Comment.Math.Univ.Carolin. 36,2 (1995)239–244

Remarks on bounded sets in $(LF)_{tv}$-spaces

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Abstract. We establish the relationship between regularity of a Hausdorff $(LB)_{tv}$-space and its properties like (K), M.c.c., sequential completeness, local completeness. We give a sufficient and necessary condition for a Hausdorff $(LB)_{tv}$-space to be an $(LS)_{tv}$-space. A factorization theorem for $(LN)_{tv}$-spaces with property (K) is also obtained.

Keywords: topological vector space, inductive limits
Classification: 46A12

1. Introduction

Let $(E_n)_n$ be an increasing sequence of vector subspaces of a vector space E, whose union is E, such that every E_n is endowed with a vector topology τ_n with $\tau_{n+1}|E_n \leq \tau_n$. The space (E, τ), where τ is the finest vector topology on E such that $\tau|E_n \leq \tau_n$, $n \in \mathbb{N}$, will be called the inductive limit space and $(E_n, \tau_n)_n$ its defining sequence. We will say that (E, τ) is an

(1) $(LM)_{tv}$, if every (E_n, τ_n) is a metrizable topological vector space (tvs);
(2) $(LN)_{tv}$-space, if every (E_n, τ_n) is a locally bounded tvs;
(3) $(LF)_{tv}$-space, if every (E_n, τ_n) is an F-space, i.e. a metrizable and complete tvs;
(4) $(LB)_{tv}$-space, if every (E_n, τ_n) is a quasi-Banach space, i.e. a locally bounded and complete tvs.

It is known cf. e.g. [11, Proposition 2.2], that if every (E_n, τ_n) is a locally convex space (lcs), then τ is locally convex. In this case the corresponding inductive limit space will be called respectively (LM), (LN), (LF), (LB). Recall that a topological vector space (tvs) E is locally bounded if E has a bounded neighbourhood of zero.

Following Floret [6], [7], and Makarov [18], an inductive limit space (E, τ) with defining sequence (E_n, τ_n) will be called

(i) *regular*, if every bounded set in (E, τ) is contained in some E_m and is bounded in (E_m, τ_m);
(ii) *sequentially retractive*, if every null-sequence in (E, τ) is contained in some E_m and is a null-sequence in (E_m, τ_m).

*This work was prepared when the author held an A. von Humboldt scholarship at the University of Saarbrücken.

Following Grothendieck, a tvs E will be said to satisfy the *Mackey convergence condition* (M.c.c.) if for every null-sequence $(x_n)_n$ in E there exists a scalar sequence $(t_n)_n$, $t_n \nearrow \infty$, with $t_n x_n \to 0$. A regular $(LM)_{tv}$-space is sequentially retractive iff it satisfies M.c.c.

Grothendieck's factorization theorem [10, p. 16], implies that a Hausdorff (LF)-space is regular iff it is locally complete. Other criteria for regularity or sequential retractivity of (LF), (LB)-spaces were obtained (among others) by Floret [6], [7], [8], Neus [15], Fernandez [5], Vogt [19]. Recently we have showed [12] (extending Gilsdorf's result of [9]) that a Hausdorff $(LB)_{tv}$-space E is regular if E has property:

(K) *Every null-sequence $(x_n)_n$ in E has a subsequence $(x_{n(k)})_k$ such that the series $\sum_{k=1}^{\infty} x_{n(k)}$ converges in E.*

Note that there exist a non-sequentially complete (metrizable) tvs with property (K) ([14, Theorem 2]), and a complete tvs without property (K), cf. e.g. [12]. On the other hand every metrizable tvs with property (K) is a Baire tvs [2, 2.2].

Developing the argument used by Gilsdorf in [9] and ideas found in [6], [7], we establish the relationship between regularity of a Hausdorff $(LB)_{tv}$-space and its properties like (K), M.c.c., sequential completeness, local completeness. We give a sufficient and necessary condition for a Hausdorff $(LB)_{tv}$-space to be an $(LS)_{tv}$-space. Moreover a factorization theorem for $(LN)_{tv}$-spaces with property (K) is obtained.

We shall need the following factorization theorem , see [1, (11), pp. 57–58], and its proof.

(0) *Let F be a Baire tvs and E a Hausdorff $(LF)_{tv}$-space with defining sequence $(E_n)_n$ of F-spaces. If $T : F \to E$ is a continuous linear map, there exists $p \in \mathbb{N}$ such that $T(F) \subset E_p$ and $T : F \to E_p$ is continuous.*

By $Bd(\tau)$ we shall denote the set of all τ-bounded subsets of a tvs (E, τ); $\mathcal{F}(\tau)$ will denote the filter of all τ-neighbourhoods of zero. A sequence $(V_n)_n$ of balanced and absorbing subsets of E will be called a *string* if $V_{n+1} + V_{n+1} \subset V_n$, $n \in \mathbb{N}$; $(V_n)_n$ is *topological* if $V_n \in \mathcal{F}(\tau)$ for all $n \in \mathbb{N}$. A subset A of E will be said *pseudo-convex* if there exists a scalar $t > 0$ such that $A + A \subset tA$.

A tvs E will be called *locally complete* if for every balanced pseudo-convex bounded and closed set B in E the linear span $[B]$ endowed with the locally bounded topology generated by B is complete. It is easy to see that for lcs this definition is equivalent to the Grothendieck's one of local completeness (cf. [3, p. 152]). E will be called *locally Baire* if every bounded subset of E is contained in a bounded set B as above such that $[B]$ is a Baire tvs, cf. e.g. [9]. Every locally bounded non-complete tvs which is Baire is locally Baire but not locally complete. Every locally complete tvs with a fundamental family of pseudo-convex balanced bounded sets is locally Baire.

All tvs given in this paper are assumed to be Hausdorff.

The author wishes to thank Professor S. Dierolf and Professor K. Floret for their remarks.

2. Results

We start with the following proposition; its proof is due to S. Dierolf [4].

Proposition 2.1. *Every sequentially retractive inductive limit space is regular.*

PROOF: Let $(E_n, \tau_n)_n$ be a defining sequence of a tvs E under which E is sequentially retractive. Let B be a bounded subset of E; we may assume that $B \subset E_1$. Assume that B is not bounded in (E_n, τ_n), $n \in \mathbb{N}$. Then for every $n \in \mathbb{N}$ there exists $U_n \in \mathcal{F}(\tau_n)$ such that B is not absorbed by U_n. Thus for every $m \geq n$ there exists $b_{n,m} \in B$ with $m^{-1} b_{n,m} \notin U_n$. Consider the following sequence

$$b_{11}, 2^{-1} b_{12}, 2^{-1} b_{22}, 3^{-1} b_{13}, 3^{-1} b_{23}, 3^{-1} b_{33}, \dots .$$

This sequence converges to zero in E, hence it converges to zero in some E_n; consequently it is residually contained in U_n, a contradiction. $\square$

Lemma 2.2. *Let (E, τ) be the inductive limit space of the sequence $(E_n, \tau_n)_n$ of tvs such that*

$$(*) \qquad Bd(\tau_n) \cap \mathcal{F}(\tau_n) \neq \emptyset, \quad n \in \mathbb{N}.$$

If (E, τ) has property (K), *then there exists for (E, τ) a defining sequence $(G_n, \gamma_n)_n$ of locally bounded Baire tvs under which (E, τ) is regular. Moreover, if every (E_n, τ_n) is locally convex, then the same is true* (*with (G_n, γ_n) normed and Baire*) *when $(*)$ is replaced by*

$$(**) \qquad Bd(\tau_{n+1}) \cap \mathcal{F}(\tau_n) \neq \emptyset, \quad n \in \mathbb{N}.$$

PROOF: Let $(S_n)_n$ be a sequence of balanced subsets of E such that $S_n + S_n \subset S_{n+1}$ and $S_n \in Bd(\tau_n) \cap \mathcal{F}(\tau_n)$, $n \in \mathbb{N}$. Set $A_n := \overline{S}_n^{\tau}$, $G_n := \operatorname{lin} A_n$, $n \in \mathbb{N}$. Let $K_j^n := (\alpha_n)^{-j} A_n$, $j \in \mathbb{N}$, where α_n are chosen such that $S_n + S_n \subset \alpha_n S_n$, $\alpha_n > 1$, $n \in \mathbb{N}$. Clearly $(K_j^n)_j$ forms a basis of neighbourhoods of zero for a locally bounded vector topology γ_n on G_n such that $\tau | G_n \leq \gamma_n$. Fix $n \in \mathbb{N}$. In order to prove that (G_n, γ_n) is Baire, it is enough to show that (G_n, γ_n) has property (K), cf. Introduction.

Let $(x_p)_p$ be a null-sequence in (G_n, γ_n). We may assume that $x_j \in K_j^n$, $j \in \mathbb{N}$. There exists a subsequence $(x_{p(k)})_k$ such that $\sum_{k=1}^{\infty} x_{p(k)}$ converges in τ. Since $y_m := \sum_{k=1}^{m} x_{p(k)}$, $m \in \mathbb{N}$, is γ_n-Cauchy, $y_m \in K_1^n + K_1^n \subset A_n$, $m \in \mathbb{N}$, and $(y_m)_m$ converges in $\tau | G_n$, the series $\sum_{k=1}^{\infty} x_{p(k)}$ converges in (G_n, γ_n). Consequently, (G_n, γ_n) is Baire, by [2, 2.2]. Let (E, γ) be the inductive limit space of the sequence (G_n, γ_n). Then $\tau \leq \gamma$. Let $U \in \mathcal{F}(\gamma)$ and $(U_n)_n$, be a γ-topological string with $U_1 + U_1 \subset U$. For every $m \in \mathbb{N}$ there exists $j_m \in \mathbb{N}$ such that $U_m \cap G_m \supset K^m_{j_m}$. Hence

$$\begin{aligned} U \supset U_1 + U_1 &\supset \bigcup_{m=1}^{\infty} (K^1_{j_1} + K^2_{j_2} + \cdots + K^m_{j_m}) \\ &\supset \bigcup_{m=1}^{\infty} ((\alpha_1)^{-j_1})S_1 + \cdots + (\alpha_m)^{-j_m} S_m). \end{aligned}$$

The last set belongs to $\mathcal{F}(\tau)$. [11, Proposition 2.2]; hence $\tau = \gamma$. To see that (E, τ) is regular with respect to the sequence $(G_n, \gamma_n)_n$, it is enough to show that $(A_n)_n$ is a fundamental sequence of τ-bounded sets; By [1, 16 (6)], the sequence $(A_n)_n$ is a fundamental sequence of bounded sets for the strongest vector topology ϑ on E which agrees with τ on every A_n. On the other hand $\tau = \vartheta$, cf. [13, proof of Theorem 2].

If every (E_n, τ_n) is locally convex and $(**)$ is satisfied, we choose absolutely convex sets $S_n \in Bd(\tau_{n+1}) \cap \mathcal{F}(\tau_n)$ such that $S_n + S_n \subset S_{n+1}$, $n \in \mathbb{N}$. Set $K^n_j := (2)^{-j} A_n$, $n, j \in \mathbb{N}$. To complete the proof of this case we proceed as above. □

Note that condition $(**)$ is satisfied when every (E_n, τ_n) is normed or when the inclusion map of (E_n, τ_n) into (E_{n+1}, τ_{n+1}) is compact (or precompact), $n \in \mathbb{N}$.

Corollary 2.3. *Let E be an $(LN)_{tv}$-space with property* (K) *and F an $(LF)_{tv}$-space with defining sequence $(F_n)_n$ of F-spaces. If $T : E \to F$ is a linear map with closed graph, then:*

(1) *T is continuous.*
(2) *For every bounded sets B in E there exists $m \in \mathbb{N}$ such that $T(B) \subset F_m$ and $T(B)$ is bounded in F_m.*

Proof: Combining our Lemma 2.2 with the closed graph theorem [1, (11), p. 57], one obtains the continuity of T. Now (2) follows from Lemma 2.2 and (0). □

For locally convex spaces we have even the following

Corollary 2.4. *Let (E, τ) be a lcs with property* (K). *Assume that at least one of the following conditions is satisfied.*

(a) *(E, τ) is bornological.*
(b) *(E, τ) is the inductive limit space of the sequence $(E_n, \tau_n)_n$ of lcs such that*

$$Bd(\tau_{n+1}) \cap \mathcal{F}(\tau_n) \neq \emptyset, \quad n \in \mathbb{N}.$$

If F, T are defined as in Corollary 2.3, *the conclusion of Corollary* 2.3 *is also true.*

Proof: (a): Since (E, τ) is bornological with property (K), it is the inductive limit space of normed Baire spaces $[B]$, where B run over the family of absolutely convex bounded and closed subsets of E. We complete the proof as in Corollary 2.3.

(b): See the proof of Corollary 2.2. □

The following extends Theorem 5.5 of [7].

Theorem 2.5. *Let E be an $(LB)_{tv}$-space and $(E_n, \tau_n)_n$ its defining sequence consisting of quasi-Banach spaces. Consider the following conditions:*

(a) *E is sequentially retractive;*
(b) *E is sequentially complete;*
(c) *E is locally complete;*
(d) *E is locally Baire;*
(e) *E is regular;*
(f) *E has property* (K).

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Leftrightarrow$ (d) $\Rightarrow$ (e) $\Leftarrow$ (f). *If E satisfies M.c.c., then all the conditions are equivalent.*

Proof: (a) $\Rightarrow$ (b): Follows from Corollary 5.3 of [7] (which also holds for $(LF)_{tv}$-spaces). (b) $\Rightarrow$ (c) $\Rightarrow$ (d) are obvious. (d) $\Rightarrow$ (c): This follows from the following: If B is a balanced pseudo-convex bounded and closed subset of E such that $[B]$ is Baire, then $[B]$ is continuously included in some (E_m, τ_m) (by using (0)). Since B is closed in E, it follows that $[B]$ is complete. (d) $\Rightarrow$ (e): Follows by using (0). (f) $\Rightarrow$ (e): Corollary 2.3. If E satisfies M.c.c., then (e) $\Rightarrow$ (f) $\Rightarrow$ (a) hold. □

An $(LB)_{tv}$-space (E, τ) with defining sequence (E_n, τ_n) of quasi-Banach spaces will be called an $(LS)_{tv}$-space if for every $n \in \mathbb{N}$ there exists $m > n$ such that the inclusion $(E_n, \tau_n) \to (E_m, \tau_m)$ is compact. By [17], an $(LS)_{tv}$-space is a regular B-complete (hence complete) space; hence such a space is Montel (= barrelled, see [1] for definition) for which every bounded closed set is compact and sequentially retractive.

The following extends Proposition 8.5.36 of [3].

Proposition 2.6. *Let (E, τ) be a tvs with an increasing sequence $(S_n)_n$ of balanced pseudo-convex bounded sets covering E. Then the following assertions are equivalent:*

(i) *(E, τ) is an $(LS)_{tv}$-space,*
(ii) *(E, τ) is Montel and satisfies M.c.c.*

Proof: We have only to show (ii) $\Rightarrow$ (i). Since (E, τ) is barrelled, then $(A_n)_n$, where $A_n := \overline{S_n}^{\tau}$, $n \in \mathbb{N}$, is a fundamental sequence of τ-bounded sets [1, 16 (6), (7)]. Since (by assumption) every A_n is τ-compact [1, 18 (8) and 18 (3)] apply to show that (E, τ) is a B-complete bornological DF-space. Let (E, ϑ) be the inductive limit space of quasi-Banach spaces $[A_n]$, $n \in \mathbb{N}$. Then $\tau \le \vartheta$. Since $Bd(\tau) = Bd(\vartheta)$ and (E, τ) is bornological, then $\tau = \vartheta$, [1, 11 (3)]. For every $n \in \mathbb{N}$ there exists $m > n$ such that A_n is compact in $[A_m]$. In fact, since every A_n is τ-compact, then (by [16]) every A_n is metrizable in τ. The assumption of Grothendieck's lemma (cf. [7, p. 86]) are satisfied for $\mathcal{F}_k := \mathcal{F}(\gamma_k) \mid A_n$, $k > n$, $\mathcal{F} := \mathcal{F}(\tau)|A_n$, where γ_k is the original topology of $[A_k]$. By Grothendieck's lemma [7, p. 86], there exists $k > n$ such that $\mathcal{F}_k$ is weaker than $\mathcal{F}$; this applies to complete the proof. □

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(*Received* February 1, 1994)