Eigenvalues of the p-Laplacian in $\boldsymbol{R}^N$ with indefinite weight

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Abstract. We consider the nonlinear eigenvalue problem

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda g(x)|u|^{p-2}u$$

in $\boldsymbol{R}^N$ with $p > 1$. A condition on indefinite weight function g is given so that the problem has a sequence of eigenvalues tending to infinity with decaying eigenfunctions in $W^{1,p}(\boldsymbol{R}^N)$. A nonexistence result is also given for the case $p \geq N$.

Keywords: eigenvalue, the p-Laplacian, indefinite weight, $\boldsymbol{R}^N$

Classification: Primary 35P30, 35J70

1. Introduction

We investigate the following nonlinear eigenvalue problem in $\boldsymbol{R}^N$

$$-\Delta_p u = \lambda g(x)|u|^{p-2}u, \tag{1}$$

where $\Delta_p u = \operatorname{div}\,(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian with $p > 1$, $\lambda \in \boldsymbol{R}$, $u \in W^{1,p}(\boldsymbol{R}^N)$, and $g \in L^\infty(\boldsymbol{R}^N)$ is an indefinite weight function, i.e. $g^\pm = \max(\pm g, 0) \not\equiv 0$. Here we consider only weak solutions, i.e. (λ, u) is a (nontrivial) solution of (1) if $\lambda \in \boldsymbol{R}$, $u \in W^{1,p}(\boldsymbol{R}^N) \setminus \{0\}$ and

$$\int |\nabla u|^{p-2}\nabla u \nabla\varphi = \lambda \int g(x)|u|^{p-2}u\varphi$$

for all $\varphi \in C_0^\infty(\boldsymbol{R}^N)$. Here and henceforth the integrals are taken over $\boldsymbol{R}^N$ unless otherwise specified.

In the case $p = 2$, the 2-Laplacian is the usual Laplace operator. The p-Laplacian with $p \neq 2$ arises in, for example, the study of non-Newtonian fluids ($p > 2$ for dilatant fluids and $p < 2$ for pseudoplastic fluids), in torsional creep problems ($p \geq 2$), as well as in glaciology ($p \in (1, 4/3]$). Eigenvalue problems of the p-Laplacian on bounded domains have been studied extensively; we mention, for example, the work of Anane [A], Azorezo and Alonso [AA], Lindqvist [Ln], and Szulkin [Sz] and references therein. When dealing with eigenvalue problems with indefinite weight on bounded domains, Otani and Teshima [OT] studied the Dirichlet boundary condition, and Huang [H] treated the Neumann case. In both papers, only the properties of the first (positive) eigenvalue and eigenfunction have been emphasized.

It is apparent that the eigenvalue problem of the p-Laplacian in $\boldsymbol{R}^N$ with definite weight does not have solutions in $W^{1,p}(\boldsymbol{R}^N)$, as we have witnessed in the case $p = 2$. Thus it is natural to study problem (1) with indefinite weight. This paper is partly motivated by recent work of Brown, Cosner and Fleckinger [BCF], and Li and Yan [LY], and partly by indefinite eigenvalue problems, and as such, is a continuation of [OT] and [H]. In Section 2 we use a variational method to prove the existence of a sequence of eigenvalues and study, in particular, some properties of the first eigenvalue and eigenfunction which are enjoyed by regular eigenvalue problems. A specific condition on the weight function g is introduced there. In Section 3 we present a nonexistence result when $p \geq N$.

2. Existence

We assume:

(H) There exist $K > 0$ and $R' > 0$ such that $g(x) \leq -K$ for $|x| \geq R'$.

We denote by G^+ the set

$$\{\, u \in W^{1,p}(\boldsymbol{R}^N) \ : \ p\Psi(u) := \int g|u|^p = 1 \,\}, \tag{2}$$

and by $B_R(x)$ the ball in $\boldsymbol{R}^N$ centered at x with radius R. We define the following functional on $W^{1,p}(\boldsymbol{R}^N)$

$$I(u) = \frac{1}{p} \int |\nabla u|^p. \tag{3}$$

Clearly, the functional I is even and is bounded below on G^+.

Lemma 1. *The functional I satisfies the Palais-Smale condition on G^+, i.e., for $\{u_n\} \subset G^+$, if $I(u_n)$ is bounded and*

$$I'(u_n) - a_n \Psi'(u_n) \to 0, \quad \text{where} \quad a_n = \frac{\langle I'(u_n), u_n\rangle}{\langle \Psi'(u_n), u_n\rangle}, \tag{4}$$

then $\{u_n\}$ has a convergent subsequence in $W^{1,p}(\boldsymbol{R}^N)$.

Proof: Let $u_n \in W^{1,p}(\boldsymbol{R}^N)$ be such a sequence. Clearly, $\{u_n\}$ is bounded in $L^p(\Omega)$ for any bounded domain $\Omega \subset \boldsymbol{R}^N$. We next show that $\{u_n\}$ is bounded in $L^p(\boldsymbol{R}^N)$. Suppose not, then there exists a sequence of bounded domains Ω_n containing $B_{R'}$, such that

$$\int_{\Omega_n} |u_n|^p \to \infty, \quad \text{and} \quad \int_{\Omega_n \setminus B_{R'}} |u_n|^p \to \infty,$$

as $n \to \infty$. Noting that $\int_{B_{R'}} g|u_n|^p$ is bounded by a constant c and using (H), we have

$$1 = \int g|u_n|^p = \int_{B_{R'}} g|u_n|^p + \int_{\Omega_n \setminus B_{R'}} g|u_n|^p + \int_{\boldsymbol{R}^N \setminus \Omega_n} g|u_n|^p$$
$$\leq c - K \int_{\Omega_n \setminus B_{R'}} |u_n|^p \to -\infty,$$

as $n \to \infty$, a contradiction. Thus $\{u_n\}$ is bounded in $W^{1,p}(\boldsymbol{R}^N)$. Hence without loss of generality, we can assume, for some $u_0 \in W^{1,p}(\boldsymbol{R}^N)$, $u_n \to u_0$ weakly in $W^{1,p}(\boldsymbol{R}^N)$, pointwise a.e. in $\boldsymbol{R}^N$, and on any bounded domain Ω, $\int_\Omega g|u_0|^p = \lim_{n\to\infty} \int_\Omega g|u_n|^p$. In particular, by (H),

$$\int_{B_{R'}} g|u_0|^p = \lim_{n\to\infty} \int_{B_{R'}} g|u_n|^p \geq 1, \tag{5}$$

which implies that $u_0 \not\equiv 0$.

It follows from (4) that for any $\varphi \in C_0^\infty(\boldsymbol{R}^N)$,

$$\int |\nabla u_n|^{p-2} \nabla u_n \nabla \varphi = a_n \int g|u_n|^{p-2} u_n \varphi + o(1). \tag{6$_n$}$$

Taking $\varphi = u_n - u_m$ in $(6)_n - (6)_m$ (via diagonal arguments if necessary) we obtain

$$\int (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m)\nabla(u_n - u_m)$$
$$\leq \int g(a_n|u_n|^{p-2}u_n - a_m|u_m|^{p-2}u_m)(u_n - u_m) + o(1)$$
$$= \int_{B_{R'}} ga_n(|u_n|^{p-2}u_n - |u_m|^{p-2}u_m)(u_n - u_m)$$
$$+ \int_{\boldsymbol{R}^N \setminus B_{R'}} ga_n(|u_n|^{p-2}u_n - |u_m|^{p-2}u_m)(u_n - u_m)$$
$$+ (a_n - a_m) \int g|u_m|^{p-2}u_m(u_n - u_m) + o(1).$$

Note here that $a_n = \int |\nabla u_n|^p$, thus is bounded. Observe that, by monotonicity of the function $|t|^{p-2}t$ and assumption (H), the integral on $\boldsymbol{R}^N \setminus B_{R'}$ is negative. Thus we have

$$\begin{aligned} &\int (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m)\nabla(u_n - u_m) \\ &\quad\leq \int_{B_{R'}} ga_n(|u_n|^{p-2}u_n - |u_m|^{p-2}u_m)(u_n - u_m) \\ &\qquad + (a_n - a_m) \int g|u_m|^{p-2}u_m(u_n - u_m) + o(1). \end{aligned} \tag{7}$$

It is clear that

$$\int_{B_{R'}} ga_n(|u_n|^{p-2}u_n - |u_m|^{p-2}u_m)(u_n - u_m) \to 0$$

as (a subsequence of) $n, m \to \infty$, since (a subsequence of) u_n converges to u_0 in $L^p(B_{R'})$. Furthermore, Hölder's inequality implies that the integral $\int g|u_m|^{p-2}u_m(u_n - u_m)$ is bounded, and we can again choose a subsequence of n, m, so that $a_n - a_m \to 0$. Therefore we conclude that the right hand side of (7) approaches 0 as (a subsequence of) $n, m \to \infty$. On the other hand, observe that for any $a, b \in \boldsymbol{R}^N$,

$$|a-b|^p \le c \cdot \{(|a|^{p-2}a - |b|^{p-2}b) \cdot (a-b)\}^{s/2} \cdot (|a|^p + |b|^p)^{1-s/2},$$

where $s = p$ if $p \in (1,2)$ and $s = 2$ if $p \ge 2$. We thus have

$$|\nabla u_n - \nabla u_m|^p \le c \cdot \{(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m)\nabla(u_n - u_m)\}^{s/2} \\ (|\nabla u_n|^p + |\nabla u_m|^p)^{1-s/2}.$$

By applying Hölder's inequality we obtain

$$\int |\nabla u_n - \nabla u_m|^p \le c_1 \cdot \Big\{\int (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m)\nabla(u_n - u_m)\Big\}^{s/2} \\ \Big(\int |\nabla u_n|^p + \int |\nabla u_m|^p\Big)^{1-s/2}.$$

We then derive from the above inequality and (7) that $u_n \to u_0$ in $W^{1,p}(\boldsymbol{R}^N)$. The lemma is thus proved. □

Write

$$\Gamma_k = \{\, A \subset G^+ : A \text{ is symmetric, compact, and } \gamma(A) = k \,\},$$

where $\gamma(A)$ is the genus of A, i.e. the smallest integer k such that there exists an odd continuous map from A to $\boldsymbol{R}^k \setminus \{0\}$.

Now, by the Ljusternik-Schnirelmann theory, see e.g. [AA], [St], [Sz], we have

Theorem 2. *For any integer $k > 0$, $\lambda_k = \inf_{A \in \Gamma_k} \sup_{u \in A} pI(u)$ is a critical value of I restricted on G^+. More precisely, there exist $u_k \in A_k \in \Gamma_k$ such that $\lambda_k = pI(u_k) = \sup_{u \in A_k} pI(u)$ and (λ_k, u_k) is a solution of (1). Moreover, $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \to \infty$ as $k \to \infty$.*

Proof: We need only to show that $\lambda_k \to \infty$ as $k \to \infty$. Since $W^{1,p}(\boldsymbol{R}^N)$ is separable, there is a biorthogonal system $\{e_m, e_m^*\}$ such that $e_m \in W^{1,p}(\boldsymbol{R}^N)$;

$e_m^* \in (W^{1,p}(\boldsymbol{R}^N))^*$, the dual space of $W^{1,p}(\boldsymbol{R}^N)$; e_m are linearly dense in $W^{1,p}(\boldsymbol{R}^N)$; and e_m^* are total for $W^{1,p}(\boldsymbol{R}^N)$, see, e.g. [Sz]. We denote

$$E_n = \text{ span } \{ e_1, e_2, \cdots, e_n \},$$

and

$$E_n^\perp = \overline{\text{ span } \{ e_{n+1}, e_{n+2}, \cdots \}}.$$

Observe that $A \cap E_{j-1}^\perp \neq \emptyset$ for any $A \in \Gamma_j$ (by (g) of Proposition 2.3 of [Sz]). Now we claim that

$$\mu_j := \inf_{A\in\Gamma_j} \sup_{A\cap E_{j-1}^\perp} pI(u) \to \infty, \quad \text{as} \quad j \to \infty.$$

Indeed, if not, then for j large, there exists a $u_j \in E_{j-1}^\perp$, with $\int g|u_j|^p = 1$, such that $\mu_j \leq pI(u_j) \leq M$ for some $M > 0$ independent of j. Thus $\int |\nabla u_j|^p$ is bounded. By our choice of $E_{j-1}^\perp$, we have $u_j \to 0$ weakly in $W^{1,p}(\boldsymbol{R}^N)$ and that contradicts the fact that $\int g|u_j|^p = 1$. (Cf. [AA] and [Sz].)

Since $\lambda_j \geq \mu_j$, the conclusion follows. □

Definition. *λ_k and u_k are called the k th (variational) eigenvalue and eigenfunction of* (1) *respectively.*

Next we establish some regularity for solutions of (1).

Lemma 3. *Let $u \in W^{1,p}(\boldsymbol{R}^N)$ be a weak solution of* (1)*. Then $u \in L^\infty(\boldsymbol{R}^N)$.*

The proof of this lemma can be carried out using a device due to Brezis and Kato [BK], and is thus omitted.

From Proposition 3.7 of Tolksdorf [T], we have

Corollary 4. *If u is a solution of* (1)*, then for any bounded domain Ω, $u \in C^{1+\alpha}(\Omega)$ for some $\alpha \in (0,1)$.*

We remark that in general $u \notin C^2$ for $p \neq 2$ (see [L] for an example). We further note that, for the eigenvalue problem of the p-Laplacian on a bounded interval, one can show that, even though the eigenfunction u may not be in C^2, $|u'|^{p-2}u' \in C^1$ (cf. [HM]), and the equation is satisfied pointwise.

Next we study properties of the first eigenvalue $\lambda_1 > 0$ and the corresponding eigenfunction u_1. Apparently u_1 is of one sign. Next we prove that u_1 can be chosen positive in $\boldsymbol{R}^N$.

Lemma 5. *If $u \geq 0$, $u \not\equiv 0$ is a solution of* (1)*, then $u > 0$ in $\boldsymbol{R}^N$.*

Proof: Suppose $u(x_0) = 0$. Take a ball B around x_0 and $u \geq 0$ in B. Clearly, u is a supersolution of the problem

$$-\Delta_p u = \lambda g(x)|u|^{p-2}u \text{ in } B,$$
$$u = 0 \text{ on } \partial B.$$

Then Theorem 1.2 of [TR] implies that $u \equiv 0$ in B, which is impossible. This completes the proof. □

From now on we can assume that $u_1 > 0$.

Lemma 6. (i) λ_1 *is simple, i.e. the positive eigenfunction corresponding to* λ_1 *is unique up to a constant multiple.*
(ii) λ_1 *is unique, i.e. if* $v \geq 0$ *is an eigenfunction associated with an eigenvalue* λ *with* $\int g|v|^p = 1$*, then* $\lambda = \lambda_1$*.*

PROOF: Let $u > 0$ and $v > 0$ be the eigenfunction associated with λ_1 and λ respectively. It is easy to see

$$\int (-\Delta_p u, \frac{u^p - v^p}{u^{p-1}}) - (-\Delta_p v, \frac{u^p - v^p}{v^{p-1}}) = (\lambda_1 - \lambda) \int g(u^p - v^p) = 0.$$

Proposition 2 of [A] then implies that $u = v$. Consequently $\lambda_1 = \lambda$ and this completes the proof. □

We now consider the asymptotic behavior of solutions of (1). A scrutiny on the proof of Theorem 3.1 (ii) of [LY] shows that the continuity requirement of $c(x)$ is not necessary (we take $f \equiv 0$), provided $u \in L^\infty$, and (H) implies that the other assumption on c is satisfied. Thus applying Theorem 3.1 (ii) of [LY] to $\boldsymbol{R}^N \setminus B_{R'}$, we have

Lemma 7. *The solution* u *of (1) satisfies*

$$|u(x)| \leq c \cdot e^{-\varepsilon |x|}, \quad |x| \geq R$$

for some $c > 0$, $\varepsilon > 0$, *and* $R > 0$.

Summarizing the above results, we can state

Theorem 8. *Assume that* $g \in L^\infty(\boldsymbol{R}^N)$, $g^+ \not\equiv 0$, *and* (H) *holds. Then*
(i) (1) *has a sequence of solutions* (λ_k, u_k) *with* $\int g|u_k|^p = 1$ *and* $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \to \infty$ *as* $k \to \infty$, *and* $|u_k|$ *decays exponentially at infinity.*
(ii) *The first eigenfunction* u_1 *can be taken positive in* $\boldsymbol{R}^N$. *Moreover,* $\lambda_1 > 0$ *is simple and unique.*

Remarks. 1. We observe that conditions (h3) and (h4) of [LY] cannot be fulfilled for our problem. In fact they only treat the bifurcation problem there.
2. Even in the case $p = 2$, this result seems new.

3. Nonexistence

In this section, we give a nonexistence result, along the line of Theorem 3.2 of [BCF].

First we give an estimate of λ_1. Define, for any bounded domain $B \subset \boldsymbol{R}^N$,

$$\delta_1(B) = \inf_{u \in G^+_{B,0}} \int |\nabla u|^p, \quad \mu_1(B) = \inf_{u \in G^+_B} \int |\nabla u|^p, \tag{9}$$

where

$G_{B,0}^+ = \{u \in W_0^{1,p}(B) : \int_B g|u|^p = 1\}$,
$G_B^+ = \{u \in W^{1,p}(B) : \int_B g|u|^p = 1\}$.

Note that δ_1 and μ_1 are well defined provided $g^+ \not\equiv 0$, and correspond to the first eigenvalue of (1) on B with Dirichlet boundary condition and Neumann boundary condition respectively. By Theorem 1 of [H], $\mu_1(B) > 0$ if and only if $\int_B g < 0$.

Lemma 9. (i) $\lambda_1 \le \delta_1(B)$. (ii) $\mu_1(B) \le \lambda_1$ *provided* $g(x) < 0$ *for all* $x \notin B$.

PROOF: (i) results from the fact that $G_{B,0}^+ \subset G^+$.

For $u \in G^+$, clearly $\int_B g|u|^p \ge 1$. Hence (ii) follows. □

Let B_n be the ball in $\boldsymbol{R}^N$ centered at the origin with radius n.

Lemma 10. $\delta_1(B_n)$ *is decreasing, and* $\lim_{n\to\infty} \delta_1(B_n) = \lambda_1$. *If moreover* (H) *holds, then* $\mu_1(B_n)$ *is increasing.*

PROOF: Monotonicity of both $\delta_1(B_n)$ and $\mu_1(B_n)$ is obvious.

Let $u_n \in G^+$ be such that $I(u_n) \to \lambda_1$ as $n \to \infty$. By standard diagonal arguments, we can select a sequence φ_n such that

$$\varphi_n \in W_0^{1,p}(B_n), \quad \int_{B_n} g|\varphi_n|^p = 1, \quad \lim_{n\to\infty} \int_{B_n} |\nabla\varphi_n|^p = \lambda_1.$$

By the definition of δ_1, we have

$$\int_{B_n} |\nabla\varphi_n|^p \ge \delta_1(B_n) \ge \lambda_1.$$

The proof is completed. □

The next lemma, which is crucial in our nonexistence result, is an extension of Lemma 3.1 of [BCF], where the case $p = 2$, $N = 1, 2$ is treated.

Lemma 11. *Assume that* $p \ge N$ *and* g *satisfies a weaker form of* (H)

(H)* *There exists* $\tilde{R} > 0$, $g(x) < 0$ *for* $|x| > \tilde{R}$.

If, in addition, $0 < \int g < \infty$, *then* $\lim_{n\to\infty} \delta_1(B_n) = 0$.

PROOF: We follow the proof of Lemma 3.1 of [BCF].

Denote $M = \min\{1, \frac{1}{2}\int g\}$. Choose $R_1 > 1$ such that

$$\int_{|x|\le R_1} g \ge M, \quad \int_{|x|\ge R_1} g^- \le M/2.$$

Fix $\varepsilon > 0$. For $R_2 > R_1$, we define a test function v as follows: $v(x) = 1$ if $|x| \le R_1$, $v(x) = 0$ if $|x| \ge R_2$, and for $R_1 \le |x| \le R_2$,

$$v(x) = \begin{cases} L - \varepsilon \ln|x|, & \text{if } \ p = N; \\ L - \varepsilon|x|^{(p-N)/(p-1)}, & \text{if } \ 1 \le N < p, \end{cases}$$

where L and R_2 are so chosen that v is continuous. It follows that

$$\varepsilon(\ln R_2 - \ln R_1) = 1, \quad \text{for} \quad p = N,$$

and

$$\varepsilon(R_2^{(p-N)/(p-1)} - R_1^{(p-N)/(p-1)}) = 1, \quad \text{for} \quad 1 \le N < p.$$

For $T > R_2$, a calculation shows that

(i) for $p = N$,

$$\int_{|x|\le T} |\nabla v|^p = c_1 \cdot \int_{R_1}^{R_2} \varepsilon^p r^{-1} dr = c_1 \cdot \varepsilon^p (\ln R_2 - \ln R_1) = c_1 \cdot \varepsilon^{p-1},$$

(ii) for $1 \le N < p$,

$$\int_{|x|\le T} |\nabla v|^p = c_3 \cdot \int_{R_1}^{R_2} \varepsilon^p \Big(\frac{p-N}{p-1}\Big)^p r^{(1-N)/(p-1)} dr = c_3 \cdot \varepsilon^{p-1} \Big(\frac{p-N}{p-1}\Big)^{p-1}.$$

On the other hand,

$$\int_{|x|\le T} g v^p = \int_{|x|\le R_1} g + \int_{R_1\le |x|\le R_2} g v^p \ge M - \int_{R_1 \le |x| \le R_2} g^- \ge M/2.$$

It then follows that for $n > T$,

$$\delta_1(B_n) \le c_4 \cdot \varepsilon^{p-1} \to 0.$$

This concludes the proof. □

As a direct consequence, we have the following nonexistence result:

Theorem 12. *Assume that $p \ge N$ and g satisfies* (H)*. *Then problem* (1) *has no positive solution in $W^{1,p}(\boldsymbol{R}^N)$ for $\lambda > 0$.*

Proof: Lemma 11 combined with Lemma 9 yields the theorem. □

Remark. In the case $1 < p < N$, Hardy's inequality

$$\Big(\int |\varphi|^p (1 + |x|^p)^{-1} dx\Big)^{1/p} \le \frac{p}{N-p} \Big(\int |\nabla \varphi|^p\Big)^{1/p}$$

holds for all $\varphi \in C_0^\infty(\boldsymbol{R}^N)$. Let V be the completion of $C_0^\infty(\boldsymbol{R}^N)$ with the norm

$$\|\varphi\|_V^p = \int |\nabla \varphi|^p + \int |\varphi|^p (1 + |x|^p)^{-1}.$$

Then we can prove, as in Lemma 1, that the functional $I(u) = \frac{1}{p} \int |\nabla u|^p$, defined on V, satisfies the Palais-Smale condition on $\tilde{G}^+ = \{\ u \in V : \int g|u|^p = 1\ \}$, provided g satisfies

(H)′ $|g(x)| \le c \cdot (1 + |x|^p)^{-\alpha}$ for some $\alpha > 1$.

(We always assume that $g^+ \not\equiv 0$.) Consequently the results in Section 2 remain valid in V for this case. We note that this result is compatible with Theorem 4.1 of [BCF].

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