Borel matrix

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Abstract. We study the Borel summation method. We obtain a general sufficient condition for a given matrix A to have the Borel property. We deduce as corollaries, earlier results obtained by G. Müller and J.D. Hill. Our result is expressed in terms belonging to the theory of Gaussian processes. We show that this result cannot be extended to the study of the Borel summation method on arbitrary dynamical systems. However, in the L^p-setting, we establish necessary conditions of the same kind by using Bourgain's entropy criterion.

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I. Introduction

The aim of this paper will be first to discuss some classical results concerning the so-called Borel matrix summation method, next to give useful criterias concerning the regularity of this particular summation method.

First we recall some well-known facts about (A,μ) uniformly distributed sequences, a topic that has motivated many researches in the last 40 years. We mainly refer to [KN] for proofs of the results we are recalling, as well as for related type of distribution. Interesting papers on this kind of uniformly distributed sequences are among many others, those of J.D. Hill [Hi], G. Müller [M], and W. Philipp [Ph].

Let $A = \{a_{n,k} : n,k \geq 1\}$ be an infinite matrix of real numbers satisfying the following regularity assumptions, in which we define a_n to be the n-th row of A,

$$\text{(1)} \qquad \begin{aligned} &\text{(i)} \quad \|A\| = \sup_{n\geq 1} \|a_n\|_1 < \infty, \\ &\text{(ii)} \quad \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} = 1. \end{aligned}$$

These conditions are to be compared with conditions for a matrix to be regular ([KN, Theorem of Silverman-Toeplitz, p. 62]) or to be a positive Toeplitz matrix ([KN, p. 60]). Let X be a compact metrizable space, and μ be a Borel probability on X. We recall that a sequence $\{x_k : k \geq 1\} \subset X$ is (A,μ) uniformly distributed ((A,μ) u.d.), if for any continuous function f on X, ($f \in \mathcal{C}(X)$),

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} f(x_k) = \int f \, d\mu.$$

When that property is satisfied for $\mu^{\mathbb{N}}$-almost every sequence $\{x_k : k \geq 1\} \subset X$, then we say that A is a Borel matrix (with respect to μ if necessary). When only studying the (A, μ) u.d. property of the sequence $\{x_k : k \geq 1\}$, we are led to a classical problem on the relative compactness of the sequence of measures $\nu_k = \sum_{j=1}^{\infty} a_{k,j}\delta_{x_j}$, $k = 1, 2, \cdots$, where δ_x stands for the Dirac measure at the point x. When considering that property for $\mu^{\mathbb{N}}$-almost all sequences $\{x_k : k \geq 1\}$, the problem takes another aspect and more on the set $\mathcal{A} = \{a_n : n \geq 1\}$ is involved. When for instance, A satisfies the Hill condition ([Hi]):

$$\forall\, \delta > 0, \quad \sum_{k=1}^{\infty} e^{-\frac{\delta}{\|a_k\|_2^2}} < \infty, \tag{2}$$

then A is a Borel matrix, whatsoever μ. Condition (2) is equivalent to

$$\lim_{\varepsilon \to 0} \varepsilon^2 \log \sharp\{n \geq 1 : \|a_n\|_2 \geq \varepsilon\} = 0. \tag{2'}$$

To show this equivalence, we can assume $\|a_k\|_2 \leq 1$ for all k. Assume $(2')$ holds. Given any fixed $0 < \delta \leq 1$, we can determine a positive integer k_δ such that

$$0 < \varepsilon \leq 2^{-k_\delta} \Rightarrow \sharp\{n \geq 1 : \|a_n\|_2 \geq \varepsilon\} \leq e^{\frac{\delta}{8\varepsilon^2}}.$$

Therefore

$$\begin{aligned}
\sum_{n : \|a_n\|_2 \leq 2^{-k_\delta}} e^{-\frac{\delta}{\|a_n\|_2^2}} &= \sum_{k=k_\delta}^{\infty} \sum \{e^{-\frac{\delta}{\|a_n\|_2^2}} : 2^{-k-1} < \|a_n\|_2 \leq 2^{-k}\} \\
&\leq \sum_{k=k_\delta}^{\infty} e^{-2^{2k}\delta} \sharp\{n \geq 1 : \|a_n\|_2 > 2^{-k-1}\} \\
&\leq \sum_{k=1}^{\infty} e^{-2^{2k}\delta} e^{2^{2k-1}\delta} \\
&< \infty.
\end{aligned}$$

Besides,

$$\sum \{e^{-\frac{\delta}{\|a_n\|_2^2}} : \|a_n\|_2 > 2^{-k_\delta}\} \leq e^{-\delta 2^{2k_\delta}} \sharp\{n \geq 1 : \|a_n\|_2 > 2^{-k_\delta}\} < \infty.$$

Hence,

$$\sum_{k=1}^{\infty} e^{-\frac{\delta}{\|a_k\|_2^2}} < \infty,$$

for arbitrary δ, which proves (2).

Now assume that (2) is satisfied and let

$$M(\delta)=\sum_{k=1}^{\infty}e^{-\frac{\delta}{\|a_k\|_2^2}}<\infty.$$

We get

$$M(\delta)\geq e^{-\frac{\delta}{\varepsilon^2}}\sharp\{n:\|a_n\|_2\geq\varepsilon\}$$

or else,

$$\log\,M(\delta)+\frac{\delta}{\varepsilon^2}\geq\log\,\sharp\{n:\|a_n\|_2\geq\varepsilon\}.$$

Multiplying both sides by ε^2, letting then ε tend to 0, gives

$$\delta\geq\limsup_{\varepsilon\to 0}\varepsilon^2\log\,\sharp\{n\geq 1:\|a_n\|_2\geq\varepsilon\}.$$

Since δ is arbitrary, we get (2').

Another, apparently different sufficient condition, is the following one due to G. Müller, ([M]), (see the Note 1 following Theorem 3)

$$\lim_{n\to\infty}\sum_{k=1}^{\infty}|a_{n,k}-a_{n,k+1}|\log\,k=0. \tag{3}$$

On the other hand, if μ is not a Dirac measure, then A is a Borel matrix relatively to μ only if

$$\lim_{n\to\infty}\|a_n\|_2=0, \tag{4}$$

([KN, Theorem 7.3, p. 211]). In the same book, p. 208, the authors notice "Unfortunately, no conditions on the matrix A are known that are both necessary and sufficient…". This has motivated us in this work. A closer look into the proof of Theorem 1.2 ([KN, p. 208]) concerning the Hill condition reveals that (2) alone is strong enough to imply

$$\mu^{\mathbb{N}}\{x=\{x_k:k\geq 1\}\ :\lim_{n\to\infty}\sum_{k=1}^{\infty}a_{n,k}f(x_k)=\int f\,d\mu\}=1, \tag{5}$$

for any bounded measurable function f on X. Further, (1) (ii) is just needed to show (5) for the constant functions. Next, the conclusion is obtained by passing to the uniform closure of the subspace generated by some countable set of functions. It is precisely at this stage that the assumption (1) (i) is needed. It is in fact no more necessary to work longer with a compact Hausdorff space; so we will assume that $(X,\mathcal{F},\mu)$ is a probability space with a countably generated σ-field $\mathcal{F}$. We replace the space of "test functions" $\mathcal{C}(X)$, by some $L^p(\mu)$, $2 \le p < \infty$, or the algebra $\mathfrak{G}_T$ defined in (8). Introduce some notations,

$$\forall\, n \ge 1, \quad S_n(f)(x) = \sum_{k=1}^{\infty} a_{n,k} f(x_k) \tag{6}$$

and define on the Bernoulli scheme $(X^{\mathbb{Z}}, \mathcal{F}^{\otimes\mathbb{Z}}, \mu^{\mathbb{Z}}, T)$ the extension of the above summation method:

$$\forall\, n \ge 1, \quad \tilde{S}_n(f) = \sum_{k=1}^{\infty} a_{n,k} f \circ T^k. \tag{7}$$

When f is only depending on the first coordinate $f(x) = f_1(x_1)$, then $\tilde{S}_n(f) = S_n(f_1)$. Let $h \in L^\infty(\mu^{\mathbb{Z}})$, or $S \subset L^\infty(\mu^{\mathbb{Z}})$; then $Alg_T(h)$ or $Alg_T(S)$ will denote the closed subalgebra of $L^\infty(\mu^{\mathbb{Z}})$ generated by the sets $\{h \circ T^n : n \in \mathbb{Z}\}$ or $\{h \circ T^n : n \in \mathbb{Z}, h \in S\}$. We view $L^\infty(\mu)$ as a closed subspace of $L^\infty(\mu^{\mathbb{Z}})$. Put

$$\mathfrak{G}_T = Alg_T(L^\infty(\mu)). \tag{8}$$

We will prove the convergence of the above extended summation method (6) on the algebra $\mathfrak{G}_T$ under weaker conditions than (2) or (3) in the Theorem 3 below. Lemma 1 below will be used (see Theorem 3 (ii) and the Note 2) to give informations concerning the algebra $\mathfrak{G}_T$. Clearly $\mathfrak{G}_T$ is the closure of S_T, where

$$\begin{aligned} S_T = \text{span}\,\{h(x) = f_1(x_{i_1}) \times \cdots \times f_p(x_{i_p}) \ : \ f_1, \cdots, f_p \in L^\infty(\mu), \\ i_1 < \cdots < i_p, p \ge 1\}. \end{aligned} \tag{9}$$

We use the majorizing measure method introduced by A.M. Garsia, G. Rodemich and H. Rumsey ([GRR]). Lemma 1 will identify some elements of the algebra $\mathfrak{G}_T$ when the space $(X, \mathcal{F}, \mu)$ is sufficiently regular, or said equivalently, has good projectors. Let (T, d) be a separable pseudo-metric space, let us denote by $\mathfrak{T}$ its Borel σ-field, and consider a probability measure $\mathfrak{m}$ on $(T, \mathfrak{T})$. We make the following assumption:

There exists a family $\Pi = \{\pi_n : n \ge 1\}$ *of finite measurable partitions of* $(T, \mathfrak{T})$, *ordered by inclusion, generating* $\mathfrak{T}$, *and some Young function* ϕ *such that*

$$\lim_{N\to\infty} \sum_{n=N}^{\infty} \sum_{\pi\in\Pi_n} \varepsilon(\pi)\phi^{-1}\Big(\frac{1}{\mathfrak{m}^2(\pi)}\Big) = 0, \tag{10}$$

where $\varepsilon(\pi) = \sup\{d(u, v) : u \in \pi, v \in \pi_1\}$, if $\pi_1 \in \Pi_{n-1}$ is such that $\pi \subset \pi_1$.

Let P_n be the conditional expectation projector with respect to the Boolean algebra generated by Π_n, $n \ge 1$. We put

$$\forall\, N \ge 1, \; R_N = \sum_{n=N}^{\infty} \sum_{\pi\in\Pi_n} \varepsilon(\pi)\phi^{-1}\Big(\frac{1}{\mathfrak{m}^2(\pi)}\Big).$$

We also define for any $f \in L^1(\mathfrak{m})$,

$$\forall\,(u,v) \in T \times T,\ \tilde{f}_d(u,v) = \frac{f(u)-f(v)}{d(u,v)}\mathbf{1}_{d(u,v)\neq 0}\,. \tag{11}$$

Let P_n be the conditional expectation operator with respect to the Boolean algebra generated by Π_n, $n \geq 1$. The following lemma is very classical and is easy to deduce from the work in [GRR]. For the sake of completeness, a very short proof of that lemma is sketched after the proof of Theorem 3 in Section III.

Lemma 1. *Assume that* (10) *is satisfied, and let $f \in L^1(\mathfrak{m})$. Then,*

$$\forall\,N \geq 1,\ \|f - P_N(f)\|_{\infty,\mathfrak{m}} \leq K\ R_N\ \|\tilde{f}_d\|_{\phi,\mathfrak{m}\times\mathfrak{m}}\,, \tag{12}$$

where K is some numerical constant, and $\|.\|_{\phi,\mathfrak{m}\times\mathfrak{m}}$ denotes the Orlicz norm on $(T\times T, \mathfrak{m}\times\mathfrak{m})$ relatively to ϕ.

It will also be necessary for us to recall for the sequel of the paper a useful Gaussian concept. Any Hilbert space H defines a Gaussian process, often called *isonormal process*, $\{Z_a : a \in H\}$ which is specified by these conditions:

$$\forall\,a,b \in H, \quad \mathbb{E}\,Z_a = 0, \qquad \mathbb{E}\,Z_a.Z_b = \langle a,b\rangle. \tag{13}$$

Referring to ([D]), we say that a non-empty subset A of a Hilbert space H is a *GB* (resp. *GC*) set, if the restriction to A of the isonormal process Z on H has a version (that we denote by Z) which is sample bounded (resp. sample norm-continuous) on A. By the 0-1 laws of the Gaussian processes, A is a *GB* set if and only if

$$\mathbb{E}\,[\sup_{a\in A} |Z_a|] < \infty.$$

We will use this property later. We refer to [T] for characterizations of *GB* and *GC* sets. Introduce now the notion of $G(1)$-set.

Definition 2. *A sequence $\{a_n : n \geq 1\}$ of elements $a_n \in l_1$ is a $G(1)$-set if for all $\varepsilon > 0$, there exists $\{b_n : n \geq 1\}$ bounded in l_1 such that* $\lim_{n\to\infty} \|b_n\|_2 = 0$, *$\{b_n : n \geq 1\}$ is a GC subset of l_2 and* $\limsup_{n\to\infty} \|a_n - b_n\|_1 \leq \varepsilon$.

II. A sufficient condition

We can state the main result of this section.

Theorem 3. (i) *Let A be a matrix satisfying* (1) (i), (4), *and*

$$\forall\,n \geq 1,\ \sum_{k=1}^{\infty} a_{n,k} = 1. \tag{14}$$

If the sequence $\mathcal{A}$ of rows of A is a $G(1)$-set, then

$$\forall\, f \in \mathfrak{G}_T,\ \lim_{n\to\infty} \tilde{S}_n(f) = \int_{X^{\mathbb{Z}}} f\, d\mu^{\mathbb{Z}} \qquad \mu^{\mathbb{Z}}\text{- a.e.} \tag{15}$$

(ii) *Let d be a pseudo-metric on $X^{\mathbb{Z}}$, such that its Borel σ-field coincides with the product σ-algebra and* (10) *is satisfied. Then, with the notations* (11)

$$\mathfrak{G}_T \supset \{f \in L^1(\mu)\ :\ \tilde{f}_d \in L^{\phi}(\mu^{\mathbb{Z}} \times \mu^{\mathbb{Z}})\}. \tag{16}$$

(iii) *Assume that A is a matrix summation method on a compact metric space, satisfying* (1) *and* (4). *Then a sufficient condition for A to have the Borel property, is that $\mathcal{A}$ be a $G(1)$-sequence.*

Note 1. If $\mathcal{A}$ is a GC subset of l_2, and if (1) and (4) hold, then clearly $\mathcal{A}$ is a $G(1)$-sequence. But this is a too strong requirement for checking the Borel property. Our result shows that it is enough for $\mathcal{A}$, to be arbitrarily close in the l_1-sense to some GC-set, also satisfying (1) and (4). Such an observation is not new. In [M], G. Müller already observed that a matrix method with a matrix A satisfying the regularity assumption (1) has the Borel property whenever (3) holds. A closer look into his proof shows that (3) allows to build an auxiliary matrix B satisfying (1) and (2), so that $\mathcal{B} = \{b_n : n \geq 1\}$ is a GC set. Moreover, $\lim_{n\to\infty} \|a_n - b_n\|_1 = 0$. We refer to [M, p. 438-440] for details. But this is now just a particular case of Theorem 3.

Note 2. We shall apply the above result to symbolic flows. Let Λ be a finite alphabet with $Card\,(\Lambda) = p$, and $X = \Lambda^{\mathbb{Z}}$. Let μ be the uniform measure on Λ. We provide the symbolic flow (X, T) with the lexicographic distance

$$d_\alpha(x, x') = \inf\{\alpha^{k+1}\ :\ x_i = x'_i \ \text{ for } \ |i| < k\}, \quad 0 < \alpha < 1.$$

Then (X, d_α) is a compact ultrametric space. Let Π_n be the collection of all closed d_α-balls of radius α^n, $n \geq 0$. We notice that it is independent of the value of α. So, we define an ordered sequence of finite measurable partitions of X. Further the $\mu^{\mathbb{Z}}$-measure of any ball of radius α^n is precisely p^{-n}. For a Young function ϕ, the condition in (10) simply reduces to

$$\sum_{n=1}^{\infty} \alpha^n \phi^{-1}(p^{2n}) < \infty. \tag{17}$$

Besides, for $f \in Lip\,(d_\alpha)$, $\|\tilde{f}_d\|_{\phi,\mu^{\mathbb{Z}}\times\mu^{\mathbb{Z}}} < \infty$ always. Since it is always possible to find ϕ such that (17) is satisfied, we deduce that the corresponding algebra $\mathfrak{G}_T$ on the symbolic flow satisfies

$$\bigcup_{0<\alpha<1} Lip\,(d_\alpha) \subset \mathfrak{G}_T.$$

III. Proof of Theorem 3

The proof of Theorem 3 relies on an adaptation of a useful lemma ([KN, Lemma 7.2, p. 209]), that we recall for the convenience of the reader. We write $\mathbf{1}_\mu^\perp$ for the one-dimensional subspace of $L^1(\mu)$ consisting of the functions $f \in L^1(\mu)$ such that $\int f\,d\mu = 0$.

Lemma 4. (a) *Let $f \in L^\infty(\mu) \cap \mathbf{1}_\mu^\perp$. Then, for all real numbers u,*

$$\int_X \exp(uf)\,d\mu \le \exp[\frac{1}{2}u^2\|f\|^2_{\infty,\mu}]\,.$$

(b) *Further, for $n, m > 0$,*

$$\int_{X^{\mathbb{Z}}} \exp[u(S_n - S_m)(f)]\,d\mu^{\mathbb{Z}} \le \exp[\frac{1}{2}u^2\|a_n - a_m\|_2^2\ \|f\|^2_{\infty,\mu}]\,.$$

(c) *For any $t > 0$,*

$$\mu\{|(S_n - S_m)(f)| > t\} \le 2\exp[-\frac{t^2}{2\|a_n - a_m\|_2^2\ \|f\|^2_{\infty,\mu}}]\,,$$

and therefore,

$$\|(S_n - S_m)(f)\|_{\Psi_2,\mu^{\mathbb{Z}}} \le K\|a_n - a_m\|_2\|f\|_{\infty,\mu}\,,$$

where $\Psi_2(x) = e^{x^2} - 1$, and K is some universal constant.

We will need the following

Lemma 5. *Under assumption* (14), *for any $f \in S_T$, there is a positive finite constant K_f, depending on f only, such that*

$$\forall\, n, m \ge 1,\ \|(\tilde S_n - \tilde S_m)(f)\|_{\Psi_2,\mu^{\mathbb{Z}}} \le K_f\|a_n - a_m\|_2\,. \tag{18}$$

Proof: The subset $\mathcal{S}_T$ of measurable functions f satisfying (18) is clearly a vector space. Condition (14) readily implies that $\mathcal{S}_T$ contains the constants, and by part (c) of Lemma 4, it also contains any function $\tilde f$ such that

$$\tilde f(x) = f(x_k),\ x = \{x_k : k \in \mathbb{Z},\},\ \text{for some fixed } k \in \mathbb{Z},\ \text{and } f \in L^\infty(\mu) \cap \mathbf{1}_\mu^\perp\,.$$

Now let us consider $F(x) = f(x_1)g(x_2)$, $f, g \in L^\infty(\mu)$. For proving that $F \in S_T$, we can assume $f, g \in L^\infty(\mu) \cap \mathbf{1}_\mu^\perp$. Writing for simplicity $b = a_n - a_m$,

$$\int_{X^{\mathbb{Z}}} \exp\ [u(\tilde S_n - \tilde S_m)(F)]\,d\mu^{\mathbb{Z}} = \int_{X^{[2,\infty(}} \exp[u(\tilde S_n - \tilde S_m)(F)]\,d\mu(x_2)\,d\mu(x_3)\cdots\ ,$$

and by part (a) of Lemma 4

$$\begin{aligned}
&\int \exp\,[u(\tilde S_n-\tilde S_m)(F)]\,d\mu(x_2)\\
&\quad=\exp\,[\sum_{k=2}^{\infty} ub_kF(T^kx)]\int \exp\,[ub_1f(x_2)g(x_3)]\,d\mu(x_2),\\
&\quad\le \exp\,[\sum_{k=2}^{\infty} ub_kF(T^kx)]\exp\,[\frac12 u^2b_1^2\|f\|_{\infty,\mu}^2 g(x_3)^2]\\
&\quad\le \exp\,[\sum_{k=2}^{\infty} ub_kF(T^kx)]\exp\,[\frac12 u^2b_1^2\|f\|_{\infty,\mu}^2\|g\|_{\infty,\mu}^2]\,.
\end{aligned}$$

By iterating that argument and applying Fatou's lemma, we get

$$\int_{X^{\mathbb{Z}}}\exp[u(\tilde S_n-\tilde S_m)(F)]\,d\mu^{\mathbb{Z}}\le \exp\,[\frac12\sum_{k=1}^{\infty}u^2b_k^2\|f\|_{\infty,\mu}^2\|g\|_{\infty,\mu}^2]\,, \tag{19}$$

for any real u. By a standard argumentation this in turn implies

$$\|(\tilde S_n-\tilde S_m)(F)\|_{\Psi_2,\mu^{\mathbb{Z}}}\le C\|f\|_{\infty,\mu}^2\|g\|_{\infty,\mu}^2\|b\|_2\,,$$

where $0<C<\infty$ is some numerical constant. Hence we have proved $F\in\mathcal{S}_T$. What is done for a product of two elements of $L^\infty(\mu)$, can be extended to any finite product:

$$F(x)=f_1(x_{i_1})\times\cdots\times f_p(x_{i_p}),\quad x=\{x_k:k\in\mathbb{Z},\},$$

where $i_1<\cdots<i_p$, $p\ge 1$, and $f_1,\cdots,f_p\in L^\infty(\mu)$, by the same method. Hence, $S_T\subset\mathcal{S}_T$. □

Proof of Theorem 3:

Step 1. Assume that $\mathcal{A}$ is a *GC* set. Let μ be any probability on $(X,\mathcal{F})$. By the above lemma, for $F\in S_T\ominus\mathbf{1}_{\mu^{\mathbb{Z}}}$, $S(F)=\{\tilde S_n(F):n\ge1\}$ defines a subgaussian process with basic probability space $(X^{\mathbb{Z}},\mathcal{F}^{\mathbb{Z}},\mu^{\mathbb{Z}})$. By assumption (1.4) and by letting $\tilde S_\infty(F)=0$, this process is mean quadratic continuous at infinity. According to a classical result (see [T], for instance), in order that $S(F)$ be sample continuous with respect to the Hilbert norm, it is enough that $\mathcal{A}$ be a *GC* set. Thus, for any $F\in S_T\ominus\mathbf{1}_{\mu^{\mathbb{Z}}}$,

$$\mu^{\mathbb{Z}}\{\lim_{n\to\infty}\tilde S_n(F)=0\}=1,$$

or, for any $F\in S_T$,

$$\mu^{\mathbb{Z}}\{\lim_{n\to\infty}\tilde S_n(F)=\int F\,d\mu^{\mathbb{Z}}\}=1.$$

Since S_T is countably dense in $\mathfrak{G}_T$, we get

$$\mu^{\mathbb{Z}}\{\forall\, F \in S_T \ : \ \lim_{n\to\infty} \tilde{S}_n(F) = \int F\, d\mu^{\mathbb{Z}}\} = 1.$$

Now, we shall make use of the assumption (1) (i). Let $F \in \mathfrak{G}_T$, and $G \in S_T$ be such that $\|F - G\|_{\infty,\mu^{\mathbb{Z}}} < \varepsilon$. Then,

$$\begin{aligned}
&\|\tilde{S}_n(F) - \int F\, d\mu^{\mathbb{Z}}\| \leq \\
&|\sum_{k=1}^{\infty} a_{n,k}(F\circ T^k - G\circ T^k) - \int (F-G)\, d\mu^{\mathbb{Z}}| + |\sum_{k=1}^{\infty} a_{n,k} G\circ T^k - \int G\, d\mu^{\mathbb{Z}}| \\
&\leq \|A\|\|F-G\|_{1,\mu^{\mathbb{Z}}} + \|F-G\|_{\infty,\mu^{\mathbb{Z}}} + |\sum_{k=1}^{\infty} a_{n,k} G\circ T^k - \int G\, d\mu^{\mathbb{Z}}| \\
&\leq [\|A\| + 1]\varepsilon + \varepsilon,
\end{aligned} \tag{20}$$

for all sufficiently large integers n. The conclusion follows.

Step 2. Let $\varepsilon > 0$ be fixed. By extracting from the sequence $\{a_n : n \geq 1\}$ a subsequence if necessary, we can assume,

$$\forall\, n \geq 1,\ \|a_n - b_n\|_1 \leq \varepsilon,$$

where $\mathcal{B} = \{b_n : n \geq 1\}$ is a GC subset of l_2 contained in $c_0(l_2) \cap l_1$. For $F \in \mathfrak{G}_T$, we have

$$\begin{aligned}
&|\sum_{k=1}^{\infty} a_{n,k} F\circ T^k - \int F\, d\mu^{\mathbb{Z}}| \\
&\leq |\sum_{k=1}^{\infty} (a_{n,k} - b_{n,k}) F\circ T^k| \ + \ |\sum_{k=1}^{\infty} b_{n,k} F\circ T^k - \int F\, d\mu^{\mathbb{Z}}| \\
&\leq \|a_n - b_n\|_1 \|F\|_{\infty,\mu^{\mathbb{Z}}} \ + \ |\sum_{k=1}^{\infty} b_{n,k} F\circ T^k - \int F\, d\mu^{\mathbb{Z}}|\,.
\end{aligned} \tag{21}$$

From the first step we deduce

$$\limsup_{n\to\infty} |\sum_{k=1}^{\infty} a_{n,k} F\circ T^k - \int F\, d\mu^{\mathbb{Z}}| \leq \varepsilon \|F\|_{\infty,\mu^{\mathbb{Z}}}\,.$$

Since ε can be in fact chosen as small as we wish, we get

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} F\circ T^k = \int F\, d\mu^{\mathbb{Z}}\,.$$

The conclusion therefore follows by arguing as along the last lines of part (a). □

Proof of Lemma 1: It is classical. Let $h \in L^1(\mu)$.

$$|P_n(h) - P_{n-1}(h)| = \sum_{\pi\in\Pi_n} |\frac{1}{\mathfrak{m}(\pi)}\int_\pi h\, d\mathfrak{m} - \frac{1}{\mathfrak{m}(\pi^*)}\int_{\pi^*} h\, d\mathfrak{m}|,$$

where π^* is defined by the relations: $\pi^* \in \Pi_{n-1}$ and $\pi \subset \pi^*$. By letting $c = \|\tilde h_d\|_{\phi,\mathfrak{m}\otimes\mathfrak{m}}$, then applying Jensen's inequality, we get

$$(22)\quad \begin{aligned} &|\int_{\pi\times\pi^*}[h(u)-h(v)]\,\frac{d\mathfrak{m}(u)d\mathfrak{m}(v)}{\mathfrak{m}(\pi)\mathfrak{m}(\pi^*)}| \\ &= |c\int_{\pi\times\pi^*} d(u,v)\phi^{-1}\circ\phi(\frac{\tilde h_d(u,v)}{c})\frac{d\mathfrak{m}(u)d\mathfrak{m}(v)}{\mathfrak{m}(\pi)\mathfrak{m}(\pi^*)}| \\ &\le \varepsilon(\pi)c\phi^{-1}\Big(\frac{\int\int_{X\times X}\phi(\frac{\tilde h_d(u,v)}{c})\,d\mathfrak{m}(u)d\mathfrak{m}(v)}{\mathfrak{m}(\pi)\mathfrak{m}(\pi^*)}\Big) \\ &\le \varepsilon(\pi)c\phi^{-1}(\frac{1}{\mathfrak{m}^2(\pi)}). \end{aligned}$$

Hence,

$$|P_n(h) - P_{n-1}(h)| \le \sum_{\pi\in\Pi_n}\varepsilon(\pi)c\phi^{-1}(\frac{1}{\mathfrak{m}^2(\pi)})\,.$$

But by the martingale convergence theorem, $\mathfrak{m}\{\lim_{n\to\infty} P_n(h) = h\} = 1$.
Thus,

$$(23)\quad |h - P_N(h)| = |\sum_{n=N+1}^{\infty} P_n(h) - P_{n-1}(h)| \le \sum_{n=N+1}^{\infty}\sum_{\pi\in\Pi_n}\varepsilon(\pi)c\phi^{-1}(\frac{1}{\mathfrak{m}^2(\pi)}).$$

We get (12) by passing to the sup-norm in the above inequality. □

One may wonder whether the GC property of $\mathcal{A}$ remains sufficient to imply for arbitrary ergodic dynamical systems, the convergence of the associated matrix method for smooth functions like the algebra $\mathfrak{G}_T$. The theorem below provides a negative answer to that question, and therefore singles out, the role of Bernoulli's schemes for this study .

Theorem 6. *Let $(X, \mathcal{A}, \mu, \tau)$ be an ergodic dynamical system, where X is a compact metrizable space, and μ a diffuse probability measure. Let us assume that*

$$(24)\quad d(\tau(u), \tau(v)) \le d(u, v)$$

for all $u, v \in X$*, and some continuous pseudo-metric* d *on* X*. Let* $f \in Lip\,(d)$ *with* $\int f\,d\mu \neq 0$ *be such that, for any infinite matrix of real numbers* $A = \{a_{n,k} : n, k \geq 1\}$ *satisfying the regularity assumptions* (1) *and such that*

$$(24') \qquad \forall\, n \geq 1, a_{n,k} = 0, \quad \text{for } k \text{ large, and } \ \|\{a_{n,k} : k \geq 1\}\|_2 = O(n^{-\frac{1}{2}}),$$

we have

$$(25) \qquad \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} f \circ \tau^k = \int f\,d\mu\,.$$

Then f *must be a constant.*

In particular, if (25) holds for all matrix summation method A satisfying (1) and such that $\mathcal{A}$ is a GC set of l_2, then f must be a constant.

Proof: Since $f \in Lip\,(d)$, the averages

$$S_J(x, y) = \frac{1}{J} \sum_{j \leq J} f(\tau^j x) f(\tau^j y)\,,$$

form a $d \times d$-equicontinuous sequence of functions on $X \times X$, we thus know (see [K, p. 12, Theorem 2.6]) that these averages do converge for *all* $(x, y) \in X \times X$ to a limit $L(x, y)$ satisfying

$$\mu \times \mu\{(x, y) : L(x, y) = \mathbb{E}\,\{f \times f \bigm| \mathcal{F}_{\tau\times\tau}\}\} = 1,$$

where $\mathcal{F}_{\tau\times\tau}$ is the σ-subalgebra of $\mathcal{F} \otimes \mathcal{F}$ generated by the $\tau \times \tau$-invariant sets of $\mathcal{F} \otimes \mathcal{F}$. That limit is further continuous. Besides, for any *fixed* x the averages $S_J(x, y)$ can be obviously considered as a matrix summation method with matrix A^x of which the coefficients are given by

$$a^x_{j,J} = \frac{1}{J \int f\,d\mu} \mathbf{1}_{[1,J]}(j) f(\tau^j x)\,,$$

where $j, J \geq 1$. Observe that

$$\|a^x_J\|_2 = O(J^{-\frac{1}{2}})$$

for μ-almost all x's. This easily follows from Birkhoff's theorem. Thus $(24')$ is fulfilled. According to our assumption on f, we deduce

$$\mu\{y : \lim_{J\to\infty} \frac{1}{J} \sum_{j \leq J} f(\tau^j x) f(\tau^j y) = [\int f\,d\mu]^2\} = 1$$

for μ-almost all x's. By applying Fubini's theorem

$$\mu \times \mu\{(x,y) : \lim_{J\to\infty} \frac{1}{J} \sum_{j\le J} f(\tau^j x) f(\tau^j y) = [\int f\, d\mu]^2\} = 1. \tag{26}$$

Hence,

$$\mu \times \mu\{(x,y) : L(x,y) = [\int f\, d\mu]^2\} = 1.$$

But L is continuous on $T \times T$ and μ is assumed to be diffuse; this implies

$$L(x,y) = [\int f\, d\mu]^2 \quad \text{everywhere.} \tag{27}$$

In particular,

$$L(x,x) = [\int f\, d\mu]^2 \quad \text{for all } x. \tag{28}$$

But,

$$\mu\{x : \lim_{J\to\infty} \frac{1}{J} \sum_{j\le J} f(\tau^j x)^2 = \int f^2\, d\mu\} = 1. \tag{29}$$

Hence,

$$L(x,x) = \int f^2\, d\mu \quad \text{for all } x,$$

and thus

$$[\int f\, d\mu]^2 = \int f^2\, d\mu. \tag{30}$$

And this implies by "Schwarz's equality" that f is a constant on a measurable set of measure one. □

IV. Necessary conditions

We are concerned in this section with the case where the space of "test" functions is an L^p-space, $2 \le p < \infty$. We get necessary conditions for the convergence of matrix summability methods defined on general dynamical systems. These conditions are of the same type as the sufficient condition stated in Theorem 3. Let $(X, \mathcal{F}, \mu)$ be a Lebesgue space and denote by $\mathfrak{T}$ the group of automorphisms on $(X, \mathcal{F}, \mu)$. Let $A = \{a_{n,k} : n, k \ge 1\}$ be as before an infinite matrix of real numbers. Set formally

$$\forall T \in \mathfrak{T},\ \forall f \in L^p(\mu),\ \forall n \ge 1,\ S_n^T(f) = \sum_{k=1}^{\infty} a_{n,k} f \circ T^k\,. \tag{31}$$

Given any sequence of real numbers $b = \{b_n : n \geq 1\}$, the formal writing

$$\forall\, f \in L^p(\mu),\ \sigma_b^T(f) = \sum_{k=1}^{\infty} b_k f \circ T^k\,, \tag{32}$$

determines, by means of the uniform bound principle, a linear continuous operator, as the L^p-limit of the partial summation (continuous) operators $\sigma_{b,N}^T(f) = \sum_{k=1}^{N} b_k f \circ T^k$, if and only if,

$$\sup_{N\geq 1} \|\sigma_{b,N}\| < \infty\,. \tag{33}$$

In the sequel, we simply assume (1) (i), so that the operators S_n^T are always L^p-continuous for the range of values $0 < p \leq \infty$. We get necessary conditions concerning the property

$$(\mathcal{B}_p^T) \qquad \forall\, f\ in\ L^p(\mu),\ \sup_{n\geq 1} |S_n^T(f)| < \infty, \qquad \mu\text{-almost surely.}$$

The results below are extensions of the Bourgain's entropy theorem ([B]). We will need in the proofs a more straightforward proof of that remarkable result (see [SW]).

Theorem 7 $(2 \leq p < \infty)$**.** *If, for some ergodic automorphism T, $(\mathcal{B}_p^T)$ is satisfied, then necessarily,*

$$\sup_{S\in\mathfrak{T}} \sup_{\substack{f\in L^p(\mu)\\ \|f\|_{2,\mu}\leq 1}} \mathbb{E}\{\sup_{n\geq 1} |Z(S_n^S(f))|\} < \infty\,, \tag{34}$$

where Z is the isonormal process on $L^2(\mu)$.
In case where $(X, \mathcal{F}, \mu)$ is a product probability space: $(Y, \mathcal{B}, \nu)^{\mathbb{Z}}$, then the set $\mathcal{A}$ is a GB set of l_2.

Before proving this result, we will specialize it in case of matrix summations methods satisfying

$$\forall\, n \geq 1, \quad S_n(f) = \sum_{k=1}^{N_n} a_{n,k} f \circ T^k\,, \tag{35}$$

where N_n are positive integers. In that case, property $(\mathcal{B}_p^T)$ has a direct translation on the coefficients $a_{n,k}$. One can indeed prove

Theorem 8 ($2 \le p < \infty$)**.** *Let us consider a matrix summation method on* $(X, \mathcal{F}, \mu)$ *satisfying* (1) (i) *and* (35)*. If, for some ergodic automorphism* T*,* $(\mathcal{B}_p^T)$ *is satisfied, then necessarily,*

(36) $\mathcal{A}$ *is a GB set of* l_2*.*

Proof of Theorem 7: By means of the Banach's principle, we may find a $K < \infty$, such that

$$\sup_{\substack{f \in L^p(\mu) \\ \|f\|_{p,\mu} \le 1}} \mu\{\sup_{n \ge 1} |S_n^T(f)| > K\} \le \frac{1}{8}\,. \tag{37}$$

Let now T^* be the conjugate class of T. By means of the conjugacy lemma ([Ha, p. 77]), T^* is weakly dense in $\mathfrak{T}$. Moreover,

$$S_n^{R^{-1}TR}(f) = R^{-1}(S_n^T(f))R. \tag{38}$$

This shows, as in [C, p. 9], with the same constant K as in (37),

$$\sup_{S \in \mathfrak{T}} \sup_{\substack{f \in L^p(\mu) \\ \|f\|_{p,\mu} \le 1}} \mu\{\sup_{n \ge 1} |S_n^S(f)| > K\} \le \frac{1}{8}\,. \tag{39}$$

By means of the evaluation (2.8) in [SW], we can now easily conclude to (34), by mimicking the proof of Theorem 3.1 in [SW].
In case of infinite products, we choose

$$f \in \mathbf{1}_\nu^\perp,\ \|f\|_{2,\nu} = 1,\ f \in L^p(\nu)\,,$$

and we observe that $\|(S_n^T - S_m^T)(f)\|_{2,\mu} = \|a_n - a_m\|_2$, which easily leads to the property for $\mathcal{A}$ by (34). □

Proof of Theorem 8: By means of the Halmos-Kakutani-Rokhlin's lemma for any $\varepsilon > 0$, any $N \ge 0$, there exists a measurable set A such that $A, TA, \cdots, T^{N-1}A$, are pairwise disjoint and $\frac{1-\varepsilon}{N} \le \mu(A) \le \frac{1}{N}$. We set $f = 1_A$. Let n, m be such that $N_n \le N_m \le N$. Then,

$$\begin{aligned}
\|S_n(f) - S_m(f)\|_{2,\mu} &= \|\sum_{k=1}^{N_n}(a_{n,k} - a_{m,k}) f \circ T^k + \sum_{k=N_n+1}^{N_m} a_{m,k} f \circ T^k\|_{2,\mu} \\
&= [\sum_{k=1}^{N_n}(a_{n,k} - a_{m,k})^2 + \sum_{k=N_n+1}^{N_m} a_{m,k}^2]^{\frac{1}{2}} \sqrt{\mu}(A) \\
&= \|a_n - a_m\|_2 \sqrt{\mu}(A).
\end{aligned}$$

By means of Proposition 1 in [B] and Slepian lemma,

$$\mathbb{E}\{\sup_{n : N_n < N} Z(a_n)\} \le C, \tag{40}$$

where C is not depending on N. The conclusion is obtained by letting N tend to infinity. □

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