A characterization of the existence of solutions to some higher order boundary value problems

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Abstract. The aim of this short note is to present a theorem that characterizes the existence of solutions to a class of higher order boundary value problems. This result completely answers a question previously set by the authors in [Differential Integral Equations **6** (1993), 1119–1123].

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The aim of this short paper is to point out Theorem 1 below, which characterizes the existence of solutions to a class of higher order boundary value problems and, moreover, completely answers a question previously investigated in [2] and [3, Section 3.2]. Related results can also be found in the extensive survey by Agarwal [1, Chapter 9].

Our notation and terminology are standard. In any case, we refer to [2]. The symbol $C([a,b]\times\mathbb{R}^{n+1})$ is used to denote the space of all continuous real-valued functions defined on $[a,b]\times\mathbb{R}^{n+1}$.

Theorem 1. *The following assertions are equivalent:*

(i) *The length of $[a,b]$ is less than $\pi/2$.*

(ii) *For every function $f\in C([a,b]\times\mathbb{R}^{n+1})$ and every bounded sequence $\{x_h\}\subseteq\mathbb{R}$, there exists an integer $\nu\geq n+1$ such that, for any $k\geq\nu$ and any $t_1,t_2,\dots,t_k\in[a,b]$, the problem*

$$\begin{cases} x^{(k)}=f(t,x,x',\dots,x^{(n)}) \\ x^{(i-1)}(t_i)=x_i\,, \qquad i=1,2,\dots,k \end{cases}$$

admits at least one solution $u\in C^k([a,b])$.

Proof: If (i) is true, so does (ii), by Theorem 1.1 of [2]. Example 1 below shows that (ii) $\Rightarrow$ (i). □

Remark 1. The implication (i) $\Rightarrow$ (ii) of Theorem 1 actually holds even if the function f satisfies Carathéodory's type conditions only (see [2, Theorem 1.1] or [3, Theorem 5]). However, in this case, one achieves generalized solutions.

Remark 2. We emphasize that in assertion (ii) of Theorem 1 no condition on the finite sequence $t_1, t_2, \dots, t_k$ is assumed. Moreover, whenever we specialize the choice of points t_i, assertion (ii) may be true also for $b - a \geq \pi/2$. As an example, this is the case if $t_1 \leq t_2 \leq \dots \leq t_k$; see [1, Theorem 9.2] and [3, Theorem 2].

Example 1. Let $f : [0, \pi/2] \times \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(t,z) = \sin t + z\,, \qquad (t,z) \in [0,\pi/2] \times \mathbb{R},$$

and let $x_h = 0$ for each $h \in \mathbb{N}$.
Then, for every positive integer ν, the problem

$$\begin{cases} x^{(4\nu)} = f(t,x) \\ x^{(i-1)}(t_i) = x_i\,, \qquad i = 1, 2, \dots, 4\nu, \end{cases}$$

where $t_i = 0$ for i odd and $t_i = \pi/2$ for i even, admits no solutions in $C^{4\nu}([0,\pi/2])$.

Indeed, if $u \in C^{4\nu}([0,\pi/2])$ is a solution to the preceding problem for some $\nu \in \mathbb{N}$, then

$$\int_0^{\pi/2} u^{(4\nu)}(t) \sin t\, dt = \int_0^{\pi/2} u(t) \sin t\, dt + \int_0^{\pi/2} \sin^2 t\, dt \tag{1}$$

and

$$u^{(i-1)}(0) = 0 \quad \text{for } i \text{ odd}, \quad u^{(i-1)}(\pi/2) = 0 \quad \text{for } i \text{ even}, \quad i = 1, 2, \dots, 4\nu. \tag{2}$$

Owing to (2), integrating by parts, one has

$$\int_0^{\pi/2} u^{(4\nu)}(t) \sin t\, dt = \int_0^{\pi/2} u^{(4\nu-2)}(t) \sin t\, dt = \dots = \int_0^{\pi/2} u(t) \sin t\, dt\,;$$

therefore, identity (1) becomes

$$\int_0^{\pi/2} u(t) \sin t\, dt = \int_0^{\pi/2} u(t) \sin t\, dt + \frac{\pi}{4}.$$

This is clearly absurd.

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References

[1] Agarwal R.P., *Boundary Value Problems for Higher Order Differential Equations*, World Sci. Publ., Singapore, 1986.

[2] Bonanno G., Marano S.A., *Higher order ordinary differential equations*, Differential Integral Equations **6** (1993), 1119–1123.

[3] Majorana A., Marano S.A., *Boundary value problems for higher order ordinary differential equations*, Comment. Math. Univ. Carolinae **35** (1994), 451-466.

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