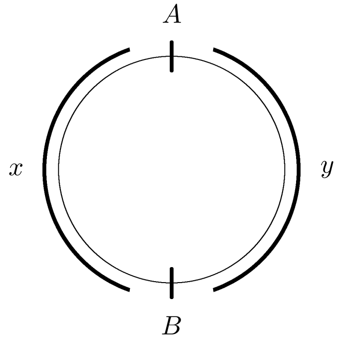

Figure 2

Let us suppose that there are two non-adjacent vertices x, y in the graph H. Consider now the CA-representation of the graph $G \oplus H$ and denote by A, B two points on the circle that lie between arcs representing vertices x, y (see Figure 2). Then for every vertex $v \in V(G)$ (which is adjacent to both x and y) the arc representing v must cross the point A or the point B, and we can divide the vertices of G into two sets (not necessarily disjoint) $V(G) = V_A \cup V_B$, such that all arcs representing the vertices from V_A cross the point A and all arcs representing the vertices from V_B cross the point B. But this means that sets V_A, V_B are independent in $-G$ and hence the graph $-G$ is bipartite, which is a contradiction. □

Lemma 3.2. *Let F be a non-complete graph such that $\dim_{CA} F \geq d$, and let $n = 2^d + 1$. Then $\dim_{CA} F \oplus D_n \geq d + 1$.*

Proof: Let us denote $V = V(F)$, $G = F \oplus D_W$ where $|W| = n$, then $V(G) = V \cup W$, $V \cap W = \emptyset$. For a contradiction let us suppose that $\dim_{CA} G \leq d$, i.e. there exist graphs $H_i \in CA$ for $i = 1, 2, \ldots, d$ such that $V(H_i) = V(G)$ and $G = \bigcap_{i=1}^{d} H_i$ (for $\dim_{CA} G < d$ we add complete graphs). Because G contains all edges between vertices from V and W, each H_i should also contain all these edges and we can write $H_i = H_i^1 \oplus H_i^2$ where $V(H_i^1) = V$, $V(H_i^2) = W$ and $\bigcap_{i=1}^{d} H_i^1 = F$, $\bigcap_{i=1}^{d} H_i^2 = D_W$.

If there exist $j \in \{1, \ldots, d\}$ such that $-H_j^2$ is not bipartite, we can use Lemma 3.1 for the graph $H_j = H_j^1 \oplus H_j^2$ and thus H_j^1 is a complete graph, which is not possible when $d = 1$ and for $d > 1$ we can write
$E(F) = \bigcap_{i \in \{1, \ldots, j-1, j+1, \ldots, d\}} E(H_i^1)$, i.e. $\dim_{CA} F \leq d - 1$, a contradiction.

Hence $-H_i^2$ is bipartite for every i. But $\bigcap_{i=1}^{d} H_i^2 = D_W$, therefore $\bigcup_{i=1}^{d} -H_i^2 = K_W$ and $n = \chi(K_W) \leq \chi(-H_1^2) \cdot \ldots \cdot \chi(-H_d^2) \leq 2^d = n-1$, a contradiction. $\square$

Theorem 3.3. $\dim_{CA} PER = \dim_{CA} CI = \dim_{CA} SEG = \infty$.

Proof: Define $G_1 = D_2$ and $G_{d+1} = G_d \oplus D_{2^d+1}$ by means of recursion. Since every discrete graph is a permutation graph, and the Zykov sum of two permutation graphs is again a permutation graph, $G_d \in PER$ for every d. On the other hand, it follows by induction from Lemma 3.2 that $\dim_{CA} G_{d+1} \geq d+1$ for every d. Hence $\dim_{CA} PER = \infty$.

$\square$

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