# On Fréchet differentiability of convex functions on Banach spaces

Wee-Kee Tang*

*Abstract.* Equivalent conditions for the separability of the range of the subdifferential of a given convex Lipschitz function $f$ defined on a separable Banach space are studied. The conditions are in terms of a majorization of $f$ by a $C^1$-smooth function, separability of the boundary for $f$ or an approximation of $f$ by Fréchet smooth convex functions.

*Keywords:* Fréchet differentiability, convex functions, variational principles, Asplund spaces

*Classification:* 46B03

It is known that on a given separable Banach space $X$ all continuous convex functions are generically Fréchet differentiable if and only if $X^*$ is separable, and if and only if $X$ admits a $C^1$-smooth bump function. In this case, every equivalent norm in $X$ can be uniformly approximated by Fréchet smooth equivalent norms on bounded sets.

The purpose of this note is to generalize these results. We give some equivalent conditions for the generic Fréchet differentiability of a given Lipschitz convex function defined on a separable Banach space in terms of the properties of the function $f$ rather than that of $X$. In this setting, we cover some continuous convex functions defined on separable non-Asplund spaces. For instance if $\|\cdot\|$ denotes the Hilbertian norm on $l_2$ and $T$ is a continuous linear map of a separable Banach space $X$ into $l_2$, then any Lipschitz convex function $f$ defined on $X$ such that $f(x) \leq \|T(x)\|^2$ for $x \in X$ satisfies the assumptions in Theorem 1. At the end of this note, we show how the methods from variational principles can be applied to find a sufficient condition for the $w^*$-lower semicontinuity of convex functions.

A standard notation is used in this paper. We denote by $\partial f(x)$ the subdifferential of a continuous convex function $f$ at $x \in X$ (cf. [DGZ], [Ph]), and $\partial f(X) = \bigcup_{x\in X} \partial f(x)$. If $f$ is defined on $X$, $f^*$ denotes the Fenchel dual (or conjugate) of $f$, i.e. $f^*(x^*) = \sup\{(x^*, x) - f(x) : x \in X\}$, for $x^* \in X^*$. A convex continuous function is said to be generically Fréchet differentiable if it is Fréchet differentiable on a dense $G_\delta$ set. A subset $B \subset \partial f(X)$ is called a boundary for $f$ if $B$ intersects $\partial f(x)$ for each $x \in X$ (see e.g. [G]). By a selector for $\partial f$ we mean a single-valued mapping $s : X \to X^*$ such that $s(x) \in \partial f(x)$ for every $x \in X$.

* Supported by NSERC (Canada).

Unless stated otherwise, all topological terms in dual Banach spaces refer to the norm topology of these spaces. We refer to [Ph] and [DGZ] for some unexplained notions and results used in this note.

A main result in this note is the following statement.

**Theorem 1.** *Let $X$ be a separable Banach space and $f$ be a Lipschitz convex function defined on $X$. The following are equivalent.*

(i) *The set $\partial f(X)$ is separable.*
(ii) *There is a selector $s$ for $\partial f$ such that $s(X) = \{s(x) : x \in X\}$ is separable.*
(iii) *There is a continuously Fréchet differentiable function $\phi$ such that $\phi \geq f$ on $X$.*
(iv) *$f$ can be approximated uniformly on $X$ by Fréchet differentiable convex functions.*
(v) *If $h$ is a convex function on $X$ such that $h \leq f$ on $X$, then $h$ is generically Fréchet differentiable on $X$.*

Proof: Clearly (i) $\Rightarrow$ (ii). We shall show (ii) $\Rightarrow$ (i) using Simons' lemma ([S]). Put $B = s(X)$ and let $\gamma = \inf\{f^*(y^*) : y^* \in B\} < \infty$. We show that $C := \operatorname{dom} f^* \subset \overline{conv} B$. If this is not so, pick $y_o^* \in C \setminus \overline{conv} B$. By separation theorem, there exist $z \in X^{**}$ and $\alpha, \beta \in \mathbb{R}$ such that $z(y_o^*) > \beta > \alpha > z(y^*)$ for each $y^* \in B$. Without any loss of generality, we assume that $\frac{\beta-\alpha}{2} > f^*(y_o^*) - \gamma$. For every $x \in X$, define a function $h_x \in l^\infty(B)$ by

$$h_x(x^*) = (x^*, x) - f^*(x^*).$$

Let $E = \{x \in X : \|x\| \leq \|z\|,\ x(y_o^*) > \beta\}$. Since $B$ is separable, there exists a sequence $\{x_n\}$ in $E$ such that $x_n$ converges to $z$ in the topology of pointwise convergence on $B$. Define a sequence $h_n \in l^\infty(C)$ by $h_n(x^*) = h_{x_n}(x^*)$ for $x^* \in C$. Note that for any $x = \sum_{k=1}^\infty \lambda_k x_k$, where $\lambda_k \geq 0$ and $\sum_{k=1}^\infty \lambda_k = 1$, we have $s(x) \in B$ and

$$\begin{aligned}\sum \lambda_k h_k(s(x)) &= h_x(s(x)) = f(x) = \sup\{(x^*, x) - f^*(x^*) : x^* \in C\} \\ &= \sup\{(x^*, \sum \lambda_k x_k) - f^*(x^*) : x^* \in C\} = \sup\{\sum \lambda_k h_k(x^*) : x^* \in C\}.\end{aligned}$$

Since $z(y^*) < \alpha$ for all $y^* \in B$, we have $\limsup x_n(y^*) \leq \alpha$ for all $y^* \in B$. Thus $\limsup h_n(y^*) \leq \alpha - f^*(y^*)$ for all $y^* \in B$ and consequently $\sup\{\limsup h_n(y^*) : y^* \in B\} \leq \alpha - \gamma$. By Simons' lemma there is $g \in conv\{h_n\}$, $g = \sum_{k=1}^N \varrho_k h_k$, $\varrho_k \geq 0$, $\sum_{k=1}^N \varrho_k = 1$ such that

$$\sup\{g(x^*) : x^* \in C\} \quad \leq \quad \frac{\alpha+\beta}{2} - \gamma.$$

On the other hand, $g(y_o^*) = \sum_{k=1}^{N} \varrho_k h_k(y_o^*) = (y_o^*, \sum_1^N \varrho_k x_k) - f^*(y_o^*) > \beta - f^*(y_o^*)$ and thus $\beta - f^*(y_o^*) < \frac{\alpha+\beta}{2} - \gamma$. Therefore $\frac{\beta-\alpha}{2} < f^*(y_o^*) - \gamma$. This contradiction shows that (ii) implies (i).

(iii) $\Rightarrow$ (i). We follow the idea in [F]. Let $x \in X$, $q \in \partial f(x)$ and $\epsilon > 0$ be given. The function $\phi - q$ is a bounded below continuous function on $X$. By Ekeland's variational principle, there is a $x_q$ such that for each $h \in X$ and $t > 0$,

$$(\phi - q)(x_q + th) \geq (\phi - q)(x_q) - \epsilon \|h\| t.$$

Hence,

$$\|\phi'(x_q) - q\|^* \leq \epsilon.$$

Therefore $\partial f(X) \subset \overline{\{\phi'(x) : x \in X\}}$. Since $\phi'$ is continuous and $X$ is separable, the set $\overline{\{\phi'(x) : x \in X\}}$ and thus also $\partial f(X)$ are separable.

(i) $\Rightarrow$ (v). By using the above argument for the functions $h$ and $f$, we see that $\partial h(X) \subset \overline{\partial f(X)}$. Therefore $\partial h(X)$ is also separable and the statement follows immediately from the proof of Theorem 1 in [Pr-Z] (see also [Ph, Theorem 2.11]).

(v) $\Rightarrow$ (i). Since $\partial f(X) \subset \operatorname{dom} f^*$, it suffices to show that $\operatorname{dom} f^*$ is separable. We split $\operatorname{dom} f^*$ into $w^*$-compact sets $C_n$ and show that all $C_n$ are norm separable. We put $C_n = \{x^* \in X^* : f^*(x^*) \leq n\}$, and note that $\operatorname{dom} f^* = \cup_{n=1}^{\infty} C_n$.

Assume for some $n \in \mathbb{N}$, the set $C_n$ is not norm separable. Since $C_n$ is compact and metrizable in the $w^*$-topology, we find a $w^*$-compact subset $A \subset C_N$ and $\epsilon > 0$ such that every $w^*$-slice has diameter greater than $\epsilon > 0$ (see the proof of [Ph, Theorem 2.19]). Define $h(x) = \sup\langle A, x\rangle - N, x \in X$. Then $h(x) \leq \sup\{\langle x^*, x\rangle - f^*(x^*) : x^* \in C_N\}$, and the function $h$ is nowhere Fréchet differentiable (see the proof of [Ph, Lemma 2.18]). As $h \leq f$ on $X$, we obtain a contradiction.

(i) $\Rightarrow$ (iv). Let $Y = \overline{span}\{\partial f(x) : x \in X\}$. Since $Y$ is norm separable, there is an equivalent norm $\|\cdot\|$ on $X$ such that its dual norm $\|\cdot\|^*$ on $X^*$ is locally uniformly rotund at points of $\operatorname{dom} f^*$. In other words, if $y \in \operatorname{dom} f^*$, $y_k \in X^*$, and $\lim(\frac{\|y_k\|^{*2}+\|y\|^{*2}}{2} - \|\frac{y_k+y}{2}\|^{*2}) = 0$, then $\lim \|y_k - y\|^* = 0$ (see e.g. the proof of [DGZ, Proposition IV.5.2]).

Now, define a sequence of functions $\{h_n\}$ on $X^*$ by $h_n(x^*) = f^*(x^*) + \frac{1}{4n^4}\|x^*\|^{*2}$. Clearly $\operatorname{dom} h_n = \operatorname{dom} f^*$. Note that if $n \in \mathbb{N}$ and $\lim(\frac{h_n(y)+h_n(y_k)}{2} - h_n(\frac{y+y_k}{2})) = 0$, then $\lim \|y_k - y\| = 0$. Define $g_n := f \Box n^4\|\cdot\|^2$, the infimal convolution of $f$ and $n^4\|\cdot\|^2$. Note that the function $g_n$ is a convex continuous function on $X$ for all $n$ and ${g_n}^* = h_n$.

Given $n \in \mathbb{N}$, $x \in X$ and $y \in \partial g_n(x)$, note that $h_n$ is rotund at $y$ with respect to $x$ in the sense of [ As-R], i.e., for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\{v : h_n(y+v) - h_n(y) - (x, v) \leq \delta\} \subset \epsilon B_{X^*}.$$

Indeed, if this is not so, there exists an $\epsilon > 0$ such that for all $k \in \mathbb{N}$, there is a $v_k$, with $\|v_k\| > \epsilon$ and

$$\frac{1}{2}h_n(y+v_k) - \frac{1}{2}h_n(y) - (x, \frac{v_k}{2}) \leq \frac{1}{2k}\,.$$

Since $h_n = {g_n}^*$, we have $x \in \partial h_n(y)$, and thus

$$(x, \frac{v_k}{2}) \leq h_n(y + \frac{v_k}{2}) - h_n(y)$$

Putting these two inequalities together, we obtain for every $k \in \mathbb{N}$,

$$\frac{h_n(y) + h_n(y+v_k)}{2} - h_n(y + \frac{v_k}{2}) \leq \frac{1}{2k}\,.$$

From the local uniform convexity of $h_n$, we have $\lim \|v_k\| = 0$, a contradiction. By [As-R, Proposition 4], $g_n$ is Fréchet differentiable at $x$ with the derivative $y$. By the proof of Lemma 2.4 in [MPVZ], one can show that $\lim g_n = f$ uniformly on $X$.

(iv) $\Rightarrow$ (iii). By (iv), there exists a Fréchet differentiable convex function $\psi$ such that $|\psi(x) - f(x)| \leq \frac{1}{2}$ for every $x \in X$. Then $\psi + 1$ is a desired function.

This completes the proof of Theorem 1. $\square$

Note that in Theorem 1, the implications (iii) $\Rightarrow$ (i) $\Rightarrow$ (v) are still valid without requiring $f$ to be Lipschitz. The assumption of separability of $X$ in the statement of Theorem 1 cannot be dropped in general. Indeed, Haydon constructed a nonseparable space $X$ where all convex continuous functions are generically Fréchet differentiable and yet no equivalent norm can be approximated uniformly on bounded sets by Fréchet differentiable convex functions (see e.g. [DGZ]).

Note also that in Theorem 1, it is crucial that the function $f$ be defined on the whole of $X$, as there may exist nowhere differentiable norms bounded on the open ball by constant functions.

The following statement shows how Ekeland's variational principle can be used in questions on $w^*$-lower semicontinuity of convex functions.

**Theorem 2.** *Let $X$ be a Banach space and $f$ be a $w^*$-lower semicontinuous Fréchet differentiable function on $X^*$. Then every norm-lower semicontinuous convex function $g$ on $X^*$ such that $g \leq f$ on $X^*$ is $w^*$-lower semicontinuous on $X^*$.*

Proof: We first note that $f'(X^*) := \cup\{f'(y) : y \in X^*\} \subset X$. Indeed, for any $y \in X^*$, $f'(y)$ is $w^*$-lower semicontinuous on $B_{X^*}$, as it is a uniform limit of $w^*$-lower semicontinuous functions on $B_{X^*}$. Since $f'(y)$ is linear, $f'(y)$ is $w^*$-continuous on $B_{X^*}$. By Banach-Dieudonné Theorem, $f'(y)$ is $w^*$-continuous on $X^*$. Hence $f'(y) \in X$ for any $y \in X^*$.

We claim that $\operatorname{dom} g^* \subset X$. Indeed, for any $h \in \operatorname{dom} g^*$, we have $\sup\{h(x^*) - g(x^*) : x^* \in X^*\} < \infty$. This implies that $f - h$ is bounded below. As in the proof of (iii) $\Rightarrow$ (i), we can show $h \in \overline{f'(X^*)}$. Therefore $h \in X$.

Since $g$ is norm-lower semicontinuous, we have $g = g^{**}{\restriction_{X^*}}$. However, $g^{**} = (g^*)^* = (g^*{\restriction_{\operatorname{dom} g^*}})^* = (g^*{\restriction_X})^*$. Hence $g$ is a dual to a function defined on $X$, therefore $g$ is $w^*$-lower semicontinuous. $\square$

**Acknowledgements.** The author would like to thank Professor V. Zizler for his guidance and Professor M. Fabian for his helpful suggestions and enlightening discussions on this paper.

## References


[As-R] Asplund E., Rockafellar R.T., *Gradients of convex functions*, Trans. Amer. Math. Soc. **139** (1968), 443–467.

[DGZ] Deville R., Godefroy G., Zizler V., *Renormings and Smoothness in Banach Spaces*, Pitman Monograph and Survey in Pure and Applied Mathematics **64**.

[F] Fabian M., *On projectional resolution of identity on the duals of certain Banach spaces*, Bull. Austral. Math Soc. **35** (1987), 363–371.

[G] Godefroy G., *Some applications of Simons' inequality*, Seminar of Functional Analysis II, Univ. of Murcia, to appear.

[MPVZ] McLaughlin D., Poliquin R.A., Vanderwerff J.D., Zizler V.E., *Second order Gateaux differentiable bump functions and approximations in Banach spaces*, Can. J. Math **45:3** (1993), 612–625.

[Ph] Phelps R.R., *Convex Functions, Monotone Operators and Differentiability*, Lect. Notes in Math., Springer-Verlag **1364** (1993) (Second Edition).

[Pr-Z] Preiss D., Zajíček D., *Fréchet differentiation of convex functions in Banach space with separable dual*, Proc. Amer. Math. Soc. **91** (1984), 202–204.

[S] Simons S., *A convergence theorem with boundary*, Pacific J. Math. **40** (1972), 703–708.


Department of Mathematics, University of Alberta, Edmonton, Alberta, T6G 2G1, Canada

(*Received* August 2, 1994)