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More on the complexity of cover graphs

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Abstract. In response to [3] and [4] we prove that the recognition of cover graphs of finite posets is an NP-hard problem.

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1. Introduction

Given a (finite) poset $P = (X, <)$ we define its *cover graph* as the undirected graph $G_P = (X, E)$ where $\{x, y\} \in E$ if y covers x in P (i.e. if $x < y$ and $x < z < y$ for no z). Sometimes the terms *Hasse diagram* or *Hasse graph* are used.

In [3] (cf. Theorem 4.1.) we claimed the following:

Theorem 1.1. *The following decision problem is NP-hard.*
Instance: A (undirected) graph G.
Problem: Is G a cover graph of a poset?

We thank B. Toft and his student J. Thostrup who turned our attention to a gap in our argument. In [4], we outlined how our argument may be saved using a recent result of Lund and Yannakakis. The purpose of this note is to present a full proof together with some remarks and comments which relate this problem to other research. Meanwhile, G. Brightwell found another proof of the main result. His proof is simpler and more transparent than ours. Yet we believe that this proof (outlined in [4]) deserves a publication as the present method is perhaps an interesting combination of nontrivial techniques and has some further applications (see a forthcoming paper by V. Rödl and L. Thoma).

Given an undirected graph $G = (V, E)$ a *Hasse orientation* of G is any orientation of G which does not contain either a cycle or a quasicycle. (A *quasicycle* in a directed graph is a sequence of vertices $x_1, x_2, \dots, x_n$ together with arcs (x_i, x_{i+1}), $i = 1 \dots, n-1$ and (x_1, x_n).) Alternatively, a Hasse orientation of G is any acyclic orientation of G which stays acyclic even after the reversal of an arbitrary arc.

It follows that non-cover graphs contain in any acyclic orientation a quasicycle and thus a monotone path. This indicates that the recognition of cover graphs may be related to chromatic number. However appealing this may seem to be, the details are quite subtle.

Particularly, we make use of explicit construction of expanders [1], efficient packing of sparse graphs [5] and, perhaps most importantly, a recent result of Lund and Yannakakis on NP hardness of approximation of chromatic number (cf. Theorem 2.5 in [2]):

Theorem 1.2 [2]. *For every constant $g > 1$ there exists a constant $c = c(g)$ such that the following problem is NP-hard:*

Given a graph G distinguish between the case that G is colorable with c colors and the case that $\chi(G) > gc$.

In fact, Lund and Yannakakis proved that distinguishing between $\chi(G) \leq c$ and $\chi(G) \geq gc$ is NP hard. This is, however, equivalent with the statement of Theorem 1.2.

In below, we use the existence of $c(4)$ and we set $k_0 = c(4)$.

While the general scheme of the proof given here is the same as in [3], the details are more technical and involved.

2. Lemmas and reduction

We start with the following two lemmas:

Lemma 2.1. *For every K there is a polynomial reduction $G \to \bar{G}$ such that*

(1) $\chi(G) = \chi(\bar{G})$ *whenever* $\chi(G) \leq K$;
(2) $\chi(\bar{G}) \leq K$ *and a K-coloring of $\bar{G}$ is known.*

Proof: Put $\overline{G} = G \times K_K$ (the direct product). Clearly (1) holds and a K-coloring of $\overline{G}$ is induced by a coloring of K_K.

$\square$

Lemma 2.2. *For every K there is a polynomial reduction which assigns to every graph $\bar{G}$ with known coloring $V(\bar{G}) = V_1 \cup V_2 \cup \cdots \cup V_K$ a cover graph G^* with the following properties:*

(1) $\chi(\bar{G}) = \chi(G^*)$;
(2) *for every $k < K$, if $\chi(\bar{G}) = k$, then there exists a Hasse orientation of G^* which does not contain an oriented path of length k (i.e. with $k+1$ vertices).*

The proof of Lemma 2.2 is given in Section 3.

Construction 2.3. Fix $k > 2$. Given a graph $H = (V, E)$ define a graph H_k as follows: For every path $P = (x_0, e_1, x_1, \dots, e_{4k}, x_{4k})$ of length $4k$ we choose vertices x_0^P, x_1^P, $x_2^P, \dots, x_{4k}^P$ and a^P, b^P (all these vertices are distinct for different paths) and add them to H together with the edges of the form:

$$\{x_i, x_i^P\} \text{ for } i = 0, 1, \dots, 4k,$$

$$\{x_{i-1}^P, x_i^P\} \text{ for } i = 1, 2, \dots, 4k,$$

$$\{x_0^P, a^P\}, \{x_k^P, a^P\}, \{x_{2k}^P, a^P\},$$

$$\{x_{2k}^P, b^P\}, \{x_{3k}^P, b^P\}, \{x_{4k}^P, b^P\}.$$

The resulting graph will be denoted H_k.

See Fig. 1.

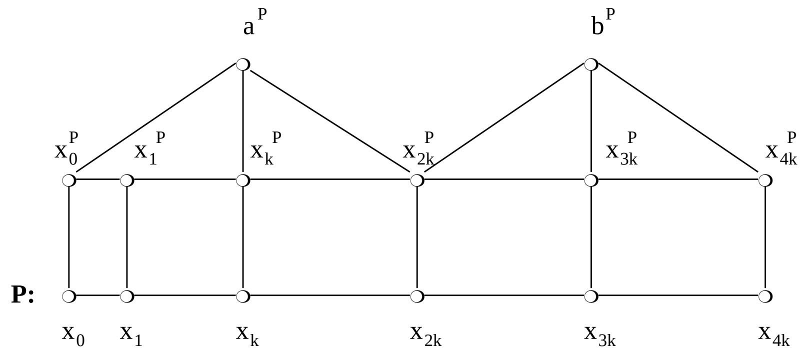

Figure 1

Clearly $\mid H_k \mid < \mid V \mid + (3+4k)\ |V|^{4k+1}$.

This construction is similar to the one used in the preprint version of [3]. Its usefulness follows from the following two lemmas.

Lemma 2.4. *Let $H = (V, E)$ be a cover graph with a Hasse-orientation of H which contains no monotone path of length k. Then H_k is a cover graph.*

Lemma 2.5. *Let $H = (V, E)$ be a cover graph and let every Hasse-orientation of H contains a directed path of length $4k$. Then H_k fails to be a cover graph.*

Proof of Lemma 2.4: Let $\vec{E}$ be a fixed Hasse orientation of H without monotone paths of length k. Put $E(H_k) = F$ and extend the orientation $\vec{E}$ to an orientation $\vec{F}$ of F by putting (for any path $P = (x_0, x_1, \ldots, x_{4k})$ in H):

$$(x_i^P, x_j^P) \in \vec{F} \quad \text{for } \{x_i, x_j\} \in E(P) \text{ such that } (x_i, x_j) \in \vec{E}$$

$$(x_i, x_i^P) \in \vec{F} \quad \text{for } x_i \in V(P)$$

$$(x_i^P, a^P) \in \vec{F} \quad \text{and } (x_j^P, b^P) \in \vec{F} \text{ whenever } \{x_i^P, a^P\} \in F,\ \{x_j^P, b^P\} \in F$$

(See Fig. 2.)

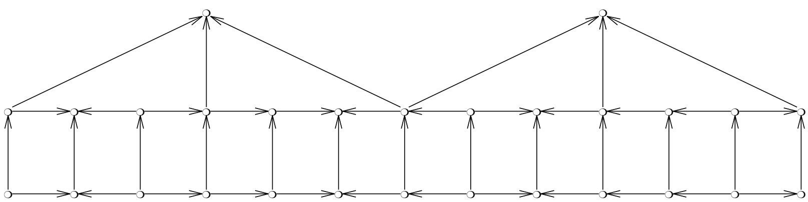

Figure 2

$\vec{F}$ is clearly an acyclic orientation. We prove that it does not contain a quasicycle.

For a contradiction, suppose that $\vec{F}$ contains a quasicycle $y_0, y_1, \ldots, y_t$, (see Fig. 3).

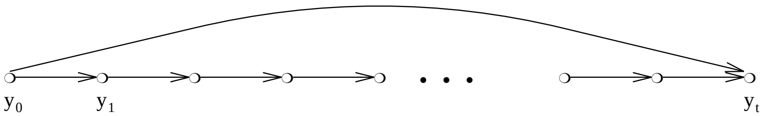

Figure 3

If $y_t \in V(H)$, then we have a quasicycle in $\vec{E}$ — a contradiction.

Suppose $y_t = x_j^P$. Then $y_i \neq a^P$, $y_i \neq b^P$ for $i = 0, 1, 2, \ldots, t$ (this follows by definition of $\vec{F}$).

We distinguish two cases: if $y_0 \notin V(H)$ then necessarily $y_0 = x_i^P$ and we get a contradiction. If $y_0 \in V(H)$, then there exists $s < t$ such that $y_0, y_1, \ldots, y_s \in V(H)$, $y_{s+1} = x_r^P, y_{s+2} = x_{r+1}^P, \ldots, y_t = x_{t+r-s-1}^P$ for some r.
$(P = (x_0, x_1, \ldots, x_{4k}))$. As $\{y_0, y_t\} \in F$ we infer that $y_0 = x_{t+r-s-1}$ and hence

$$x_{t+r-s-1} = y_0, y_1, \ldots, y_s = x_r$$

is an oriented path from $x_{t+r-s-1}$ to x_r while

$$x_r, x_{r+1}, \ldots, x_{t+r-s-1}$$

is an oriented path from x_r to $x_{t+r-s-1}$ and thus there is an oriented cycle in $\vec{E}$ — a contradiction.

The last possible position of y_t is $y_t = a^P$ or $y_t = b^P$. Assume without loss of generality that $y_t = b^P$; then all $y_0, y_1, \ldots, y_{t-1}$ are of the form $y_i = x_{i+j}^P$ for some integer j. In fact $\{x_j, x_{j+1}, \ldots x_{j+t-1}\}$ is a set of consecutive vertices of path P. But then these vertices form a directed path which (by definition of $\vec{F}$) gives a directed path of length k or $2k$ in $\vec{E}$ (depending on position of vertices y_0 and y_{t-1} — a contradiction. $\square$

Proof of Lemma 2.5: Let H and H_k be as above. Put $F = E(H_k)$ and for contradiction assume that $\vec{F}$ is a Hasse orientation. Let $\vec{E}$ be the orientation induced by $\vec{F}$ on E. By the assumption, E contains a monotone path

$$P = (x_0, x_1, \ldots, x_{4k})$$

with the orientation, say from x_0 towards x_{4k}.

We observe that if $(x_i^P, x_i) \in \vec{F}$, then,

$(x_i^P, x_{i+1}^P) \in \vec{F}$ and consequently, $(x_{i+1}^P, x_{i+1}) \in \vec{F}$, $\dots$, $(x_{4k}^P, x_{4k}) \in \vec{F}$. Hence, it follows that there exists i_0 such that

$$\begin{array}{ll} (x_i, x_i^P) \in \vec{F} & \text{if } i < i_0 \\ (x_i^P, x) \in \vec{F} & \text{if } i \geq i_0 \\ (x_i^P, x_{i+1}^P) \in \vec{F} & i = 0, 1, \dots i_0 - 2 \\ & \text{and } i = i_0, \dots 4k - 1 \\ \text{while } (x_{i_0}^P, x_{i_0-1}^P) \in \vec{F} & \end{array}$$

(See Fig. 4.)

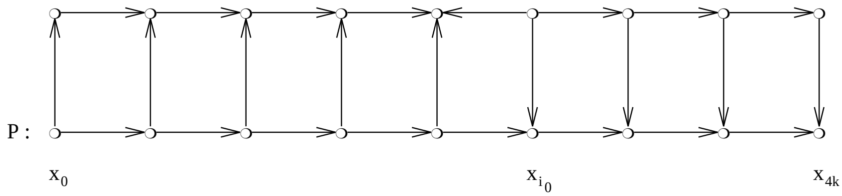

Figure 4

Now, if $i_0 \leq 2k$, then we cannot complete the Hasse orientation for edges incident with b^P and if $i_0 > 2k$ we cannot find an orientation for edges incident to a^P. A contradiction. □

The above lemmas may be combined to yield the main result:

Proof of Theorem 1.1: Put $k_0 = c(4)$ and set $K = 4k_0 + 1$.
Combining Theorem 1.2 and Lemma 2.1, we infer that the following problem is NP-hard:

Given a graph $\bar{G}$ with $\chi(\bar{G}) \leq K$ and a known K-coloring $V(\bar{G}) = V_1 \cup V_2 \cup \dots \cup V_K$

distinguish between $\chi(\bar{G}) \leq k_0$ and $\chi(\bar{G}) = K$.

Using Lemma 2.2, we then infer the following:

$(*)$ Given a cover graph G^* with $\chi(G^*) \leq K$ and with an (unknown) Hasse orientation having no oriented path of length $\chi(G^*)$, the problem distinguishing between $\chi(G^*) \leq k_0$ and $\chi(G^*) = K$ is NP-hard.

We also observe that any acyclic directed graph of chromatic number at least $4k + 1$ contains a directed path of length $4k$.

Finally, setting $H = G^*$ and applying Construction 2.3 to H we obtain a graph $F = H_{k_0}$ which by Lemmas 2.4, 2.5 and (*) has the following properties:

if $\chi(G^*) \leq k_0$ then F is a cover graph,

if $\chi(G^*) = K$ then F fails to be a cover graph,

and hence, by (*) above, deciding if F is a cover graph is NP-hard. □

3. Proof of Lemma 2.2

Let K be fixed and the graph $\bar{G}$ with k-coloring $V(\bar{G}) = V_1 \cup \ldots, \cup V_K$ be given. We will use the well known result of O. Gabber and Z. Galil [1], who, for any $m = t^2$, $t \geq t_0$ gave an explicit construction of $(m, 5, d_0)$-expander graphs Γ_m, $d_0 = (2 - \sqrt{3})/4$. Explicitly, they constructed graphs Γ_m with the following properties:

(1) Γ_m is a 5-regular bipartite graph with m (white) vertices $X = \{x_1, \ldots, x_m\}$ and m (black) vertices $Y = \{y_1, \ldots, y_m\}$. (Thus $\Gamma_m = (X, Y, E)$.)

(2) For any $X' \subset X$ the set $\Gamma(X')$ of neighbors of X' (in Y) satisfies:

$$\mid \Gamma(X') \mid \geq (1 + d_0(1 - \frac{\mid X' \mid}{m})) \mid X' \mid .$$

We are going to introduce the $j - th$ power Γ_m^j of the graphs Γ_m:

Construction of Γ_m^j

For each $j = 1, 2 \ldots$ consider the bipartite graph $\Gamma_m^j = (X, Y, F)$ where $\{x, y\} \in F$ if in Γ_m there exists a path of length at most $2j + 1$ between x and y and $x \in X$, $y \in Y$.

Denoting by $\Delta(G)$ the maximal degree of a graph, we have clearly

(a) $\Delta(\Gamma_m) = 5$ and
$\Delta(\Gamma_m^j) < 5^{2j+2}$ for $j = 1, 2, \ldots$.

We will also use the following:

Fact 3.1. *For every $\epsilon > 0$ there exists j such that for each m, the bipartite graph $\Gamma_m^j = (X, Y, F)$ satisfies*

(b) *For any $X' \subset X, Y' \subset Y$, $|X'| > \epsilon|X|$, $|Y'| > \epsilon|Y|$ there is an edge $\{x, y\} \in F$ where $x \in X'$, $y \in Y'$.*

Proof: For given $\epsilon > 0$ take j so large that $\epsilon(1 + d_0\epsilon)^{j+1} > 1 - \epsilon$. For a contradiction, assume that there are sets X', Y' contradicting the statement (b).

For $i \leq 2j + 1$ let $X_i \subset X (Y_i \subset Y)$ be the set of vertices that can be reached from X' by a path of length at most i in Γ_m. Due to our assumption, we have $\mid Y_{2j+1} \mid < (1 - \epsilon)m$ (otherwise, $Y' \cap Y_{2j+1} \neq 0$, meaning that there is an edge $\{x, y\} \in F$, $x \in X'$, $y \in Y'$). The graph Γ_m is regular and hence it contains a perfect matching, thus also Γ_m^j contains this perfect matching. This means for $i < j$ that $\Gamma_m(Y_{2i+1}) = X_{2i+2}$ satisfies $\mid X_{2i+2} \mid \geq \mid Y_{2i+1} \mid$.

Summarizing, we infer (due to our assumption that $\mid X_{2i} \mid \leq \mid Y_{2j+1} \mid < (1-\epsilon)m$)

$$\mid Y_{2i+1} \mid = \mid \Gamma(X_{2i}) \mid \geq$$
$$\geq \left[1 + d_0 \left(1 - \frac{\mid X_{2i} \mid}{m}\right)\right] \mid X_{2i} \mid \geq (1 + \epsilon d_0) \mid X_{2i} \mid \geq (1 + \epsilon d_0) \mid Y_{2i-1} \mid$$

for any $i = 1, 2, \dots j$ and hence also

$$| Y_{2j+1} | \geq (1 + \epsilon d_0)^{j+1} | X' | \geq \epsilon(1 + d_0\epsilon)^{j+1} | X | \geq (1 - \epsilon) | X | = (1 - \epsilon)m,$$

contradicting our assumption.

Next, we will use the following modification of a result of N. Sauer and J. Spencer [5].

Fact 3.2. *Let $G_1 = (V, F_1)$ and $G_2 = (V, F_2)$ be two bipartite graphs with the same vertex set V. Assume $2\Delta(G_1)\Delta(G_2) < m =| V | /2$.*

Let $V = X \cup Y$, $|X| = |Y| = m$. Then there exists an algorithm which is polynomial in m and which finds a permutation $\pi : V \to V$, $\pi(X) = X$, $\pi(Y) = Y$ so that $F_1 \cap F_2(\pi) = \emptyset$, where

$$F_2(\pi) = \{\{\pi(v), \pi(v')\}; \{v, v'\} \in F_2\}.$$

Proof: We will construct a sequence of permutations π_i, $i = 0, 1, \dots$ such that if

$$F_1 \cap F_2(\pi_i) \neq \emptyset \tag{1}$$

then

$$| F_1 \cap F_2(\pi_{i+1}) | < | F_1 \cap F_2(\pi_i) | . \tag{2}$$

Suppose that (1) holds for $i \geq 0$ and let $V(G_1) = V(G_2) = \{v_1, v_2, \dots, v_{2m}\}$ and let $\{v_1, v_2\} \in F_1 \cap F_2(\pi_i)$; without loss of generality assume that $v_1 \in X$ and $v_2 \in Y$. Let R be the set of all indexes r for which there exists an index s such that $\{v_2, v_s\} \in F_1$ and $\{v_s, v_r\} \in F_2(\pi_i)$. Similarly, $r \in \bar{R}$ iff $\{v_2, v_s\} \in F_2(\pi_i)$ and $\{v_s, v_r\} \in F_1$ for some s. We have $| R \cup \bar{R} | \leq 2\Delta(G_1)\Delta(G_2) \leq m - 1$ and hence, there exists $p \notin R \cup \bar{R}$ with $v_p \in Y$.

We will construct π_{i+1} satisfying (2) as follows:

$$\pi_{i+1}(v) = \pi_i(v) \quad \text{if } \pi_i(v) \notin \{v_2, v_p\}$$
$$\pi_{i+1}(v) = v_p \quad \text{if } \pi_i(v) = v_2$$

and

$$\pi_{i+1}(v) = v_2 \quad \text{if } \pi_i(v) = v_p.$$

Note that $p \neq 2$. Interchanging v_2 and v_p, in graph $F_2(\pi_i)$ the edges $\{v_1, v_2\}$ in F_1 and $\{v_1, v_2\}$ in $F_{2(\pi_{i+1})}$ no longer coincide. Moreover, due to $p \notin R \cup \bar{R}$, we do not place any other edge of G_2 on an edge of G_1 and thus, the total number of common edges decreases. □

Note that for each $i = 0, 1, 2 \dots$ we perform at most $\binom{2m}{2}$ steps to find out if $F_1 \cap F_2(\pi_i) \neq \emptyset$.

If so, we make additional $\leq m - 2$ steps to find a vertex v_p. As the total number of iterations $i = 0, 1, 2 \dots$ is clearly bounded by $\binom{2m}{2}$, we get that the algorithm runs in time not exceeding $O(m^4)$ steps.

It is crucial for our proof of Lemma 2.2 that we may iterate the above packing Lemma 3.2.

Proposition 3.3. *For every $\epsilon > 0$ and integers K and n, there is an integer $m, m = O(n^{2K})$ (for K fixed) and an $O(m^4 n)$ algorithm that constructs $n+1$ edge disjoint copies $\Gamma_{(0)} = (X, Y, E_0)$, $\Gamma_{(1)} = (X, Y, E_1), \ldots, \Gamma_{(n)} = (X, Y, E_n)$ of the graph Γ_m^j (from Fact 3.1) in such a way that there is no rainbow cycle of length $l \leq 2K$.*

Here a cycle C is said to be a *rainbow cycle* if $|E(C) \cap E_i| = 1$ for all $i > 0$.

Proof: Let $\epsilon > 0$ (and *thus also j from Fact* 3.1), $K \geq 2$ and n be fixed. Set $m = (n \cdot 5^{2j+2})^{2K+2}$. We will construct bipartite graphs $\Gamma_{(0)}, \Gamma_{(1)}, \ldots, \Gamma_{(n)}$ on the set $V = X \cup Y$, $|X| = |Y| = m$ as follows:

Put $\Gamma_{(0)} = \Gamma_m^j$. In the induction step, suppose that the graphs $\Gamma_{(0)}, \ldots, \Gamma_{(a-1)}$ have been constructed in such a way that there is no rainbow cycle of length $\leq 2K$.

Let $G_1 = (V, F_1)$ be the graph defined by $\{u, v\} \in F_1$ if in $\bigcup_{i=0}^{a-1} E_i$ there is a path from u to v of length at most $2K$. Put $G_2 = \Gamma_m^j$. We have $\Delta(G_2) < 5^{2j+2}$

$$\Delta(G_1) < (a \cdot 5^{2j+2})^{2K} \leq (n \cdot 5^{2j+2})^{2K}.$$

Thus, $2\Delta(G_1) \cdot \Delta(G_2) < m$ and hence by Fact 3.2 there is an $0(m^4)$ algorithm which places G_2 on G_1 with no common edge and which preserves the sets X and Y. This however gives a placement of a copy $\Gamma_{(a)}$ of $\Gamma_m^j = G_2$ in such a way that $\Gamma_{(0)}, \Gamma_{(1)}, \ldots, \Gamma_{(a)}$ do not contain a rainbow cycle of length $\leq 2K$. □

Now we are ready to finish the proof of Lemma 2.2:

Let $\bar{G}$ be a given graph with a K-coloring $V(\bar{G}) = V_1 \cup \cdots \cup V_K$. Let $E(\bar{G}) = \{e_1, \ldots, e_n\}$.

Set $\epsilon = K^{-1}$ and consider the graphs $\Gamma_{(0)} = (X, Y, E_0)$, $\Gamma_{(1)} = (X, Y, E_1), \ldots,$ $\Gamma_{(n)} = (X, Y, E_n)$ constructed in Proposition 3.3. Put $X = \{x_1, \ldots, x_m\}, Y = \{y_1, \ldots, y_m\}$ and assume without loss of generality that the bipartite graph $\Gamma_{(0)}$ contains a matching $\{x_i, y_i\}$, $i = 1, 2, \ldots, m$.

To construct G^*, we replace each vertex $v \in V(\bar{G})$ by an m-element set $X_v = \{x_1^v, \ldots, x_m^v\}$ with the sets X_v disjoint for distinct $v \in V(\bar{G})$.

For each edge $\{v, v'\} = e_a$ (for $a \in \{1, \ldots, n\}$) we consider a copy $\hat{\Gamma}_{(a)} = (X_v, X_{v'}, \hat{E_a})$ of $\Gamma_{(a)} = (X, Y, E_a)$ in such a way that the mapping $\varphi_{e_a} : \hat{\Gamma}_{(a)} \to \Gamma_{(a)}$ defined by $x_i^v \to x_i, x_i^{v'} \to y_i, i = 1, \ldots, m$ is an isomorphism. In this way, we obtain a graph G^*. (More formally, we set $V(G^*) = \bigcup_{v \in V(\bar{G})} X_v$ and $E(G^*) = \bigcup_{a=1}^{n} \hat{E_a}$.) As $\bar{G}$ is a homomorphic image of G^*, we have clearly $\chi(\bar{G}) \geq \chi(G^*)$. To prove the opposite inequality, assume that $\chi(G^*) = r < \chi(\bar{G}) \leq K$ and consider an r-coloring of G^*. For each $v \in V(\bar{G})$ let $Y_v \subseteq X_v$ be a set, $| Y_v | \geq \frac{|X_v|}{r}$, colored by the most frequent (in X_v) color.

As $\chi(\bar{G}) > r$ there exists an edge $e_a = \{v, v'\} \in E(\bar{G})$ with Y_v and $Y_{v'}$ colored by the same color. Using the expanding properties of the graph $\hat{\Gamma}_{(a)}$ (isomorphic to Γ^j_m) proved in Fact 3.1, we get that there exist vertices $y \in Y_v$, $y' \in Y'_{v'}$ such that $\{y, y'\} \in E(\hat{\Gamma}_{(a)}) \subseteq E(G^*)$.

Hence, G^* does not have a proper r-coloring and $\chi(G^*) = \chi(\bar{G})$.

To prove (2) of Lemma 2.2, consider a k-coloring $V(\bar{G}) = W_1 \cup W_2 \cup \cdots \cup W_k$. We define the following acyclic orientation A of G^*:

for $\{x^v_s, x^{v'}_t\} \in E(G^*), x^v \in X_v, x^{v'} \in X_{v'}$

set $(x^v_s, x^{v'}_t) \in A$ iff $v \in W_i, v' \in W_j$ and $i < j$.

First, observe that A does not contain an oriented path of length k.

For a contradiction, suppose now that A contains a quasicycle with edges $(x^{v_1}_{i_1}, x^{v_2}_{i_2}), \ldots, (x^{v_{p-1}}_{i_{p-1}}, x^{v_p}_{i_p}), (x^{v_1}_{i_1}, x^{v_p}_{i_p})$. Put $e^1 = \{v_1, v_2\}$, $e^2 = \{v_2, v_3\}, \ldots,$ $e^{p-1} = \{v_{p-1}, v_p\}, e^p = \{v_1, v_p\}$. As A is acyclic, we infer that all the vertices $v_1, \ldots, v_p$ and thus all the edges $e^1, \ldots, e^p$ are distinct. Clearly, $p \leq k \leq K$.

Now consider the vertices $\varphi_{e^1}(x^{v_1}_{i_1}), \varphi_{e^1}(x^{v_2}_{i_2}), \varphi_{e^2}(x^{v_2}_{i_2}), \varphi_{e^2}(x^{v_3}_{i_3}), \ldots$
$\ldots, \varphi_{e^{p-1}}(x^{v_{p-1}}_{i_{p-1}}), \varphi_{e^{p-1}}(x^{v_p}_{i_p}), \varphi_{e^p}(x^{v_p}_{i_p}), \varphi_{e^p}(x^{v_1}_{i_1})$.

These vertices (all of which need not be distinct) form a rainbow cycle of length $\leq 2p$ in the graph Γ_a which contradicts the nonexistence of short rainbow cycles guaranteed by Proposition 3.3 and thus A is a Hasse orientation.

Concluding remarks.

1. The above proof gives a stronger statement about the hardness of the recognition of cover graphs with bounded chromatic number ($\leq 4k_0$). One could ask if this can be further improved.

Indeed, it was proven by G. Brightwell [6] that recognition of cover graphs with chromatic number 4 is hard.

2. Another interesting open problem is the recognition of cover graphs of finite lattices. This is connected with the following problem:

The proof of Lemma 2.2 yields that for every K there exists a polynomial construction which to every graph G constructs a graph G^* with the following properties:

(1) G^* is triangle free;

(2) $\chi(G) = \chi(G^*)$ providing $\chi(G) < K$.

It would be interesting to refine this in both directions; for graphs of girth > 4 and for graphs with (unbounded) chromatic number.

In [7], the positive answer to this problem is given. This can be used to show that the problem of recognition of cover graphs of finite lattices is NP-hard as well.

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