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k-Ramsey classes and dimensions of graphs

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Abstract. In this note, we introduce the notion of k-Ramsey classes of graphs and we reveal connections to intersection dimensions of graphs.

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1. Intersection dimensions

All graphs considered are finite, undirected and without loops or multiple edges. The key notion is the *intersection dimension* of a graph: Given a class $\mathcal{A}$ of graphs and a graph $G = (V, E)$, the $\mathcal{A}$*-dimension* of G is

$$dim_{\mathcal{A}}G = \min\Big\{k \mid \exists\, E_1, \dots, E_k \subset \tbinom{V}{2} \text{ s.t. } (V, E_i) \in \mathcal{A} \text{ for each } i \text{ and } E = \bigcap_{i=1}^{k} E_i\Big\}.$$

This general definition was introduced in [1], but several authors have studied the same for particular graph classes $\mathcal{A}$ (e.g. [3], [12] for circular dimension and [13], [14] for interval dimension, also called boxicity). In [8], we have introduced and studied *relative* intersection dimension: Given two classes $\mathcal{A}$ and $\mathcal{B}$, we set

$$dim_{\mathcal{A}}(\mathcal{B}) = \sup_{G \in \mathcal{B}} dim_{\mathcal{A}}G \left(= \sup_{G} \frac{dim_{\mathcal{A}}G}{dim_{\mathcal{B}}G} \right).$$

The purpose of this definition is clarified by the following observation ([8]): if $dim_{\mathcal{A}}(\mathcal{B}) < \infty$ then it is the smallest real number x such that $dim_{\mathcal{A}}G \leq x \cdot dim_{\mathcal{B}}G$ holds for any graph G, otherwise no such x exists. In [8], we have considered the relative dimensions for several classes of graphs listed below (for most of these classes the intersection dimension was studied before, others are important generalizations of such classes). See [4] as a general reference to most of these

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classes.

INT	= interval graphs (intersection graphs of intervals on a line),
CA	= circular-arc graphs (intersection graphs of intervals on a circle),
CI	= circle graphs (intersection graphs of chords of a circle),
SP	= split graphs (graphs whose vertex sets split into a clique and a stable set),
CHOR	= chordal graphs (also called triangulated or rigid circuit graphs),
PER	= permutation graphs (intersection graphs of straight line segments with end points on two parallel lines),
FUN	= function (= co-comparability) graphs (intersection graphs of graphs of continuous real functions on a closed interval).

The results on relative dimensions which were obtained in [8], [5] are summarized in the following table (the number $dim_{\mathcal{A}}(\mathcal{B})$ is placed in the intersection of the $\mathcal{A}$-th row and $\mathcal{B}$-th column, the items marked by asterix will be proved in this paper).

	INT	CA	CI	SP	CHOR	PER	FUN
INT	1	∞	∞	∞	∞	∞	∞
CA	1	1	∞	∞	∞	∞	∞
CI	∞	∞	1	∞	∞	1	∞
SP	∞	∞	∞	1	∞	∞	∞
CHOR	1	∞	∞	1	1	∞	∞
PER	∞	∞	∞	∞	∞	1	∞
FUN	1	2*	2*	2*	2*	1	1

Table 1: Relative dimensions

When we started to work on relative dimensions, we hoped to find cases of nontrivial dependence between the dimensions. It turned out, however, that in most of the cases either $dim_{\mathcal{A}}(\mathcal{B}) = 1$ (i.e. $\mathcal{A} \subset \mathcal{B}$), or $dim_{\mathcal{A}}(\mathcal{B}) = \infty$. The only exception to this 'rule of triviality' is the FUN dimension. The purpose of this note is to shed light on this fact.

In the next section, we introduce the notion of k-Ramsey classes and we reveal the connection between this notion and the relative dimension. In technical Section 3, we prove that complements of most of the considered classes are 1- or 2-Ramsey. This explains why only numbers 1, 2 or ∞ occur in our table. Note also that these lines lead to simplification of some of the proofs from [8]. Finally, in Section 4, we prove that the FUN dimension of all considered classes is bounded.

2. Induced Ramsey property

We say that a class $\mathcal{A}$ of graphs is *k-Ramsey* if for every $G \in \mathcal{A}$ and every positive integer m, there is a graph $H \in \mathcal{A}$ such that for each coloring ϕ of the edges of H by m colors, H contains an induced subgraph G' isomorphic to G such that ϕ uses at most k distinct colors on G'. A 1-Ramsey class is called simply Ramsey.

Ramsey classes of graphs were studied by Erdös, Deuber, Nešetřil, Rödl and others. Let us remark that already the proof that the class of all graphs is Ramsey is quite nontrivial [2], [9]. Nešetřil and Rödl [9] applied Ramsey arguments to product dimensions of graphs.

For every positive integer k, there are classes that are k-Ramsey but not $(k-1)$-Ramsey. For an easy example, consider the class of disjoint unions of at most k complete graphs.

Given a class $\mathcal{M}$ of graphs, we denote by $\overline{\mathcal{M}}$ the class of the complements of the graphs in $\mathcal{M}$, i.e. $\overline{\mathcal{M}} = \{\overline{G} \mid G \in \mathcal{M}\}$.

Theorem 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be classes of graphs such that* (i) *$\mathcal{A}$ is hereditary (i.e. closed on induced subgraphs), and* (ii) *$\overline{\mathcal{B}}$ is k-Ramsey. Then $dim_{\mathcal{A}}(\mathcal{B}) \leq k$ or $dim_{\mathcal{A}}(\mathcal{B}) = \infty$.*

PROOF: Suppose $dim_{\mathcal{A}}(\mathcal{B}) > k$ and let G be a graph such that $G \in \overline{\mathcal{B}}$ and $dim_{\mathcal{A}}(\overline{G}) > k$. Fix a positive integer m, and let H be a big graph guaranteed by the k-Ramsey property of $\overline{\mathcal{B}}$. Set $H' = \overline{H}$. Obviously, $H' \in \mathcal{B}$. We claim that $dim_{\mathcal{A}} H' > m$.

Suppose that $dim_{\mathcal{A}} H' \leq m$. Then there are sets $E_1, E_2, \dots, E_m \subset \binom{V(H')}{2}$ such that (a) $E(H') = \bigcap_{i=1}^{m} E_i$, and (b) each graph $G_i = (V(H'), E_i)$ is in $\mathcal{A}$. Define a coloring of the nonedges of H' (that is, of the edges of H) by setting $\phi(xy) = i$ s.t. $xy \notin E_i$ (the existence of at least one such i follows from (a), in case of ambiguity we choose e.g. the smallest possible i). By the k-Ramsey Property, there exist an $I \subset \{1, 2, \dots, m\}$, $|I| \leq k$ and an induced subgraph G' of H isomorphic to G s.t. $\phi(E(G')) = I$. Hence, its complement $G'' = \overline{G'}$ is an induced subgraph of H' isomorphic to $\overline{G}$, and $E(G'') = \bigcap_{i \in I} E(G_i) \cap \binom{V(G'')}{2} = \bigcap_{i \in I} E(G_i \mid V(G''))$. It follows from (b) that $G_i \in \mathcal{A}$, and since $\mathcal{A}$ is closed on induced subgraphs, $G_i \mid V(G'') \in \mathcal{A}$ and $dim_{\mathcal{A}} G'' \leq k$. This contradicts the choice of G such that $dim_{\mathcal{A}}(\overline{G}) > k$. □

Proposition 2.2 ([9])**.** *The classes $\overline{INT}$, $\overline{PER}$ and $\overline{FUN}$ are Ramsey.*

Corollary 2.3. *All entries of Table* 1 *in columns labeled INT, PER and FUN are either* 1 *(if $\mathcal{A} \subset \mathcal{B}$) or* ∞ *(otherwise).*

We will prove in Theorem 3.1 that the classes $\overline{CA}$, $\overline{CI}$ and $\overline{SP}$ are 2-Ramsey. Thus we obtain the following corollary.

Corollary 2.4. *All entries of Table* 1 *labeled CA, CI and SP are either* 1 *or* 2 *or* ∞.

It follows that in order to show $dim_{\mathcal{A}}(\mathcal{B})=\infty$ for some $\mathcal{A}$ and $\mathcal{B} \in \{CA, CI, SP\}$, it suffices to give one example of a graph $G \in \mathcal{B}$ such that $dim_{\mathcal{A}} G \geq 3$. This observation provides a simplification to some of the proofs in [8].

The only column which is so far not taken care of by Corollaries 2.3 and 2.4 is the CHOR column. Here Theorem 2.1 does not apply, since one can show that complements of chordal graphs are not k-Ramsey (for any k). However, SP $\subset$ CHOR and hence $dim_{\mathcal{A}}(SP) = \infty$ implies $dim_{\mathcal{A}}(CHOR) = \infty$ for $\mathcal{A} \in \{INT, CA, CI, PER\}$ and the only 'nontrivial' infinity in the CHOR column stands for $dim_{SP}(CHOR)$.

3. Circular arc graphs and circle graphs

Theorem 3.1. *The classes $\overline{SP}$, $\overline{CI}$ and $\overline{CA}$ are 2-Ramsey.*

PROOF: **1. Split graphs.** Since the class of split graphs is closed under complementation, it is enough to prove that SP is 2-Ramsey. Suppose we are given a number m and a split graph $G = (Q \cup S, E)$ such that Q induces a clique and S a stable set. Denote by $G' = (Q \cup S, E')$ the bipartite graph with color classes Q and S induced by G. Since bipartite graphs are Ramsey ([9]), there exists a bipartite graph $H' = (Q' \cup S', F')$ such that for every m-coloring of the edges of H', there are $Q'' \subset Q'$ and $S'' \subset S'$ such that $H' \mid (Q'' \cup S'')$ is isomorphic to G' and monochromatic.

Denote $q' = |Q'|$ and $s' = |S'|$ and set $q'' = R^2_m(q')$, i.e. q'' is the least number such that in every m-coloring of the edges of a complete graph on q'' vertices, one always finds a monochromatic clique of size q'. Construct a graph $H = (Q''' \cup S''', F)$ by taking a clique on Q''' of size q'', and hanging a copy of H' on each q'-element subset of Q''' (thus H has $q'' + \binom{q''}{q'} s'$ vertices). Given an m-coloring of the edges of H, we first find a subset $Q' \subset Q'''$ of size q' which induces a monochromatic clique (say of black color), and then in the particular copy of H' hanging on this Q', we find an induced bipartite subgraph G‘ on Q‘ $\subset Q'$ and S‘ $\subset S'''$ which is isomorphic to G' and monochromatic (say of white color). Then $H \mid (Q$‘ $\cup\, S$‘$)$ is isomorphic to G and its edges are colored by at most two colors.

2. Circle graphs. It is useful to view circle graphs as overlap graphs (cf. e.g. [4]): Every circle graph $G = (V, E)$ with n vertices has an interval representation such that (i) every vertex v is represented by a closed interval I_v with endpoints among $\{1, 2, \dots, 2n\}$, (ii) intervals corresponding to distinct vertices have distinct endpoints, and (iii) vertices u and v are adjacent in G if and only if the intervals I_u, I_v overlap (i.e. their intersection is nonempty, but none is a subinterval of the other one).

We denote by OV_n the graph whose vertices are all intervals with distinct endpoints in $\{1, 2, \dots, n\}$ and edges connect intervals $[a, b], [c, d]$ such that $a \le c \le b \le d$ or $c \le a \le d \le b$. Then $OV_n \in CI$ for every n, and each circle graph on n vertices is an induced subgraph of OV_{2n}. Thus it suffices to prove that for every n and m, there exists an N such that for every m-coloring of the nonedges of OV_N, one can find a bichromatic $\overline{OV_n}$.

We show that $N \le R^n_{m^{e(n)}}(2n-4)$, where $e(n)$ denotes the number of nonedges of OV_n ($e(n) < n^4$) and $R^p_k(q)$ is the least number R such that for every k-coloring of the p-element subsets of an R-element set X, one can find a q-element subset of X whose all p-tuples receive the same color (this R is finite by the Ramsey theorem).

Consider an m-coloring f of the edges of $\overline{OV_N}$, with $N \geq R^n_{m^{e(n)}}(2n-4)$. For a subset $A = \{a_1 < a_2 < \ldots < a_n\} \subset \{1, 2, \ldots, N\}$, define a coloring f_A of $\overline{OV_n}$ by $f_A([i,j][r,s]) = f([a_i, a_j][a_r, a_s])$ (there are at most $m^{e(n)}$ different colorings of $\overline{OV_n}$), and define a coloring ϕ of n-element subsets of $\{1, 2, \ldots, N\}$ by $\phi(A) = f_A$. It follows that there is a coloring g of $\overline{OV_n}$ and a $(2n-4)$-element subset $B = \{b_1 < b_2 < \ldots < b_{2n-4}\} \subset \{1, 2, \ldots, N\}$ such that $f_A = g$ for every $A \subset B, |A| = n$. We show that in $\overline{OV_N}$, $C = \{b_1, b_2, \ldots, b_n\}$ induces a bichromatic copy of $\overline{OV_n}$. Indeed, if $i < j, r < s$ are such that $i \leq r$ and $[b_i, b_j][b_r, b_s] \in E(\overline{OV_N \mid C})$, then either $j < r$, or $i < r$ and $s < j$. Consider $A = \{b_i, b_j, b_r, b_s, b_{n+1}, \ldots, b_{2n-4}\}$. Since $f_A = g$, we have $f([b_i, b_j][b_r, b_s]) = g([1,2][3,4])$ in the former case and $f([b_i, b_j][b_r, b_s]) = g([1,4][2,3])$ in the latter case.

3. Circular arc graphs. Here the argument is similar to the previous one. Denote CA_n the graph whose vertices are intervals with endpoints in $\{1, 2, \ldots, n\}$, and complements of such intervals. Two vertices of CA_n are adjacent if the corresponding sets are not disjoint. Every CA graph on n vertices is an induced subgraph of CA_{2n} (we obtain a representation of this kind from an ordinary circular arc representation by cutting the circle in one point, arcs which are not cut correspond to intervals, arcs which contain the cut point correspond to the complements of intervals).

We again set $N = R^n_{m^{e(n)}}(2n-4)$, where now $e(n)$ denotes the number of nonedges of CA_n and we show that for every m-coloring of the nonedges of CA_N, one can find a bichromatic $\overline{CA_n}$. The proof is identical to the proof above, the two kinds of nonedges in CA_n are $[i,j][r,s]$ for $i < j < r < s$ and $[i,j]\overline{[r,s]}$ for $r < i < j < s$ (here $\overline{[r,s]} = [1,r] \cup [s,n]$), there are no nonedges of type $\overline{[i,j]}\overline{[r,s]}$. $\square$

4. FUN dimension

Before we prove the FUN row of Table 1, we introduce another class of graphs which generalizes several previously considered ones. This is the class CP, so called *circle polygon* graphs, which are intersection graphs of convex polygons inscribed in a circle. These graphs were suggested by M. Fellows [private communication] as a natural generalization of circle graphs. A polynomial time recognition algorithm was given by Koebe [7].

Lemma 4.1 ([6]). *We have CI $\subset$ CP, CA $\subset$ CP, CHOR $\subset$ CP and PER $\subset$ CP, but CA $\not\subset$ FUN, CI $\not\subset$ FUN and SP $\not\subset$ FUN (and, consequently, CHOR $\not\subset$ FUN and CP $\not\subset$ FUN).*

The remaining entries (in the FUN row) of Table 1 will be justified by the following:

Proposition 4.2. $dim_{FUN}(CP) \leq 2$.

Proof: Suppose we are given a graph $G = (V, E) \in \mathrm{CP}$ and a representation $(M_u, u \in V)$, where each M_u is a convex polygon with vertices lying on a given circle C. Choose a point x on the circle which does not belong to any of the polygons. For two disjoint polygons M_u, M_v, we consider the connected components obtained by removing the vertices of M_u and M_v from the circle C. We say that u is *above* v if both endpoints of the component containing x are vertices of M_u, and we say that u and v are *indifferent* if one endpoint of the component containing x is a vertex of M_u and the other one is a vertex of M_v.

Define $E_1 = \{uv \mid uv \notin E$ and u and v are indifferent$\}$ and $E_2 = \{uv \mid uv \notin E$ and u and v are not indifferent$\}$. Then $G_1 = (V, E \cup E_2)$ is an interval (and hence function) graph. Also $G_2 = (V, E \cup E_1)$ is a function graph, since $\overline{G_2} = (V, E_2)$ is a comparability graph ('being above' is a partial order). Thus $G = G_1 \cap G_2$ has FUN dimension at most 2. □

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