Estimators for epidemic alternatives

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Abstract. We introduce and study the behavior of estimators of changes in the mean value of a sequence of independent random variables in the case of so called epidemic alternatives which is one of the variants of the change point problem.

The consistency and the limit distribution of the estimators developed for this situation are shown. Moreover, the classical estimators used for 'at most change' are examined for the studied situation.

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1. Introduction

Consider the following model:

$$\begin{aligned} X_i &= \theta_0 + e_i, & i &= 1, \dots, m_1, \\ &= \theta_0 + \delta_n + e_i, & i &= m_1+1, \dots, m_2, \\ &= \theta_0 + e_i, & i &= m_2+1, \dots, n, \end{aligned}$$

where $\theta_0, \delta_n, 1 \le m_1 < m_2 < n$ are unknown parameters, $e_1, \dots, e_n$ are i.i.d. random variables, $Ee_i = 0$ and $0 < \operatorname{var} e_i = \sigma^2 < \infty$ with σ^2 unknown. The model describes the situation, where the normal state with the mean value θ_0 runs up to the m_1-th observation then it changes to the epidemic one with the mean value $\theta_0 + \delta_n$ that goes from $m_1 + 1$-st through m_2-nd observation and the normal state is restored afterwards. This model is called the epidemic alternative.

The testing problem $H_0 : \delta_n = 0$ against $H_1 : \delta_n \neq 0$ was first considered by Kline and Levin [13] for the case when e_i's have normal distribution. Yao [16] published a survey of the available test procedures together with their comparison. Lombard [14] and Gombay [9] deal with rank test procedures. Brodsky and Darkhovsky [6] constructed estimators (see (2.3) below) of the change points m_1, m_2 in a series of dependent observations and proved their consistency.

The object of the present paper is to develop estimators of the change points m_1, m_2 and to derive their asymptotic properties for local changes ($\delta_n \to 0$). Namely, we shall study an estimator related to the maximum likelihood one when

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the errors e_i's are normally distributed (see (2.2) below) and the estimator introduced by Brodsky and Darkhovsky [6] based on the argmax of differences of certain averages (see (2.3) below). Moreover, the performance of three types of estimators used for the case 'at most one change' (see (2.4)–(2.9) below) is examined in the model (1.1). It is shown that all considered estimators have the same rate of consistency and have the limit distribution as the argmax of certain Gaussian processes. The main results are formulated in Theorem 2.1 and Theorem 2.2. Results of simulation study will be published in [3].

2. Main results

The estimators of the change points are based on the partial sums

$$S_k = \sum_{i=1}^{k} (X_i - \overline{X}_n), \quad k = 1, ..., n, \tag{2.1}$$

where

$$\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i.$$

The estimator related to the maximum likelihood estimator when the errors e_i's have normal distribution is defined as follows:

$$\begin{aligned}(\hat{m}_{11}(\epsilon), \hat{m}_{21}(\epsilon)) = \ \text{argmax}\ \{ & \sqrt{\frac{n}{(k_2 - k_1)(n - k_2 + k_1)}} |S_{k_2} - S_{k_1}|; \\ & 1 \le k_i \le n,\ i = 1, 2,\ n\epsilon \le k_2 - k_1 \le (1 - \epsilon)n\},\end{aligned} \tag{2.2}$$

where $0 < \epsilon < 1/2$.

Darkhovsky and Brodsky [6] introduced the estimator

$$\begin{aligned}(\hat{m}_{12}(\epsilon), \hat{m}_{22}(\epsilon)) = & \ \text{argmax}\ \{ \frac{n^2}{(k_2 - k_1)(n - k_2 + k_1)} |S_{k_2} - S_{k_1}|; \\ & 1 \le k_i \le n,\ i = 1, 2,\ n\epsilon \le k_2 - k_1 \le (1 - \epsilon)n\} \\ = & \ \text{argmax}\ \{ | \frac{1}{(k_2 - k_1)} S_{k_2} - \frac{1}{(n - k_2 + k_1)} (S_{k_1} + S_n - S_{k_2})|; \\ & 1 \le k_i \le n,\ i = 1, 2,\ n\epsilon \le k_2 - k_1 \le (1 - \epsilon)n\},\end{aligned} \tag{2.3}$$

where $0 < \epsilon < 1/2$. They investigated the consistency when $\delta_n = \delta \ne 0$ is fixed (not depending on n) and X_i, $i = 1, ..., n$, need not be independent.

The following three estimators for the case 'at most one change' will be investigated:

$$\begin{aligned}\hat{m}_{13}(\epsilon) = \min\{ & \ \text{argmax}\ \{ \sqrt{\frac{n}{k(n-k)}} S_k; n\epsilon \le k \le n(1 - \epsilon)\}, \\ & \text{argmin}\ \{ \sqrt{\frac{n}{k(n-k)}} S_k; n\epsilon \le k \le n(1 - \epsilon)\}\},\end{aligned} \tag{2.4}$$

$$(2.5)\quad \hat{m}_{23}(\epsilon) = \max\{ \text{ argmax } \{\sqrt{\frac{n}{k(n-k)}} S_k; n\epsilon \le k \le n(1-\epsilon)\}, \text{ argmin } \{\sqrt{\frac{n}{k(n-k)}} S_k; n\epsilon \le k \le n(1-\epsilon)\}\},$$

$$(2.6)\quad \hat{m}_{14} = \min\{ \text{ argmax } \{S_k; 1 \le k_i \le n\}, \text{ argmin } \{S_k; 1 \le k \le n\}\},$$

$$(2.7)\quad \hat{m}_{24} = \max\{ \text{ argmax } \{S_k; 1 \le k_i \le n\}, \text{ argmin } \{S_k; 1 \le k \le n\}\}$$

and

$$(2.8)\quad \hat{m}_{15}(G) = \min\{ \text{ argmax } \{S_{k+G} - 2S_k + S_{k-G}; G < k < n-G\}, \text{ argmin } \{S_{k+G} - 2S_k + S_{k-G}; G < k < n-G\}\},$$

$$(2.9)\quad \hat{m}_{25}(G) = \max\{ \text{ argmax } \{S_{k+G} - 2S_k + S_{k-G}; G < k < n-G\}, \text{ argmin } \{S_{k+G} - 2S_k + S_{k-G}; G < k < n-G\}\},$$

where $0 < \epsilon < 1/2$ and G should be small w.r.t. n (see assumption (2.17) below). The behavior of the estimators $\hat{m}_{i3}(\epsilon)$ and $\hat{m}_{i4}$, $i = 1, 2$, in the case of at most one change was deeply studied in [4]. The behavior of $\hat{m}_{i5}(G)$, $i = 1, 2$, both in the case of at most one change and more changes was studied in a more general framework in [2].

The following expectations give a simple transparent picture on the behavior of the estimators

$$(2.10)\quad \begin{aligned} ES_k &= -\delta_n k\frac{m_2 - m_1}{n} && 1 \le k \le m_1,\\ &= \delta_n(k\frac{n - m_2 + m_1}{n} - m_1) && m_1 < k \le m_2,\\ &= -\delta_n(n-k)\frac{n - m_2 + m_1}{n} && m_2 < k \le n, \end{aligned}$$

and

$$(2.11)\quad \begin{aligned} E(S_{k+G} - 2S_k + S_{k-G}) &= 0 && G < k \le m_1 - G,\\ &= \delta_n(k + G - m_1) && m_1 - G < k \le m_1\\ &= \delta_n(G - k + m_1) && m_1 < k \le m_1 + G,\\ &= 0 && m_1 + G < k \le m_2 - G,\\ &= -\delta_n(k + G - m_2) && m_2 - G < k \le m_2,\\ &= -\delta_n(G - k + m_2) && m_2 < k \le m_2 + G,\\ &= 0 && m_2 + G < k < n - G. \end{aligned}$$

We see that ES_k, $k = 1, ..., n$ and $E(S_{k+G} - 2S_k + S_{k-G})$, $k = G+1, ..., n-G$, are piece-wise linear functions in k with extremes at $k = m_1, m_2$.

Now, we shall state two main results.

Theorem 2.1. *Let $X_1, ..., X_n$ follow the model* (1.1) *and let, as $n \to \infty$,*

$$\frac{m_i}{n} \to \gamma_i, \quad i = 1, 2, \quad 0 < \gamma_1 < \gamma_2 < 1 \tag{2.12}$$

$$\delta_n \to 0, \qquad |\delta_n|\sqrt{n} \to \infty \tag{2.13}$$

then

$$\frac{\delta_n^2}{\sigma^2}(\hat{m}_{ij}(\epsilon) - m_i) \xrightarrow{d} \operatorname{argmax}\{W_j(s) - |s| g_{ij}(s),\, s \in R\} \tag{2.14}$$

and

$$\frac{\delta_n^2}{\sigma^2}(\hat{m}_{i4} - m_i) \xrightarrow{d} \operatorname{argmax}\{W_4(s) - |s| g_{i4}(s),\, s \in R\} \tag{2.15}$$

for $i = 1, 2$; $j = 1, 2, 3$ and $0 < \epsilon < \min(\gamma_1, 1 - \gamma_2, \gamma_2 - \gamma_1, 1 - \gamma_2 + \gamma_1)$, where

$$\begin{aligned} W_j(s) &= W_{j1}(s) \quad s < 0 \\ &= W_{j2}(s) \quad s > 0, \end{aligned} \tag{2.16}$$

W_{j1} and W_{j2} are independent Wiener processes, $j = 1, ..., 4$,

$$\begin{aligned} g_{j1}(s) &= 1/2 && s \in R,\ j = 1, 2 \\ g_{12}(s) &= 1 - \gamma_2 + \gamma_1 && s < 0 \\ &= \gamma_2 - \gamma_1 && s > 0 \\ g_{13}(s) &= \frac{1}{2}(1 - \frac{1 - \gamma_2}{1 - \gamma_1}) && s < 0 \\ &= \frac{1}{2}(1 + \frac{1 - \gamma_2}{1 - \gamma_1}) && s > 0 \\ g_{23}(s) &= \frac{1}{2}(1 + \frac{\gamma_1}{\gamma_2}) && s < 0 \\ &= \frac{1}{2}(1 - \frac{\gamma_2}{\gamma_1}) && s > 0 \end{aligned}$$

and

$$g_{12}(s) = g_{22}(-s) = g_{14}(-s) = g_{24}(s) \quad s \in R$$

Proof: is postponed to Section 3. □

Theorem 2.2. *If the assumptions of Theorem* 2.1 *are satisfied and if, as $n \to \infty$,*

$$G/n \to 0, \quad |\delta_n|^{-2} G \ln \frac{n}{G} \to 0 \tag{2.17}$$

then

$$\frac{\delta_n^2}{\sigma^2}(\hat{m}_{i5}(G) - m_i) \xrightarrow{d} \operatorname{argmax}\,\{W_5(s) - |s|/6,\, s \in R\}, \tag{2.18}$$

where W_5 is the two-sided Wiener process described in (2.16).

Proof: is a consequence of Theorem 4.1 in [2], where we put $\psi(x, \theta) = x - \theta$, $x \in R$, $\theta \in R$. □

Remarks.

1. *If δ_n and σ^2 are replaced by consistent estimators, the assertions of Theorem 2.1 and Theorem 2.2 remain true.*

2. *Khakhubia* [12] *and Gombay and Horváth* [10] *derived the distribution of* argmax $\{W_1(s) - |s|/2,\ s \in R\}$, *which together with the previous item enables us to construct the confidence interval for m_1 and m_2.*

3. *Going through the proof of Theorem 2.1 we find that $\hat{m}_{1i}(\epsilon)$ and $\hat{m}_{2i}(\epsilon)$ are asymptotically independent, $i = 1, 2, 3$. The pairs $(\hat{m}_{14}, \hat{m}_{24})$ and $(m_{15}(G), \hat{m}_{25}(G))$ have the same property.*

4. *The M-type analogs of the estimators can be constructed as follows. Replace the residuals $X_i - \overline{X}_n$, $i = 1, ..., n$ by the M-residuals $\psi(X_i - \theta_n(\psi))$, $i = 1, .., n$, where ψ is a suitable score generating function and $\hat{\theta}_n(\psi)$ is the M-estimator of θ_0 with the score function ψ in the model* (1.1) *with $\delta_n = 0$ (i.e. $\hat{\theta}_n(\psi)$ is a solution of the equation $\sum_{i=1}^n \psi(X_i - \theta) = 0$). The respective properties can be obtained along the line of [2].*

5. *Since γ_1 and γ_2 are unknown, it is hardly to check the assumption* (2.12). *We should choose $\epsilon > 0$ sufficiently small in order the assumption* (2.12) *is met. If we put $\epsilon = 0$, the estimators need not be consistent since the maximum can be reached for $k_i \notin (m_i - n\epsilon^*, m_i + n\epsilon^*)$, $i = 1, 2$, for some $\epsilon^* > 0$ with probability larger than some positive constant. For instance, if e_i has the density*

$$
\begin{aligned}
f(x) &= \frac{2+\Delta}{2}|x|^{-3-\Delta} \qquad && |x| \geq 1 \\
&= 0 && |x| < 1
\end{aligned}
$$

with $\Delta > 0$ then

$$
\begin{aligned}
\max\{\sqrt{\frac{n}{(k_2-k_1)(n-k_2+k_1)}}|S_{k_2} - S_{k_1}|;& \\
1 \leq k_i \leq n,\ i = 1, 2,\ n\epsilon \leq k_2 - k_1 \leq (1-\epsilon)n\}& \\
= \delta_n^2 n(\gamma_2 - \gamma_1)(1 - \gamma_2 + \gamma_1)(1 + o_p(1)),&
\end{aligned}
$$

while

$$
\begin{aligned}
P(\max\{\sqrt{\frac{n}{(k_2-k_1)(n-k_2+k_1)}}|S_{k_2} - S_{k_1}|;& \\
1 \leq k_1 < k_2 \leq n\} > Q\sqrt{\log n}) \to 1&
\end{aligned}
$$

for some $Q > 0$, which means that if $|\delta_n| = o_p(n^{-1/2}\sqrt{\log n})$ the maximum need not be reached by (k_1, k_2) close to (m_1, m_2).

3. Proof of Theorem 2.1

Because of a similarity of the proofs of the assertions on $\hat{m}_{ij} - m_i$ for $i = 1, 2$, $j = 1, 2, 3, 4$ we shall treat in detail $\hat{m}_{11}(\epsilon)$ and $\hat{m}_{21}(\epsilon)$ and give an outline of the others.

The proof is divided into three steps. We start with auxiliary results, then show that the rate of consistency of the estimators $\hat{m}_{i1}(\epsilon)$ is δ_n^{-2}, i.e. $\hat{m}_{i1}(\epsilon) - m_i = O_p(\delta_n^{-2})$, $i = 1, 2$, and in the last step we derive the limit distribution of the estimator.

In the rest of the paper we shall assume $\delta_n > 0$, $n \geq 1$. The case $\delta_n < 0$ for some $n \geq 1$ can be treated quite analogously.

The estimators $(\hat{m}_{11}(\epsilon), \hat{m}_{21}(\epsilon))$ can be defined equivalently as

$$\begin{aligned} \text{argmax}\, \{ & \frac{n}{(k_2 - k_1)(n - k_2 + k_1)} (S_{k_2} - S_{k_1})^2 \\ & - \frac{n}{(m_2 - m_1)(n - m_2 + m_1)} (S_{m_2} - S_{m_1})^2; \\ & \qquad 1 \leq k_i \leq n,\ i = 1, 2;\ n\epsilon \leq k_2 - k_1 \leq n(1 - \epsilon) \}. \end{aligned} \tag{3.1}$$

Moreover, noticing

$$S_k = \sum_{i=1}^{k} (e_i - \overline{e}_n) + \delta_n \sum_{i=1}^{k} (I\{m_1 < i \leq m_2\} - \frac{m_2 - m_1}{n}), \tag{3.2}$$

where $I\{A\}$ denotes the indicator of the set A, we may write for $1 \leq k_1 < k_2 \leq n$

$$\begin{aligned} & \frac{n}{(k_2 - k_1)(n - k_2 + k_1)} (S_{k_2} - S_{k_1})^2 \\ & \qquad = A_1(k_1, k_2) + 2\delta_n A_2(k_1, k_2) + \delta_n^2 A_3(k_1, k_2), \end{aligned} \tag{3.3}$$

where

$$A_1(k_1, k_2) = \frac{n}{(k_2 - k_1)(n - k_2 + k_1)} \Big(\sum_{i=k_1+1}^{k_2} (e_i - \overline{e}_n) \Big)^2,$$

$$\begin{aligned} A_2(k_1, k_2) = \frac{n}{(k_2 - k_1)(n - k_2 + k_1)} & \sum_{i=k_1+1}^{k_2} (e_i - \overline{e}_n) \\ & \sum_{j=k_1+1}^{k_2} (I\{m_1 < j \leq m_2\} - \frac{m_2 - m_1}{n}), \end{aligned}$$

$$A_3(k_1, k_2) = \frac{n}{(k_2 - k_1)(n - k_2 + k_1)} \Big(\sum_{j=k_1+1}^{k_2} (I\{m_1 < j \leq m_2\} - \frac{m_2 - m_1}{n}) \Big)^2.$$

Useful results on $A_i(k_1, k_2)$, $i = 1, 2, 3$, are proved in the following three lemmas.

Lemma 3.1. *If the assumptions of Theorem* 1.1 *are satisfied then, as* $n \to \infty$,

$$\max\{A_1(k_1,k_2); 1 \le k_i \le n, i=1,2,\ n\epsilon \le k_2-k_1 \le n(1-\epsilon)\} = O_p(1) \tag{3.4}$$

$$\max\{|A_1(k_1,k_2)-A_1(m_1,m_2)|; |k_i-m_i| \le n\epsilon_n,\ 1 \le k_i \le n,\ i=1,2\} = O_p(\sqrt{\epsilon_n}) \tag{3.5}$$

for any $0<\epsilon<\min\{\gamma_2-\gamma_2, 1-\gamma_2+\gamma_1\}$, *for any* $\epsilon_n \ge 0$ *satisfying* $\epsilon_n \to 0$ *and* $n\epsilon_n \to \infty$.

PROOF: The first assertion is an easy consequence of the Kolmogorov inequality.

The assertion (3.5) is implied by the following relations

$$\begin{aligned} A_1(k_1,k_2)-A_1(m_1,m_2) &= \Big(\sum_{i=k_1+1}^{k_2}(e_i-\overline{e}_n)\Big)^2 \\ &\quad \frac{n(m_2-k_2-m_1+k_1)(n-m_2+m_1-k_2+k_1)}{(k_2-k_1)(m_2-m_1)(n-k_2+k_1)(n-m_2+m_1)} \\ &\quad + \frac{n}{(m_2-m_1)(n-m_2+m_1)}\Big(\sum_{i=k_1+1}^{k_2}(e_i-\overline{e}_n)-\sum_{i=m_1+1}^{m_2}(e_i-\overline{e}_n)\Big) \\ &\quad \Big(\sum_{i=k_1+1}^{k_2}(e_i-\overline{e}_n)+\sum_{i=m_1+1}^{m_2}(e_i-\overline{e}_n)\Big) \\ &= O_p(\epsilon_n)+O_p(\sqrt{\epsilon_n}) = O_p(\sqrt{\epsilon_n}). \end{aligned}$$

which holds uniformly for $|k_i-m_i| \le n\epsilon_n$, $1 \le k_i \le n$, $i=1,2$. □

Lemma 3.2. *If the assumptions of Theorem* 1.1 *are satisfied then, as* $n \to \infty$,

$$\max\{A_2(k_1,k_2), 1 \le k_i \le n, i=1,2,\ n\epsilon \le k_2-k_1 \le n(1-\epsilon)\} = O_p(\sqrt{n}) \tag{3.6}$$

$$\begin{aligned} \max\{&|A_2(k_1,k_2)-A_2(m_1,m_2)|(|k_2-m_2|+|k_1-m_1|)^{-1}; \\ &q_n\delta_n^{-2} \le |k_i-m_i| \le n\epsilon_n,\ 1 \le k_i \le n,\ i=1,2\} = O_p(\delta_n q_n^{-1/2}) \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \max\{&|A_2(k_1,k_2)-A_2(m_1,m_2)-\sum_{k_1+1}^{k_2}e_i+\sum_{m_1+1}^{m_2}e_i| \\ &(|k_2-m_2|+|k_1-m_1|)^{-1}; |k_i-m_i| \le n\epsilon_n,\ 1 \le k_i \le n,\ i=1,2\} = O_p(\epsilon_n\sqrt{n}) \end{aligned} \tag{3.8}$$

for any $0 < \epsilon < \min\{\gamma_2 - \gamma_1, 1 - \gamma_2 + \gamma_1\}$, *for any* $q_n \to \infty$ *and for any* $\epsilon_n > 0$, $n \geq 1$ *such that* $\epsilon_n \to 0$ *and* $n\epsilon_n \to \infty$.

Proof: The first assertion is a direct consequence of the Kolmogorov inequality.

Since

$$A_2(k_1,k_2) - A_2(m_1,m_2) - \sum_{k_1+1}^{k_2}(e_i - \overline{e}_n) + \sum_{m_1+1}^{m_2}(e_i - \overline{e}_n)$$

$$= \sum_{k_1+1}^{k_2}(e_i - \overline{e}_n)\frac{n}{(k_2-k_1)(n-k_2+k_1)}$$

$$\sum_{j=k_1+1}^{k_2}(I\{m_1 < j \leq m_2\} - 1 - \frac{m_2 - k_2 - m_1 + k_1}{n})$$

$$= O_p(n^{-1/2}(|k_1 - m_1| + |k_2 - m_2|))$$

holds uniformly for $|k_i - m_i| \leq n\epsilon_n$, $i = 1,2$, the assertion (3.8) is valid. Finally, by the Hájek-Rényi inequality (see [8, p. 230]) we have for any $\lambda > 0$

$$P(\max\{\frac{1}{|k_i - m_i|}|\sum_{j=1}^{k_i} e_j - \sum_{j=1}^{m_i} e_j|;$$

$$\delta_n^{-2} q_n \leq |k_i - m_i| \leq n\epsilon_n, 1 \leq k_i \leq n,\, i = 1,2\} \geq \lambda)$$

$$\leq 2\lambda^{-2}\sum^{*}(\frac{1}{j^2} - \frac{1}{(j+1)^2})j\sigma^2 \leq D_1\lambda^{-2}\delta_n^2 q_n^{-1}$$

where $\sum^*$ denotes the summation over the set $\{k_i, \delta_n^{-2}q_n \leq |k_i - m_i| \leq n\epsilon_n, 1 \leq k_i \leq n,\, i = 1,2\}$, for some $D_1 > 0$, which together with (3.8) implies (3.7). $\square$

Lemma 3.3. *Under the assumptions of Theorem* 1.1 *there exist* $B_1 > 0$ *and* $B_2 > 0$ (*not depending on* n) *such that*

(3.9) $$\max\{A_3(k_1,k_2) - A_3(m_1,m_2), n\epsilon_n \leq |k_i - m_i|,\, i = 1,2,$$
$$n\epsilon \leq k_2 - k_1 \leq n(1-\epsilon),\, i = 1,2, 1 \leq k_1 < k_2 \leq n\} \leq -B_1 n\epsilon_n$$

and

(3.10)
$$|A_3(k_1,k_2) - A_3(m_1,m_2) + |m_2 - k_2| + |m_1 - k_1||(|m_2 - k_2|^2 + |m_1 - k_1|^2)^{-1};$$
$$|k_i - m_i| \leq n\epsilon_n, 1 \leq k_i \leq n,\, i = 1,2$$

for any $0 < \epsilon < \min\{\gamma_2 - \gamma_1, 1 - \gamma_2 + \gamma_1\}$, *for any* $\epsilon_n > 0$, $n = 1,2,\ldots$ *satisfying* $\epsilon_n \to 0$ *and* $n\epsilon_n \to \infty$.

Proof: To show (3.9) we decompose the set of indices (k_1,k_2) as follows:

$$\{(k_1,k_2); |k_i - m_i| \geq n\epsilon_n, i = 1,2, n\epsilon \leq k_2 - k_1 \leq n(1-\epsilon),\, 1 \leq k_1 < k_2 \leq n\}$$
$$= C_{1n} \cup C_{2n} \cup C_{3n} \cup C_{4n} \cup C_{5n} \cup C_{6n},$$

where

$$C_{1n} = \{(k_1, k_2); 1 \le k_1 < k_2 \le m_1,\ n\epsilon \le k_2 - k_1 \le (1-\epsilon)n\}$$
$$C_{2n} = \{(k_1, k_2); m_2 < k_1 < k_2 \le n,\ n\epsilon \le k_2 - k_1 \le (1-\epsilon)n\}$$
$$C_{3n} = \{(k_1, k_2); 1 \le k_1 \le m_1 - n\epsilon_n, m_2 + n\epsilon_n \le k_2 \le n, n\epsilon \le k_2 - k_1\}$$
$$C_{4n} = \{(k_1, k_2); m_1 + n\epsilon_n \le k_1 <, k_2 \le m_2 - n\epsilon_n, n\epsilon \le k_2 - k_1\},$$
$$C_{5n} = \{(k_1, k_2); m_1 + n\epsilon_n \le k_1 \le m_2, m_2 + n\epsilon_n < k_2 \le n, n\epsilon \le k_2 - k_1 \le n(1-\epsilon)\},$$
$$C_{6n} = \{(k_1, k_2); 1 \le k_1 \le m_1 - n\epsilon_n, m_1 < k_2 \le m_2 - n\epsilon_n, n\epsilon \le k_2 - k_1 \le n(1-\epsilon)\}.$$

Direct computations give that

$$\max\{A_3(k_1, k_2) - A_3(m_1, m_2); (k_1, k_3) \in C_{1n}\} \le -\frac{m_2 - m_1}{n - m_1 + 1}(n - m_2 + m_1) \tag{3.11}$$

and

$$\max\{A_3(k_1, k_2) - A_3(m_1, m_2); (k_1, k_3) \in C_{2n}\} \le -\frac{m_2 - m_1}{m_2 + 1}(m_1 + 1) \tag{3.12}$$

Further,

$$A_3(k_1, k_2) = \frac{(m_2 - m_1)^2}{n}(-1 + \frac{n}{k_2 - k_1}) \quad \text{for} \quad (k_1, k_2) \in C_{3n}$$

and hence

$$\max\{A_3(k_1, k_2) - A_3(m_1, m_2); (k_1, k_3) \in C_{3n}\} \le \frac{(m_2 - m_1)^2}{n}(-1 + \frac{n}{m_2 - m_1 + 2n\epsilon_n}) - A_3(m_1, m_2) \le -D_2 n\epsilon_n. \tag{3.13}$$

for some $D_2 > 0$. Similarly,

$$A_3(k_1, k_2) = \frac{(n - m_2 + m_1)^2}{n}(-1 + \frac{n}{n - k_2 + k_1}) \quad \text{for} \quad (k_1, k_2) \in C_{4n}$$

which implies

$$\max\{A_3(k_1, k_2) - A_3(m_1, m_2); (k_1, k_3) \in C_{4n}\} \le \frac{(n - m_2 + m_1)^2}{n}(-1 + \frac{n}{n - m_2 - m_1 + 2n\epsilon_n}) - A_3(m_1, m_2) \le -D_3 n\epsilon_n \tag{3.14}$$

for some $D_3 > 0$. Next, for $(k_1, k_2) \in C_{5n}$

$$A_3(k_1,k_2) = A_3(m_1,m_2) - (k_2 - m_2)(1 - \frac{m_2 - k_2}{k_2 - k_1}) - (k_1 - m_1)(1 - \frac{m_1 - k_1}{n - k_2 + k_1})$$

and hence

$$\max\{A_3(k_1,k_2) - A_3(m_1,m_2); (k_1,k_2) \in C_{5n}\} \le -D_4 n\epsilon_n \tag{3.15}$$

for some $D_4 > 0$. Quite analogously, we have

$$\max\{A_3(k_1,k_2) - A_3(m_1,m_2); (k_1,k_2) \in C_{6n}\} \le -D_5 n\epsilon_n \tag{3.16}$$

for some $D_5 > 0$. Combining (3.11)–(3.16) we receive (3.9). The validity of (3.10) can be checked by direct computations. □

Denoting

$$Z_n(k_1,k_2) = \frac{n}{(k_2 - k_1)(n - k_2 + k_1)}(S_{k_2} - S_{k_1})^2, \quad 1 \le k_1 < k_2 \le n$$

we find that

$$\begin{aligned} Z_n(m_1,m_2) &= \frac{n}{(m_2 - m_1)(n - m_2 + m_1)}\Big(\sum_{i=k_1+1}^{k_2}(e_i - \overline{e}_n)\Big)^2 \\ &\quad + 2\delta_n \sum_{i=k_1+1}^{k_2}(e_i - \overline{e}_n) + \delta_n^2 \frac{(m_2 - m_1)(n - m_2 + m_1)}{n} \\ &= O_p(1) + O_p(|\delta_n|\sqrt{n}) + \delta_n^2 \frac{(m_2 - m_1)(n - m_2 + m_1)}{n} \\ &= \delta_n^2 n(\gamma_2 - \gamma_1)(1 - \gamma_2 + \gamma_1)(1 + o_p(1)). \end{aligned} \tag{3.17}$$

Next, by (3.4), (3.6), (3.7) and (3.9) we have

$$\begin{aligned} &\max\{Z_n(k_1,k_2) - Z_n(m_1,m_2); |k_i - m_i| \ge n\epsilon_n, 1 \le k_i \le n,\ i = 1,2, \\ &\quad n\epsilon \le k_2 - k_1 \le n(1-\epsilon)\} O_p(1) + O_p(\sqrt{n}|\delta_n|) - B_1 n\delta_n^2\epsilon_n \\ &\quad = -B_1 n\epsilon_n(1 + o_p(1)). \end{aligned} \tag{3.18}$$

By (3.5), (3.7) and (3.10) we obtain

$$\begin{aligned} &\max\{Z_n(k_1,k_2) - Z_n(m_1,m_2);\ \delta_n^{-2} q_n \le |k_i - m_i| < n\epsilon_n, 1 \le k_i \le n,\ i = 1,2, \\ &\quad n\epsilon \le k_2 - k_1 \le n(1-\epsilon)\} \\ &= O_p(\sqrt{|\epsilon_n|}) - B_1 q_n(1 + o(1) + o_p(q_n^{-1/2})). \end{aligned} \tag{3.19}$$

Now, (3.17), (3.18) and (3.19) imply that as $n \to \infty$ with probability tending to 1 the maximum of $Z_n(k_1, k_2) - Z_n(m_1, m_2)$ is reached for $k_i \in (m_i - n\epsilon_n, m_i + n\epsilon_n)$, $i = 1, 2$, i.e. as $n \to \infty$,

$$P(|\hat{m}_{i1} - m_i| \le q_n \delta_n^{-2}) \to 1 \quad i = 1, 2$$

for any $q_n \to \infty$.

Thus the limit distribution of

$$\operatorname{argmax}\{Z_n(k_1, k_2); 1 \le k_i \le n, i = 1, 2, n\epsilon_n \le k_2 - k_1 \le n(1 - \epsilon_n)\}$$

is the same as that

$$\operatorname{argmax}\{Z_n(k_1, k_2) - Z_n(m_1, m_2); |k_i - m_i| \le q_n \delta_n^{-2}, 1 \le k_i \le n, i = 1, 2\}$$

Moreover, noticing that by (3.5), (3.8) and (3.10) we get

$$\begin{aligned} \max\{Z_n(k_1, k_2) - Z_n(m_1, m_2) - 2\delta_n \sum_{i=k_1+1}^{k_2} e_i + 2\delta_n \sum_{i=m_1+1}^{m_2} e_i \\ + \delta_n^2 |k_1 - m_1| + \delta_n^2 |k_2 - m_2|; \\ |k_i - m_i| < \delta_n^{-2} q_n, 1 \le k_i \le n, i = 1, 2\} = o_p(1), \end{aligned}$$

which finally implies that the limit distribution of

$$\operatorname{argmax}\ \{Z_n(k_1, k_2); 1 \le k_i \le n, i = 1, 2, n\epsilon_n \le k_2 - k_1 \le n(1 - \epsilon_n)\}$$

is the same as that of

$$(\operatorname{argmax}\ \{2V_{1n}(s) - |s|, s \in R\},\ \operatorname{argmax}\ \{2V_{2n}(s) - |s|, s \in R\}),$$

where

$$\begin{aligned} V_{in}(s) &= (-1)^{i+1} \delta_n \sum_{j=m_i+[\delta_n^{-2} s]}^{m_i} e_j && s < 0 \\ &= (-1)^{i} \delta_n \sum_{j=m_i+1}^{m_i+[\delta_n^{-2} s]} e_j && s > 0 \\ &= 0 && s = 0 \end{aligned}$$

$i = 1, 2$. The processes $\{V_{1n}(s); s \in R\}$ and $\{V_{2n}(s); s \in R\}$ are independent. According to Theorem 16.1 of [5] they converge in distribution to the two-sided Wiener processes $\{W_1(s); s \in R\}$ and $\{W_2(s); s \in R\}$, respectively, described by

(2.16). This implies that the limit distribution of $\hat{m}_{i1}(\epsilon)$ is the same as that of argmax $\{W_i(s) - |s|/2; s \in R\}$, $i = 1, 2$, respectively, and that $\hat{m}_{11}(\epsilon)$ and $\hat{m}_{21}(\epsilon)$ are asymptotically independent. The proof of (2.13) for $j = 1$ is finished.

To prove the assertion (2.14) for $j = 2$ we proceed similarly as above. Therefore we give some crucial relations only. For $1 \le k_1 < k_2 \le n$ we have, as $n \to \infty$,

$$\frac{n^2}{(k_2-k_1)(n-k_2+k_1)}(S_{k_2} - S_{k_1}) = \frac{n^2}{(k_2-k_1)(n-k_2+k_1)}$$
$$\Big(\sum_{i=k_1+1}^{k_2}(e_i - \overline{e}_n) + \delta_n \sum_{i=k_1+1}^{k_2}(I\{m_1 < i \le m_2\} - \frac{m_2-m_1}{n})\Big)$$

and hence, as $n \to \infty$,

$$\frac{n^2}{(m_2-m_1)(n-m_2+m_1)}(S_{m_2} - S_{m_1}) = O_p(n^{-1/2}) + n\delta_n = n\delta_n(1 + o_p(1))$$
$$\frac{n^2}{(k_2-k_1)(n-k_2+k_1)}(S_{k_2} - S_{k_1}) \le D_6 n\epsilon_n\delta_n(1 + o_p(1))$$

uniformly for $|k_i - m_i| \ge n\epsilon_n$, $\epsilon_n \to 0$ and $\sqrt{n}\epsilon_n\delta_n \to \infty$, $i = 1, 2$ and for some $D_6 > 0$.

Finally,

$$\frac{n^2}{(k_2-k_1)(n-k_2+k_1)}(S_{k_2} - S_{k_1}) - \frac{n^2}{(m_2-m_1)(n-m_2+m_1)}(S_{m_2} - S_{m_1})$$
$$= \frac{1}{(\gamma_2-\gamma_1)(1-\gamma_2+\gamma_1)}\Big(\sum_{i=k_1+1}^{k_2} e_i - \sum_{i=m_1+1}^{m_2} e_i$$
$$+\min(k_2, m_2) - k_2 - \max(k_1, m_1) + k_1 + (\gamma_2-\gamma_1)(k_2 - m_2 - k_1 + m_1)\Big)(1 + o_p(1))$$

for $|k_i - m_i| \le n\epsilon_n$, $\epsilon_n \to 0$ and $\sqrt{n}\epsilon_n\delta_n \to \infty$, $i = 1, 2$.

The assertion (2.14) for $j = 3$ and (2.15) can be proved along the same line as Theorem 2 in [4].

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