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Topological properties of the solution set of integrodifferential inclusions

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Abstract. In this paper we examine nonlinear integrodifferential inclusions in $\mathbb{R}^N$. For the nonconvex problem, we show that the solution set is a retract of the Sobolev space $W^{1,1}(T,\mathbb{R}^N)$ and the retraction can be chosen to depend continuously on a parameter λ. Using that result we show that the solution multifunction admits a continuous selector. For the convex problem we show that the solution set is a retract of $C(T,\mathbb{R}^N)$. Finally we prove some continuous dependence results.

Keywords: retract, absolute retract, path-connected, Vietoris continuous, h-continuous, orientor field

Classification: 34A60

1. Introduction

In two recent papers the authors considered integrodifferential inclusions of the Volterra type and established two existence results, one for convex orientor fields and the other for nonconvex ones (see Avgerinos [1] and Papageorgiou [19]).

In this paper we carry further our investigation of this class of set-valued systems by examining the topological structure of the solution set. So we show that the solution set of the "nonconvex" problem is a retract of the Sobolev space $W^{1,1}(T,\mathbb{R}^N)$, while that of the "convex" problem is a retract of the space $C(T,\mathbb{R}^N)$. In particular then in the nonconvex case we get the path-connectedness of the solution set in $W^{1,1}(T,\mathbb{R}^N)$, while in the convex case we have that the solution set is path-connected and compact in $C(T,\mathbb{R}^N)$.

Using the above topological result, we are able to prove the existence of a continuous selector for the solution multifunction, extending this way to a broader class of set-valued systems, a recent similar result of Cellina [3], which was proved with a different, more involved and less general method.

Finally we turn our attention to continuous dependence results and prove that the solution set of a parametric integrodifferential inclusion depends continuously in both the Vietoris and Hausdorff hyperspace topologies, on the parameter.

The topological structure of the solution set of differential inclusions was studied by various authors in the past. We refer to the fundamental works of Himmelberg-Van Vleck [10], DeBlasi-Myjak [4] and Gorniewicz [8]. More recently Staicu [22] and DeBlasi-Pianigiani-Staicu [6] obtained structural results similar to some of the theorems proved in this paper, for a class of the hyperbolic differential inclusions.

2. Mathematical preliminaries

Let (Ω, Σ) be a measurable space and X a separable Banach space. Throughout this paper we will be using the following notations:

$$P_{f(c)}(X) = \{A \subseteq X: \text{ nonempty, closed, (convex)}\}$$
$$\text{and } \; P_{(w)k(c)}(X) = \{A \subseteq X: \text{ nonempty, } (w\text{-})\text{compact, (convex)}\}.$$

A multifunction (i.e. a set-valued function) $F : \Omega \to P_f(X)$ is said to be measurable if for all $z \in X$ the $\mathbb{R}_+$-valued function $\omega \to d(z, F(\omega)) = \inf\{\|z - x\| : x \in F(\omega)\}$ is measurable. Also a multifunction $G : \Omega \to 2^X \setminus \{\emptyset\}$ is said to be "graph measurable", if $GrG = \{(\omega, x) \in \Omega \times X : x \in G(\omega)\} \in \Sigma \times B(X)$, where $B(X)$ is the Borel σ-field of X. We mention that for closed valued multifunctions, the measurability implies graph measurability. The converse is true if there exists a complete σ-finite measure $\mu(\cdot)$ on (Ω, Σ). For details we refer to Levin [13] and Wagner [24].

Now let $\mu(\cdot)$ be a finite measure defined on (Ω, Σ) and $F : \Omega \to 2^X \setminus \{\emptyset\}$ a multifunction. We use S_F^1 to denote the set of selectors of $F(\cdot)$ that belongs in the Lebesgue-Bochner space $L^1(X)$ i.e. $S_F^1 = \{f \in L^1(\Omega, X) : f(\omega) \in F(\omega)$ μ-a.e.$\}$. This set may be empty. When $F(\cdot)$ is L^1-integrably bounded (i.e. $F(\cdot)$ is measurable and $\omega \to |F(\omega)| = \sup\{\|z\| : z \in F(\omega)\} \in L^1(\Omega)$) then S_F^1 is nonempty. If $GrF \in \Sigma \times B(X)$, where $B(X)$ is the Borel σ-field of X (i.e. $F(\cdot)$ is graph measurable), then S_F^1 is nonempty if and only if $\omega \to \inf\{\|z\| : z \in F(\omega)\} \in L_+^1$.

Furthermore if $F(\cdot)$ is $P_f(X)$-valued, then S_F^1 is closed in $L^1(\Omega, X)$ and if $F(\cdot)$ is convex valued, then so is S_F^1.

In addition S_F^1 is decomposable; i.e. if $(f_1, f_2, A) \in S_F^1 \times S_F^1 \times \Sigma$, then $\chi_A f_1 + \chi_{A^c} f_2 \in S_F^1$. For further details on these and related issues, we refer to Hiai-Umegaki [9] and Papageorgiou [18].

On $P_f(X)$ we can define a generalized metric, known in the literature as the Hausdorff metric by setting $h(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}$.

It is well known that $(P_f(X), h)$ is a complete metric space. If V is a metric space, a multifunction $F : V \to P_f(X)$ is said to be Hausdorff continuous (h-continuous), if it is continuous from V into $(P_f(X), h)$.

Let Y, Z, be Hausdorff topological spaces and let $F : Y \to 2^Z \setminus \{\emptyset\}$ be a multifunction. We say that $F(\cdot)$ is upper semicontinuous (u.s.c.) if for all $V \subset Z$ open, the set $F^+(V) = \{y \in Y : F(y) \subseteq V\}$ is open in Y. Also we say that $F(\cdot)$ is lower semicontinuous (l.s.c.) if for all $U \subseteq Z$ open, the set $F^-(U) = \{y \in Y : F(y) \cap U \neq \emptyset\}$ is open in Y. Observe that when $F(\cdot)$ is single valued, then the notions above coincide with the continuity of $F(\cdot)$. If $F(\cdot)$ is both u.s.c. and l.s.c., we say that $F(\cdot)$ is Vietoris continuous (since on $2^Z \setminus \{\emptyset\}$ we consider the Vietoris hyperspace topology). Suppose Y, Z, are metric spaces. On $P_k(Z)$ the Vietoris and Hausdorff hyperspace topology coincide (see

Klein-Thompson [11, Corollary 4.2.3, p. 41]). So a multifunction $F : Y \to P_k(Z)$ is Vietoris continuous if and only if it is h-continuous. For a comprehensive introduction to these continuity concepts and additional results we refer to DeBlasi-Myjak [5] and Klein-Thompson [11].

Let Y be a Hausdorff topological space. A subset A of Y is said to be a "retract" of Y, if there is a continuous map $f : Y \to A$ such that $f(a) = a$ for all $a \in A$ (equivalently, if the identity map $i : A \to A$ is extendable to a continuous $f : Y \to A$).

Any such map $f(\cdot)$ is called a retraction of Y onto A. It is easy to check that a retraction A of Y is closed. A metrizable space Y is said to be an "absolute retract" (for metrizable spaces) if it can be substituted for $\mathbb{R}$ in Tietze's extension theorem; i.e. if for every metrizable X, every $A \in P_f(X)$ and each continuous map $f : A \to Y$, $f(\cdot)$ admits a continuous extension $\overline{f} : X \to Y$. An absolute retract is path-connected, a retract of an absolute retract is an absolute retract and finally thanks to Dugundji's extension theorem (see Dugundji [7, Theorem 6.1, p. 188]), a closed, convex set of a normed space is an absolute retract (and of course a retract). For details we refer to Kuratowski [12, pp. 339–371].

Finally if X is a Banach space and $\{A_n, A\}_{n\geq 1} \subseteq 2^X \setminus \{\emptyset\}$, we define:

$$s\text{-}\underline{\lim}\, A_n = \{x \in X : x = s\text{-}\lim x_n, x_n \in A_n, n \geq 1\},$$

$$s\text{-}\overline{\lim}\, A_n = \{x \in X : x = s\text{-}\lim x_{n_k}, x_{n_k} \in A_{n_k}, n_1 < n_2 < \cdots < n_k < \dots\},$$

and $w\text{-}\overline{\lim}\, A_n = \{x \in X : x = w\text{-}\lim x_{n_k}, x_{n_k} \in A_{n_k}, n_1 < n_2 < \cdots < n_k < \dots\}$,

(with s- denoting the strong topology on X and w- the weak topology on X).

We say that the A_n's converge to A in the Kuratowski sense by $A_n \xrightarrow{K} A$ if $s\text{-}\underline{\lim}\, A_n = s\text{-}\overline{\lim}\, A_n$. We say that A_n's converge to A in the Kuratowski-Mosco sense, denoted by $A_n \xrightarrow{K-M} A$, if $s\text{-}\underline{\lim}\, A_n = w\text{-}\overline{\lim}\, A_n = A$.

3. Topological structure of the solution set – nonconvex case

Let $T = [0, b]$ and Λ a complete metric space (the parameter space). On $T \times \mathbb{R}^N \times \Lambda$ we consider the following integrodifferential inclusions of the Volterra type:

$$\begin{aligned} &\dot{x}(t) \in F(t, x(t), (Vx)(t), \lambda) \text{ a.e.}, \\ &x(0) = x_0(\lambda). \end{aligned} \tag{1}$$

Here $V : C(T, \mathbb{R}^N) \to C(T, \mathbb{R}^N)$ is the nonlinear Volterra integral operator defined by $(Vx)(t) = \int_0^t K(t, s)g(s, x(s))\, ds$, $t \in T$.

By a solution of (1), we mean a function $W^{1,1}(T, \mathbb{R}^N)$ such that

$$\begin{aligned} &\dot{x}(t) = f(t) \text{ a.e.}, \\ &x(0) = x_0(\lambda) \text{ and } f(\cdot) \in L^1(T, \mathbb{R}^N), \\ &f(t) \in F(t, x(t), (Vx)(t), \lambda) \text{ a.e. (i.e. } f \in S^1_{F(\cdot,x(\cdot),(Vx)(\cdot),\lambda)}). \end{aligned}$$

Here $W^{1,1}(T,\mathbb{R}^N)$ denotes the Sobolev space of all functions $x : T \to \mathbb{R}^N$ such that $x(\cdot)$ is absolutely continuous (hence $\dot{x} \in L^1(T,\mathbb{R}^N)$) and is equipped with the Sobolev norm $\|x\|_{W^{1,1}(T,\mathbb{R}^N)} = \|x\|_{L^1(T,\mathbb{R}^N)} + \|\dot{x}\|_{L^1(T,\mathbb{R}^N)}$.

We will denote the solution set of (1) by $S(\lambda) \subseteq W^{1,1}(T,\mathbb{R}^N)$. We will make the following hypotheses on the data of (1):

H(F)$_\mathbf{1}$: $F : T \times \mathbb{R}^N \times \mathbb{R}^N \times \Lambda \to P_k(\mathbb{R}^N)$ is a multifunction such that
(1) $t \to F(t,x,y,\lambda)$ is measurable,
(2) $h(F(t,x,y,\lambda), F(t,x',y',\lambda)) \le k(t)(\|x-x'\| + \|y-y'\|)$ a.e., $k(\cdot) \in L^1_+$,
(3) $|F(t,x,y,\lambda)| = \sup\{\|v\| : v \in F(t,x,y,\lambda)\} \le a(t) + c(t)(\|x\| + \|y\|)$ a.e. with $a, c \in L^1_+$,
(4) $\lambda \to F(t,x,y,\lambda)$ is h-continuous.

H(K): $K : \Delta = \{(t,s) : 0 \le s \le t \le b\} \to \mathcal{L}(\mathbb{R}^N,\mathbb{R}^N)$ is a continuous map.

H(g): $g : T \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuous map such that
(1) $t \to g(t,x)$ is measurable,
(2) $\|g(t,x) - g(t,y)\| \le k_1(t)\|x-y\|$ a.e. with $k_1 \in L^1_+$,
(3) $\|g(t,x)\| \le a_1(t) + c_1(t)\|x\|$, a.e. with $a_1, c_1 \in L^1_+$.

H$_\mathbf{0}$: $\lambda \to x_0(\lambda)$ is continuous from Λ into $\mathbb{R}^N$.

Under the above hypotheses, we know that for every $\lambda \in \Lambda$, $S(\lambda) \neq \emptyset$ (see Theorem 3.3 of Papageorgiou [19]).

The next theorem determines the topological structure of $S(\lambda)$.

Theorem 3.1. *If the hypotheses $H(F)_1$, $H(K)$, $H(g)$ and H_0 hold, then there exists a continuous map $\gamma : W^{1,1}(T,\mathbb{R}^N) \times \Lambda \to W^{1,1}(T,\mathbb{R}^N)$ such that $\gamma(x,\lambda) \in S(\lambda)$ for all $(x,\lambda) \in W^{1,1}(T,\mathbb{R}^N) \times \Lambda$ and $\gamma(x,\lambda) = x$ for $x \in S(\lambda)$.*

Proof: Let $R : L^1(T,\mathbb{R}^N) \times \Lambda \to P_f(L^1(T,\mathbb{R}^N))$ be the multifunction defined by $R(h,\lambda) = S^1_{F(\cdot,\eta(f,\lambda)(\cdot),V(\eta(f,\lambda))(\cdot),\lambda)}$, where $\eta(f,\lambda)(t) = x_0(\lambda) + \int_0^t f(s)\,ds$, $t \in T$.

We introduce the following norm on $L^1(T,\mathbb{R}^N)$:
$|h| = \int_0^b \exp(-L\theta(s))\|h(s)\|\,ds$, $L > 1$, with $\theta(t) = \int_0^t \widehat{k}(s)\,ds$, where $\widehat{k}(s) = k(s)(1 + M\|k_1\|_1)$ and $M > 0$ is such that $\|K(t,s)\| \le M$ (cf. hypothesis $H(K)$).

Clearly this norm is equivalent to the usual one. By $d_1(\cdot,\cdot)$ we will denote the distance function corresponding to this norm and by $h_1(\cdot,\cdot)$ the Hausdorff metric that it generates. We will show that $h \to R(h,\lambda)$ is an h_1-contraction.

So let $h_1, h_2 \in L^1(T,\mathbb{R}^N)$ and let $f_1 \in R(h_1,\lambda)$.

Let
$$\begin{aligned}\Delta_\lambda(t) = \big\{v \in F(t,\eta(h_2,\lambda)(t),(V\eta(h_2,\lambda))(t),\lambda) : \|f_1(t) - v\| \\ = d(f_1,(t),F(t,\eta(h_2,\lambda)(t),(V\eta(h_2,\lambda))(t),\lambda))\big\}.\end{aligned}$$
From hypotheses $H(F)_1$ (1) and (2) and Theorem 3.3 of Papageorgiou [17], we know that $(t,x,y) \to F(t,x,y,\lambda)$ is measurable $\Longrightarrow$
$$\Longrightarrow t \to F(t,\eta(h_2,\lambda)(t),\ (V\eta(h_2,\lambda)(t),\lambda)) \text{ is measurable}$$

and so

$$t \to d(f_1, (t),\ F(t, \eta(h_2, \lambda)(t),\ (V\eta(h_2, \lambda))(t), \lambda)) \quad \text{is measurable.}$$

Hence $Gr\Delta_\lambda \in B(T) \times B(\mathbb{R}^N)$ with $B(T)$ (resp. $B(\mathbb{R}^N)$) being the Borel σ-field of T (resp. of $\mathbb{R}^N$).

Apply Aumann's selection theorem (see Wagner [24, Theorem 5.10]) to get $f_2 : T \to \mathbb{R}^N$ measurable such that $f_2(t) \in \Delta_\lambda(t)$a.e.

Then we have

$$\begin{aligned}
&d_1(f_1(t), R(h_2, \lambda)) \le |f_1 - f_2| \\
&= \int_0^b \exp(-L\theta(t)) \|f_1(t) - f_2(t)\| \, dt \\
&= \int_0^b \exp(-L\theta(t))\, d\big(f_1(t), F(t, \eta(h_2, \lambda)(t), (V\eta(h_2, \lambda))(t), \lambda)\big)\, dt \\
&\le \int_0^b \exp(-L\theta(t)) h\big(F(t, \eta(h_1, \lambda)(t), (V\eta(h_1, \lambda))(t), \lambda), \\
&\qquad F(t, \eta(h_2, \lambda)(t), (V\eta(h_2, \lambda))(t), \lambda)\big)\, dt \\
&\le \int_0^b \exp(-L\theta(t)) k(t) \big[\|\eta(h_1, \lambda)(t) - \eta(h_2, \lambda)(t)\| \\
&\qquad + \|(V\eta(h_1, \lambda))(t) - (V\eta(h_2, \lambda))(t)\|\big]\, dt.
\end{aligned}$$

Observe that $\|\eta(h_1, \lambda)(t) - \eta(h_2, \lambda)(t)\| \le \int_0^t \|h_1(s) - h_2(s)\| \, ds$.

Also we have

$$\begin{aligned}
&\|(V\eta(h_1, \lambda))(t) - (V\eta(h_2, \lambda))(t)\| \le \int_0^t M\ k_1(s) \|\eta(h_1, \lambda)(s) - \eta(h_2, \lambda)(s)\| \, ds \\
&\le M \int_0^t k_1(s) \int_0^s \|h_1(\tau) - h_2(\tau)\| \, d\tau \, ds \le M\|k\|_1 \int_0^t \|h_1(s) - h_2(s)\| \, ds.
\end{aligned}$$

So finally we get

$$\begin{aligned}
&d_1(f_1(t), R(h_2, \lambda)) \\
&\le \int_0^b \exp(-L\theta(t)) k(t) \Big[\int_0^t \|h_1(s) - h_2(s)\| \, ds + M\|k\|_1 \int_0^t \|h_1(s) - h_2(s)\| \, ds\Big]\, dt \\
&= \int_0^b \exp(-L\theta(t)) \widehat{k}(t) \int_0^t \|h_1(s) - h_2(s)\| \, ds \, dt \\
&\qquad \text{(recall that } \widehat{k}(t) = k(t)(1 + M\|k_1\|_1)) \\
&\le (-l/L) \int_0^b \big(\int_0^t \|h_1(s) - h_2(s)\| \, ds\big)\, d\exp(-L\theta(t)) \\
&= (-l/L) \int_0^b \|h_1(t) - h_2(t)\| \exp(-L\theta(t)) \, dt = (1/L)|h_1 - h_2|.
\end{aligned}$$

Interchanging the roles of h_1, h_2, in the above argument, we also get for $f_2 \in R(h_2, \lambda)$ that $d_1(f_2(t), R(h_1, \lambda)) \le (1/L)|h_1 - h_2|$.

Therefore we conclude that $h_1(R(h_1, \lambda), R(h_2, \lambda)) \le (1/L)|h_1 - h_2|$.

Next we will show that $(h, \lambda) \to R(h, \lambda)$ is h_1-continuous. To this end let $(h_n, \lambda_n) \to (h, \lambda)$ in $(L^1(T, \mathbb{R}^N), |\cdot|) \times \Lambda$. Then we have

$$\begin{aligned} &h_1(R(h_n, \lambda_n), R(h, \lambda)) \le h_1(R(h_n, \lambda_n), R(h, \lambda_n)) + h_1(R(h, \lambda_n), R(h, \lambda)) \\ &\quad \le (1/L)|h_n - h| + h_1(R(h, \lambda_n), R(h, \lambda)). \end{aligned}$$

Note that if $u \in R(h, \lambda_n)$, we have

$$\begin{aligned} &d_1(u, R(h, \lambda)) = \inf\{|u - v| : v \in R(h, \lambda)\} \\ &= \inf\Big\{\int_0^b \exp(-L\theta(t))\|u(t) - v(t)\|\, dt : v \in R(h, \lambda)\Big\} \\ &= \int_0^b \exp(-L\theta(t)) \inf\big\{\|u(t) - v\| : v \in F(t, \eta(h, \lambda)(t), (V\eta(h, \lambda))(t), \lambda)\big\}\, dt \\ &\qquad \text{(cf. Theorem 2.2 of Hiai-Umegaki [9])} \\ &= \int_0^b \exp(-L\theta(t))\, d(u(t), F(t, \eta(h, \lambda)(t), (V\eta(h, \lambda))(t), \lambda))\, dt. \end{aligned}$$

Similarly for $u \in R(h, \lambda)$, we can get that

$$d_1(u, R(h, \lambda)) = \int_0^b \exp(-L\theta(t))\, d(u(t), F(t, \eta(h, \lambda_n)(t), (V\eta(h, \lambda_n)))(t), \lambda_n)\, dt.$$

So recalling the definition of the Hausdorff metric (see Section 2), we conclude that

$$\begin{aligned} &h_1(R(h, \lambda_n), R(h, \lambda)) \\ &= \int_0^b \exp(-L\theta(t)) h\big(F(t, \eta(h_n, \lambda)(t), (V\eta(h, \lambda_n))(t), \lambda_n), \\ &\qquad F(t, \eta(h, \lambda)(t), (V\eta(h, \lambda))(t), \lambda)\big)\, dt \\ &\Longrightarrow h_1(R(h, \lambda_n), R(h, \lambda)) \to 0 \text{ as } n \to \infty \quad \text{(cf. hypothesis } H(F)_1\,(4)) \\ &\Longrightarrow h_1(R(h_n, \lambda_n), R(h, \lambda)) \to 0 \text{ as } n \to \infty;\ \text{i.e. } R(\cdot, \cdot) \text{ is } h_1\text{-continuous.} \end{aligned}$$

So we can apply Theorem 1 of Bressan-Cellina-Fryszkowski [2] to get a continuous function $u : L^1(T, \mathbb{R}^N) \times \Lambda \to L^1(T, \mathbb{R}^N)$ such that $u(h, \lambda) \in \Gamma(\lambda) = \{h \in L^1(T, \mathbb{R}^N) : h \in R(h, \lambda)\}$ for all $(h, \lambda) \in L^1(T, \mathbb{R}^N) \times \Lambda$ and $u(h, \lambda) = h$ for $h \in \Gamma(\lambda)$.

Remark that since $R(\cdot, \lambda)$ is an h_1-contraction, from Nadler's fixed point theorem (see [14]) we have that $\Gamma(\lambda) \neq \emptyset$ for every $\lambda \in \Lambda$.

Then let $\gamma : W^{1,1}(T,\mathbb{R}^N)\times\Lambda\to W^{1,1}(T,\mathbb{R}^N)$ be defined by

$$\gamma(x,\lambda)=x_0(\lambda)+\int_0^t u(\dot{x},\lambda)(s)\,ds.$$

Exploiting the continuity of $u(\cdot,\cdot)$, we can easily check that $\gamma(\cdot,\cdot)$ is continuous. Next we will show that for all $(x,\lambda)\in W^{1,1}(T,\mathbb{R}^N)\times\Lambda$, we have $\gamma((x,\lambda)\in S(\lambda)$. Indeed note that

$$\begin{aligned}
&u(\dot{x},\lambda)\in R(u(\dot{x},\lambda),\lambda)\\
&\Longrightarrow u(\dot{x},\lambda)(t)\in F\big(t,\eta(u(\dot{x},\lambda),\lambda)(t),(V\eta(u(\dot{x},\lambda),\lambda))(t),\lambda\big)\text{ a.e.}\\
&\Longrightarrow \eta(u(\dot{x},\lambda))\in S(\lambda)\\
&\Longrightarrow \gamma(x,\lambda)\in S(\lambda).
\end{aligned}$$

Furthermore if $x\in S(\lambda)$, then $\dot{x}(t)\in F(t,x(t),(Vx)(t),\lambda))$ a.e. $\Rightarrow \dot{x}\in\Gamma(\lambda)\Rightarrow u(\dot{x},\lambda)=\dot{x}\Rightarrow \eta(\dot{x},\lambda)(\cdot)=\gamma(x,\lambda)(\cdot)=x(\cdot)$. So $\gamma(\cdot,\cdot)$ is the desired map. □

Remark. In particular Theorem 3.1 tells us that for every $\lambda\in\Lambda$, $S(\lambda)$ is a retract of $W^{1,1}(T,\mathbb{R}^N)$ and the retraction can be chosen to depend continuously on the parameter $\lambda\in\Lambda$. Also $S(\lambda)\in P_f(W^{1,1}(T,\mathbb{R}^N))$ (cf. Section 2).

Using Theorem 3.1, we can establish the following useful topological property of the set $S(\lambda)$.

Theorem 3.2. *If the hypotheses $H(F)_1$, $H(K)$, $H(g)$ and H_0 hold, then for every $\lambda\in\Lambda$, $S(\lambda)$ is path-connected in $W^{1,1}(T,\mathbb{R}^N)$.*

Proof: From Dugundji's extension theorem (see Dugundji [7, Theorem 6.1, p. 188]), we know that $L^1(T,\mathbb{R}^N)$ is an absolute retract. Also from the proof of Theorem 3.1, we know that $\Gamma(\lambda)$ is a retract of $L^1(T,\mathbb{R}^N)$. From Theorem 6, p. 341 of Kuratowski [12], we know that the retract of an absolute retract is also an absolute retract. So $\Gamma(\lambda)$ is an absolute retract.

But from Kuratowski [12, p. 339], we know that an absolute retract is path-connected. Hence $\Gamma(\lambda)$ is path-connected in $L^1(T,\mathbb{R}^N)$. Finally since continuous maps preserve path-connectedness (see Dugundji [7, p. 115]), we have that $\eta(\Gamma(\lambda),\lambda)=S(\lambda)$ is path-connected in $W^{1,1}(T,\mathbb{R}^N)$. □

Also via Theorem 3.1, we can establish the existence of a continuous selector for the solution multifunction $\lambda\to S(\lambda)$. Our result extends the theorem of Cellina [3].

Theorem 3.3. *If the hypotheses $H(F)_1$, $H(K)$, $H(g)$ and H_0 hold, then there exists $\xi:\Lambda\to W^{1,1}(T,\mathbb{R}^N)$ continuous map such that $\xi(\lambda)\in S(\lambda)$ for every $\lambda\in\Lambda$.*

Proof: Let $p(\lambda)=\eta(0,\lambda)=x_0(\lambda)$. From the hypothesis H_0, we have that $\lambda\to p(\lambda)$ is continuous from Λ into $W^{1,1}(T,\mathbb{R}^N)$. Let $\gamma : W^{1,1}(T,\mathbb{R}^N)\times\Lambda\to W^{1,1}(T,\mathbb{R}^N)$ be the continuous map of Theorem 3.1. Then $\lambda\to\xi(\lambda)=\gamma(p(\lambda),\lambda)$ is continuous and $\xi(\lambda)\in S(\lambda)$ for every $\lambda\in\Lambda$ (cf. Theorem 3.1). □

4. Topological structure of the solution set – convex case

In this section we consider the case of a convex-valued orientor field $F(t,x,y)$ ("convex" problem) and establish another topological result concerning the solution set. Here there is no parameter λ. So our problem is

$$(2)\qquad \begin{aligned} &\dot{x}(t) \in F(t, x(t), (Vx)(t)) \text{ a.e.},\\ &x(0) = x_0. \end{aligned}$$

We will make the following hypothesis on the orientor field $F(t,x,y)$.

$\boldsymbol{H(F)_2}$: $F : T \times \mathbb{R}^N \times \mathbb{R}^N \to P_{kc}(\mathbb{R}^N)$ is a multifunction such that

(1) $t \to F(t,x,y)$ is measurable,
(2) $h(F(t,x,y), F(t,x',y')) \le k(t)[\|x - x'\| + \|y - y'\|]$ a.e., with $k(\cdot) \in L^1_+$,
(3) $|F(t,x,y)| = \sup\{\|v\| : v \in F(t,x,y)\} \le a(t) + c(t)(\|x\| + \|y\|)$ a.e. with $a, c \in L^1_+$.

We have the following result concerning the topological structure of the solution set $S(x_0) \subseteq C(T, \mathbb{R}^N)$ of (2).

Theorem 4.1. *If the hypotheses $H(F)_2$, $H(K)$ and $H(g)$ hold, then $S(x_0)$ is a compact retract of $C(T, \mathbb{R}^N)$.*

Proof: Let $R : C(T, \mathbb{R}^N) \to P_{fc}(L^1(T, \mathbb{R}^N))$ be the multifunction defined by

$$R(x) = \{h \in L^1(T, \mathbb{R}^N) : h(t) \in F(t, x(t), (Vx)(t)) \text{ a.e.}\}.$$

Then set $H(x) = \{\eta(h) : h \in R(x)\}$. Via a straightforward application of the Arzelà-Ascoli theorem, we can get that for every $x \in C(T, \mathbb{R}^N)$, $H(x) \in P_{kc}(C(T, \mathbb{R}^N))$. So we have that $H : C(T, \mathbb{R}^N) \to P_{kc}(C(T, \mathbb{R}^N))$.

On $C(T, \mathbb{R}^N)$ consider the following norm

$$|x| = \sup_{t \in T} \exp(-L\theta(t)) \|x(t)\|, \; L > 0, \text{ with } \int_0^t (k(s) + M\, k_1(s))\, ds.$$

Clearly this is equivalent to the usual norm on $C(T, \mathbb{R}^N)$.

As before, let $d_1(\cdot,\cdot)$ denote the distance function corresponding to this norm and $h_1(\cdot,\cdot)$ the Hausdorff metric generated by it.

We will show that for $L > 2$, $H(\cdot)$ is an h_1-contraction. So let $x_1, xc_2 \in C(T, \mathbb{R}^N)$ and take $z_1 \in H(x_1)$. Then by definition $z_1 = \eta(h_1)$ for some $h_1 \in R(x_1) = S^1_{F(\cdot, x_1(\cdot), (Vx_1)(\cdot))}$.

As before via Aumann's selection theorem, we can find $h_2 \in R(x_2)$ such that $\|h_1(t) - h_2(t)\| = d(h_1(t), F(t, x_2(t), (Vx_2)(t)))$ a.e. Set $z_2 = \eta(h_2) \in H(x_2)$. We have:

$$\begin{aligned}\|z_1(t)-z_2(t)\| &\le \int_0^t \|h_1(s)-h_2(s)\|\,ds\\ &= \int_0^t d(h_1(s),F(s,x_2(s),(Vx_2)(s)))\,ds\\ &\le \int_0^t h(F(s,x_1(s),(Vx_1)(s)),F(s,x_2(s),(Vx_2)(s)))\,ds\\ &\le \int_0^t k(s)(\|x_1(s)-x_2(s)\|+\|(Vx_1)(s)-(Vx_2)(s)\|)\,ds.\end{aligned}$$

Recall that

$$\|(Vx_1)(s)-(Vx_2)(s)\| \le \int_0^s M\,k_1(\tau)\|x_1(\tau)-x_2(\tau)\|\,d\tau.$$

So we get

$$\|z_1(t)-z_2(t)\| \le \int_0^t k(s)(\|x_1(s)-x_2(s)\|+\int_0^s M\,k_1(\tau)\|x_1(\tau)-x_2(\tau)\|\,d\tau)\,ds.$$

We have

$$\begin{aligned}\int_0^t k(s)\|x_1(s)-x_2(s)\|\,ds &= \int_0^t k(s)\exp(-L\theta(s))\exp(L\theta(s))\|x_1(s)-x_2(s)\|\,ds\\ &= \int_0^t k(s)\exp(L\theta(s))|x_1-x_2|\,ds\\ &\le (1/L)|x_1-x_2|\int_0^t d\exp(L\theta(s))\\ &\le (1/L)|x_1-x_2|\exp(L\theta(t)).\end{aligned}$$

Also we have

$$\begin{aligned}&\int_0^t k(s)\int_0^s M\,k_1(\tau)\|x_1(\tau)-x_2(\tau)\|\,d\tau\,ds\\ &= \int^t k(s)\int_0^s M\,k_1(\tau)\exp(-L\theta(\tau))\exp(L\theta(\tau))\|x_1(\tau)-x_2(\tau)\|\,d\tau\,ds\\ &\le \int_0^t k(s)|x_1-x_2|\int_0^s \widehat{k}(\tau)\exp(L\theta(\tau))\,d\tau\,ds\\ &= \int_0^t k(s)|x_1-x_2|(1/L)\int_0^s d\exp(L\theta(\tau))\\ &\le \int_0^t \widehat{k}(s)|x_1-x_2|(1/L)\exp(L\theta(s))\,ds\\ &= (1/L^2)|x_1-x_2|\int_0^t d\exp(L\theta(s))\\ &\le (1/L^2)|x_1-x_2|\exp(L\theta(t)).\end{aligned}$$

So finally we have

$$\begin{aligned}&\|z_1(t)-z_2(t)\| \le [(1/L)+(1/L^2)]|x_1-x_2|\exp(L\theta(\tau))\\ &\quad\Longrightarrow \exp(-L\theta(t))\|z_1(t)-z_2(t)\| \le [(1/L)+(1/L^2)]|x_1-x_2|\\ &\quad\Longrightarrow d_1(z_1,H(x_2)) \le [(1/L)+(1/L^2)]|x_1-x_2|\\ &\quad\Longrightarrow h_1(H(x_1),H(x_2)) \le [(1/L)+(1/L^2)]|x_1-x_2|.\end{aligned}$$

So if $L>2$, then $[(1/L)+(1/L^2)]<1$ and so $H(\cdot)$ is an h_1-contraction.

Let $\Gamma=\{x\in C(T,\mathbb{R}^N): x\in H(x)\}$. From Nadler's fixed point theorem (see [14]) we have $\Gamma\neq\emptyset$. Also from Ricceri [20], we have that Γ is a retract of $C(T,\mathbb{R}^N)$. Observe that $\Gamma=S(x_0)$.

So it remains to prove the compactness of $S(x_0)$ in $C(T,\mathbb{R}^N)$. To this end let $x\in S(x_0)$. Then we have

$$\begin{aligned}&\|x(t)\| \le \|x_0\|+\int_0^t \|f(s)\|\,ds,\ t\in T,\ f\in S^1_{F(\cdot,x(\cdot),(Vx)(\cdot))}\\ &\Longrightarrow \|x(t)\| \le \|x_0\|+\int_0^t (a(s)+c(s)(\|x(s)\|+\|(Vx)(s)\|))\,ds\\ &\le \|x_0\|+\|a\|_1+\int_0^t c(s)\|x(s)\|\,ds+\int_0^t c(s)\int_0^s c_1(\tau)\|x(\tau)\|\,d\tau\,ds.\end{aligned}$$

Applying Pachpatte's inequality [15], we can find $M_1>0$ such that $\|x(t)\|<M$ for all $t\in T$. So $\|(Vx)(t)\| \le \int_0^t M(a_1(s)+c_1(s)M_1)\,ds \le M_2$ for some $M_2>0$ and all $(t,x)\in T\times S(x_0)$. Thus without any loss of generality, we may assume that

$$|F(t,x,y)| \le a(t)+(M_1+M_2)c(t)=\varphi(t) \text{ a.e.},\ \ \varphi(\cdot)\in L^1_+$$

(otherwise in what follows replace $F(t,x,y)$ by $F(t,r_{M_1}(x),r_{M_2}(y))$, with $r_{M_1}(\cdot)$ (resp. $r_{M_2}(\cdot)$) being the M_1- (resp. M_2-) radial retraction on $\mathbb{R}^N$).

Let $W=\{h\in L^1(T,\mathbb{R}^N): \|h(t)\|\le\varphi(t) \text{ a.e.}\}$ and set $V=\eta(W)$. A simple application of the Arzelà-Ascoli theorem tells us that V is compact in $C(T,\mathbb{R}^N)$. Since $S(x_0)\subseteq V$, if we show that $S(x_0)$ is closed in $C(T,\mathbb{R}^N)$, then we will be done. Hence let $x_n\in S(x_0)$, $n\ge1$, and assume $x_n\to x$ in $C(T,\mathbb{R}^N)$. By definition $x_n=\eta(f_n)$ with $f_n\in S^1_{F(\cdot,x_n(\cdot),(Vx_n)(\cdot))}$.

Since $\|f_n(t)\le\varphi(t)\|$ a.e., from the Dunford-Pettis theorem and by passing to a subsequence, if necessary, we may assume that $f_n \xrightarrow{w} f$ in $L^1(T,\mathbb{R}^N)$.

Using Theorem 3.1 of Papageorgiou [16], we get

$$\begin{aligned}&f(t)\in\overline{\mathrm{conv}}\,\overline{\lim}\{f_n(t)\}_{n\ge1}\\ &\subseteq\overline{\mathrm{conv}}\,\overline{\lim}F(t,x_n(t),(Vx_n)(t))=F(t,x(t),(Vx)(t)) \text{ a.e.}\end{aligned}$$

Also $\eta(f_n)(t)\to\eta(f)(t)\Rightarrow x=\eta(f)\Rightarrow x\in S(x_0)\Rightarrow S(x_0)$ is indeed compact in $C(T,\mathbb{R}^N)$. $\square$

Remark. From Theorem 4.1, we get that $S(x_0) \subseteq C(T, \mathbb{R}^N)$ is an absolute retract, hence path-connected in $C(T, \mathbb{R}^N)$. Also Theorem 4.1 is not valid if $F(t, x, y)$ is not convex-valued.

Indeed consider the simple multivalued Cauchy problem $\dot{x}(t) \in \{-1, 1\}$ a.e., $x(0) = 0$. Its solution set is not closed in $C(T, \mathbb{R}^N)$ (in fact is dense in the solution set of $\dot{x}(t) \in [-1, 1]$ a.e., $x(0) = 0$ and so cannot be a retract of $C(T, \mathbb{R}^N)$ (see Section 2)).

5. Continuous dependence results

In this section, we return to the parametric multivalued Cauchy problem (1) and examine the continuity properties of the solution multifunction $\lambda \to S(\lambda)$.

We will say that a multifunction $G : \Lambda \to P_f(\mathbb{R}^N)$ is d-continuous (also known as Wijsman continuous or W-continuous) if and only if for every $x \in \mathbb{R}^N$, $\lambda \to d(x, G(\lambda))$ is continuous. If $G(\cdot)$ is Vietoris or h-continuous, then it is d-continuous but the converse is not in general true. It is true if $\overline{G(\Lambda)}$ is compact.

We will make the following hypothesis on the orientor field $F(t, x, y, \lambda)$.

$\boldsymbol{H(F)_3}$: $F : T \times \mathbb{R}^N \times \mathbb{R}^N \times \Lambda \to P_{kc}(\mathbb{R}^N)$ is a multifunction such that

(1) $t \to F(t, x, y, \lambda)$ is measurable,
(2) $h(F(t, x, y, \lambda), F(t, x', y', \lambda)) \leq k_B(t)[\|x - x'\| + \|y - y'\|]$ a.e., for all $\lambda \in B \subseteq \Lambda$ compact, with $k_B(\cdot) \in L^1_+$,
(3) $|F(t, x, y, \lambda)| = \sup\{\|v\| : v \in F(t, x, y, \lambda)\} \leq a_B(t) + c_B(t)(\|x\| + \|y\|)$ a.e. for all $\lambda \in B \subseteq \Lambda$ and with $a_B, c_B \in L^1_+$,
(4) $\lambda \to F(t, x, y, \lambda)$ is d-continuous.

Theorem 5.1. *If the hypotheses $H(F)_3$, $H(K)$, $H(g)$ and H_0 hold, then $\lambda \to S(\lambda)$ is Vietoris continuous from Λ into $P_k(C(T, \mathbb{R}^N))$.*

Proof: From the a priori bounds obtained in the second half of the proof of Theorem 4.1 we may assume that for $B \subseteq \Lambda$ compact and for all $\lambda \in B$, we have

$$|F(t, x, y, \lambda)| \leq a_B(t) + c_B(t)(M_{1B} + M_{2B}) = \varphi_B(t) \text{ a.e., } \varphi_B \in L^1_+.$$

Let $W_B = \{h \in L^1(T, \mathbb{R}^N) : \|h(t)\| \leq \varphi_B(t) \text{ a.e.}\}$. From the Dunford-Pettis theorem, we know that $W_B \in P_{wkc}(L^1(T, \mathbb{R}^N))$. Let $R : W_B \times B \to P_{wkc}(W_B)$ be defined by $R(h, \lambda) = S^1_{F(\cdot, \eta(h,\lambda), (V\eta(h,\lambda))(\cdot), \lambda)}$. Let $|\cdot|$ be the equivalent $L^1(T, \mathbb{R}^N)$-norm introduced and used in the proof of Theorem 3.1. As in that proof, we can show that for all $\lambda \in B$, $R(\cdot, \lambda)$ is an h_1-retraction with the same constant $\frac{1}{L} < 1$.

Next we will show that if $\lambda_n \to \lambda$ in Λ, $B = \{\lambda_n, \lambda\}_{n \geq 1}$, on W_B we consider the norm $|\cdot|$ and $(h_n, \lambda_n) \to (h, \lambda)$ in $W_B \times B$, then $R(h_n, \lambda_n) \xrightarrow{K-M} R(h, \lambda)$ as $n \to \infty$. To this end let $f \in R(h, \lambda)$ and set

$v_n(t) = d(f(t), F(t, \eta(h_n, \lambda_n)(t), (V\eta(h_n, \lambda_n))(t), \lambda_n))$. Then we have

$$\begin{aligned} v_n(t) &\le d(f(t), F(t, \eta(h, \lambda)(t), (V\eta(h, \lambda))(t), \lambda_n)) \\ &\quad + h(F(t, \eta(h, \lambda)(t), (V\eta(h, \lambda))(t), \lambda_n), \\ &\qquad F(t, \eta(h_n, \lambda_n)(t), (V\eta(h_n, \lambda_n))(t), \lambda_n)) \\ &\le d(f(t), F(t, \eta(h, \lambda)(t), (V\eta(h, \lambda))(t), \lambda_n)) \\ &\quad + k_B(t)(\|\eta(h, \lambda)(t) - \eta(h_n, \lambda_n)(t)\| \\ &\qquad + \|(V\eta(h, \lambda))(t) - (V\eta(h_n, \lambda_n))(t)\|) \to 0 \end{aligned}$$

(cf. the hypothesis $H(F)_3\,(1)$). So $v_n(t) \to 0$ as $n \to \infty$ for all $t \in T$.

Via Aumann's selection theorem, we can find $f_n \in R(h_n, \lambda_n)$, $n \ge 1$, such that $v_n(t) = \|f(t) - f_n(t)\|$ a.e. for all $n \ge 1$. So $f_n \xrightarrow{H} f$ and $f_n \in R(h_n, \lambda_n)$, $n \ge 1$. Thus we have proved

$$R(h, \lambda) \subseteq s\text{-}\underline{\lim}\, R(h_n, \lambda_n). \tag{3}$$

Next let $f \in w\text{-}\overline{\lim}\, R(h_n, \lambda_n)$. Then by definition (and by denoting, for economy in the notation, subsequences with the same index as the original sequences), we know that we can find $f_n \in R(h_n, \lambda_n)$ such that $f_n \xrightarrow{w} f$ in $L^1(T, \mathbb{R}^N)$.

From Theorem 3.1 of [16], we get $f(t) \in \overline{\text{conv}}\ \overline{\lim} F(t, \eta(h_n, \lambda_n)(t), (V\eta(h_n, \lambda_n))(t), \lambda_n)$ a.e. Let $z \in \mathbb{R}^N$. Using the hypothesis $H(F)_3\,(2)$ we get

$$\begin{aligned} &d(z, F(t, \eta(h, \lambda)(t), (V\eta(h, \lambda))(t), \lambda_n)) \\ &\le d(z, F(t, \eta(h_n, \lambda_n)(t), (V\eta(h_n, \lambda_n))(t), \lambda_n)) \\ &+ k_B(t)(\|\eta(h, \lambda)(t) - \eta(h_n, \lambda_n)(t)\| + \|(V\eta(h, \lambda))(t) - (V\eta(h_n, \lambda_n))(t)\|) \\ &\Longrightarrow d(z, F(t, \eta(h, \lambda)(t), (V\eta(h, \lambda))(t), \lambda)) \\ &\le \underline{\lim}\, d(z, F(t, \eta(h_n, \lambda_n)(t), (V\eta(h_n, \lambda_n))(t), \lambda_n)) \quad \text{(cf. hypothesis } H(F)_3\,(4)). \end{aligned}$$

Using Theorem 2.2 of Tsukada [23], we get that

$$\begin{aligned} &\overline{\lim}\, F(t, \eta(h_n, \lambda_n)(t), (V\eta(h_n, \lambda_n))(t), \lambda_n) \\ &\subseteq F(t, \eta(h, \lambda)(t), (V\eta(h, \lambda))(t), \lambda) \text{ a.e.} \\ &\Longrightarrow f \in R(h, \lambda) \\ &\Longrightarrow w\text{-}\overline{\lim}\, R(h_n, \lambda_n) \subseteq R(h, \lambda). \end{aligned} \tag{4}$$

From (3) and (4) above, we get $R(h_n, \lambda_n) \xrightarrow{K-M} R(h, \lambda)$.

Let $\Gamma(\lambda_n) = \{h \in L^1(T, \mathbb{R}^N) : h \in R(h, \lambda_n)\}$ and $\Gamma(\lambda) = \{h \in L^1(T, \mathbb{R}^N) : h \in R(h, \lambda)\}$. These sets are nonempty and closed by Nadler's fixed point theorem (see [14]). Then Theorem 1 of Rybinski [21] tells us that $\Gamma(\lambda_n) \xrightarrow{K} \Gamma(\lambda) \Rightarrow$

$\eta(\Gamma(\lambda_n), \lambda_n) = S(\lambda_n) \xrightarrow{K} \eta(\Gamma(\lambda), \lambda) = S(\lambda)$ (since $\eta(\cdot, \cdot)$ is continuous). Because $S(\lambda_n) \subseteq \eta(W_B) = V_B \in P_{kc}(C(T, \mathbb{R}^N))$, $n \geq 1$, we conclude (see for example DeBlasi-Myjak [5]) that $\lambda \to S(\lambda)$ is Vietoris continuous from Λ into $P_k(C(T, \mathbb{R}^N))$. □

Recall (see Section 2) that on $P_k(C(T, \mathbb{R}^N))$ the Vietoris and Hausdorff topologies coincide. So we also have

Theorem 5.2. *If the hypotheses $H(F)_3$, $H(K)$, $H(g)$ and H_0 hold, then $\lambda \to S(\lambda)$ is h-continuous from Λ into $P_k(C(T, \mathbb{R}^N))$.*

In particular from Theorems 5.1 and 5.2, we have:

Theorem 5.3. *If the hypotheses $H(F)_2$, $H(K)$ and $H(g)$ hold, then $x_0 \to S(x_0)$ is Vietoris and h-continuous from $\mathbb{R}^N$ into $P_k(C(T, \mathbb{R}^N))$.*

From Theorem 4.1, we know that $S(x_0)$ is path-connected, hence connected in $C(T, \mathbb{R}^N)$. Combining this fact with Theorem 5.3 and Theorems 7.4.3 and 7.4.4 of Klein-Thompson [11, p. 90], we have:

Theorem 5.4. *If the hypotheses $H(F)_2$, $H(K)$ and $H(g)$ hold and $C \in P_k(\mathbb{R}^N)$ and is connected, then $S(C) = \bigcup_{x_0 \in C} S(x_0)$ is a connected and compact subset of $C(T, \mathbb{R}^N)$.*

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