Monotone normality and extension of functions

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Abstract. We provide a characterisation of monotone normality with an analogue of the Tietze-Urysohn theorem for monotonically normal spaces as well as answer a question due to San-ou concerning the extension of Urysohn functions in monotonically normal spaces. We also extend a result of van Douwen, giving a characterisation of K_0-spaces in terms of semi-continuous functions, as well as answer another question of San-ou concerning semi-continuous Urysohn functions.

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1. Introduction, basic definitions and notation

The concept of a continuous function is fundamental to the study of general topology and much work has been done concerning the existence and extension of continuous functions and the collection of continuous (bounded) real-valued functions on a space (see for instance [5]). Amongst this work one finds the famous results of Tietze and Urysohn which are central to the theory of normal spaces.

In 1973 Heath, Lutzer and Zenor [6] introduced the concept of monotone normality which is a strengthening of normality (indeed, monotonically normal spaces are hereditarily collectionwise normal). An obvious question therefore is, given the results mentioned above for normal spaces; does there exist an analogous theory for monotonically normal spaces? Part of this question was answered by Borges in 1973 [1] when he gave an analogue of Urysohn's Lemma for monotonically normal spaces and in the original paper [6] the question was asked as to whether a version of the Tietze-Urysohn theorem held in monotonically normal spaces. In particular it was shown that monotonically normal spaces satisfy the monotone extension property and the question—does the converse hold?—was posed. In his thesis [2] van Douwen proved that not only did the monotone extension property not characterise monotone normality but a wide range of extension properties failed also. With hindsight the reason for this failing is clear. None of the considered extension properties linked functions defined on different closed subspaces. In this paper, we provide an analogue of the Tietze-Urysohn theorem for monotonically normal spaces as well as answer two questions of San-ou concerning Urysohn functions in monotonically normal spaces. We also provide a characterisation of K_0-spaces. The author would like to thank Professor J.E. Vaughan for many helpful suggestions concerning this paper.

By a space we will mean a topological space. All spaces considered will be T_1 and our terminology will generally follow that of [3]. We will denote the topology on a space X by τX.

In what follows, $C(X)$ $[C^*(X)]$ will denote the continuous [continuous and bounded], real-valued functions on X. If F is a subspace of a space X and Φ is a function from $C(F)$ to $C(X)$, then we say that Φ is an *extender* if, for all $f \in C(F)$, $\Phi(f)$ extends f. A space X is said to have the *monotone extension property* if for every non-empty, closed subspace F of X there is an extender Φ from $C^*(F)$ to $C^*(X)$ such that $\Phi(f) \le \Phi(g)$ whenever $f \le g$, for $f, g \in C^*(F)$. Alternatively, we sometimes say in this case that the extender is *monotone*. The following lemma is well-known and is used several times in the sequel.

Lemma 1.1 ([5, 3.12, p. 43]). *Let X be an arbitrary space, and let R_0 be any dense subset of the real line $\mathbb{R}$. Suppose that open sets U_r of X are defined, for all $r \in R_0$, such that $\bigcup_r U_r = X$, $\bigcap_r U_r = \emptyset$ and $\overline{U_r} \subseteq U_s$ whenever $r < s$. Then the formula $f(x) = \inf\{r \in R_0 : x \in U_r\}$ defines f as a continuous function on X.*

Definition 1.2 ([6]). *A space X is said to be monotonically normal if there is a function G (a monotone normality operator) which assigns to each ordered pair (A, U) of subsets of X, with A closed, U open and $A \subseteq U$, an open set $G(A, U)$ such that*

(a) $A \subseteq G(A, U) \subseteq \overline{G(A, U)} \subseteq U$
(b) *if B is closed, V is open, $B \subseteq V$ and $A \subseteq B$ and $U \subseteq V$, then $G(A, U) \subseteq G(B, V)$.*

Theorem 1.3 ([6]). *A space X is monotonically normal if and only if there is a function H which assigns to each ordered pair (x, U), with U an open set containing x, an open set $H(x, U)$ such that,*

(i) $x \in H(x, U) \subseteq U$,
(ii) *if V is open and $x \in U \subseteq V$ then $H(x, U) \subseteq H(x, V)$,*
(iii) *if $x \neq y$ are points of X then $H(x, X \setminus \{y\}) \cap H(y, X \setminus \{x\}) = \emptyset$.*

Monotone normality is an hereditary property. Metric spaces are monotonically normal as are GO spaces [6]. Related to monotone normality is the notion of K_n-spaces defined by van Douwen [2]. Essentially, a K_n-space is a space X such that, given a subspace F of X, the open sets in F can be extended to open sets in X in a manner which respects set inclusion in some sense.

Definition 1.4. *A space X is said to be a K_n-space, for an integer n, if for each non-empty closed subspace F of X, there exists a map $\kappa : \tau F \to \tau X$ (a K_n-function) such that,*

(i) $\kappa(U) \cap F = U$ *for all* $U \in \tau F$,
(ii) $\kappa(U) \subseteq \kappa(V)$ *for* $U \subseteq V \in \tau F$,
(iii) $(n = 0)$ $\kappa(\emptyset) = \emptyset$ *and* $\kappa(U) \cap \kappa(V) = \kappa(U \cap V)$ *for all* $U, V \in \tau F$,
$(n \ge 1)$ *if* $U_i \in \tau F$, *for* $0 \le i \le n$, *are such that* $U_i \cap U_j = \emptyset$ *for* $0 \le i < j \le n$, *then* $\bigcap_{i=0}^{n} \kappa(U_i) = \emptyset$.

Without loss of generality, we may assume that $\kappa(F) = X$. It is a straightforward exercise to show that a K_n-space is a K_m-space for $n \leq m$, and that K_1-spaces are hereditarily collectionwise normal. It is known that monotonically normal spaces are K_1-spaces [2]. The link between monotonically normal spaces and K_1-spaces is, however, stronger still. We have the following:

Theorem 1.5 ([4]). *A space X is monotonically normal if and only if for each closed subspace F of X, there exists a K_1-function $\kappa_F : \tau F \to \tau X$, such that,*

(MK) *If F_i (for $i = 0, 1$) are subspaces of X with $U_i \in \tau F_i$, such that $U_0 \subseteq U_1$ and $F_1 \setminus U_1 \subseteq F_0 \setminus U_0$, then $\kappa_{F_0}(U_0) \subseteq \kappa_{F_1}(U_1)$.*

2. Monotone normality and continuous functions

Urysohn's Lemma shows that the existence of continuous functions separating pairs of disjoint, closed sets characterises normality. The question now arises as to whether it is possible to characterise monotone normality in a similar way to this. It is natural to look at the proof of Urysohn's Lemma and see what extra conditions monotone normality gives us. In this way, we have the following analogue of Urysohn's Lemma for monotonically normal spaces. This result was first proved by Borges [1]. The proof, which is my own, is included here for later use.

Lemma 2.1. *If X is a monotonically normal space, then to each ordered pair (A, U), where A is closed in X and U is open in X and $A \subseteq U$, we can assign a continuous function $f_{A,U} : X \to [0, 1]$ such that $f_{A,U}(x) = 0$ if $x \in A$ and $f_{A,U}(x) = 1$ if $x \in X \setminus U$, and such that if $A \subseteq B$ and $U \subseteq V$, then $f_{B,V}(x) \leq f_{A,U}(x)$ for all $x \in X$.*

Proof: Assume X is a monotonically normal space with monotone normality operator G and A is a closed subset of X and U is an open subset of X with $A \subseteq U$. We define open sets $U(r)$ for all $r \neq 0$ in the set of dyadic rationals. Let $U(r) = \emptyset$ if $r < 0$ and $U(r) = X$ if $r > 1$ and let D denote the dyadic rationals in the interval $(0, 1]$.

Now put $U(1) = U$, an open set, and let $U(1/2) = G(A, U)$. By definition of G, $U(1/2)$ is an open set such that $A \subseteq U(1/2) \subseteq \overline{U(1/2)} \subseteq U(1)$. Next, let $U(1/4) = G(A, U(1/2))$ and $U(3/4) = G(\overline{U(1/2)}, U(1))$. Iterating, we get a sequence $U(r)$ for $r \in D$ such that,

$$A \subseteq U(r) \subseteq \overline{U(r)} \subseteq U(s) \subseteq \overline{U(s)} \subseteq U(1) \quad \text{for all} \quad r < s \in D.$$

Moreover, if $\frac{p}{2^q}$ is an element of D in its lowest terms, and $U(0) = A$ for ease of notation, (and denoting $U(p/2^q)$ by $U_{p,q}$) then

$$U_{p,q} = G\left(\overline{U_{p-1,q}}, U_{p+1,q}\right).$$

Let $f_{A,U}(x) = \inf\{r : x \in U(r)\}$. By Lemma 1.1 this defines a continuous function from X to $[0, 1]$ and clearly we have $f_{A,U}(x) = 0$ if $x \in A$ and $f_{A,U}(x) = 1$

if $x \in X \setminus U$. It remains to verify the second part of the lemma, i.e. that if $A \subseteq B$ and $U \subseteq V$ then $f_{B,V}(x) \le f_{A,U}(x)$ for all $x \in X$. It is enough to show that for all $r \in D$, $U(r) \subseteq W(r)$ where the $U(r)$ are as above and the $W(r)$ are the corresponding open sets in the construction of $f_{B,V}$.

$D = \{\frac{p}{2^q} \mid 0 < p \le 2^q,\ p, q \in \mathbb{N}\}$, so proceed by induction on q. If $q = 0$ then $A \subseteq B$ and $U(1) = U \subseteq V = W(1)$. Assume the result holds for $q \le s$ and let $p/2^{s+1}$ be an element of D in its lowest terms. This forces p odd so $p-1 = 2n$ for some n and therefore $(p-1)/2^{s+1} = n/2^s$ and $(p+1)/2^{s+1} = (n+1)/2^s$. So, by the inductive hypothesis, we have $U_{p-1,s+1} \subseteq W_{p-1,s+1}$ and $U_{p+1,s+1} \subseteq W_{p+1,s+1}$ (these sets have already been accounted for since the subscripts are of the form $r/2^s$ for some r). Consequently,

$$U_{p,s+1} = G\left(\overline{U_{p-1,s+1}}, U_{p+1,s+1}\right) \subseteq G\left(\overline{W_{p-1,s+1}}, W_{p+1,s+1}\right) = W_{p,s+1}$$

and we have completed the inductive step and the proof. □

We also note that Lemma 2.1 actually provides a characterisation of monotone normality in the same way that Urysohn's Lemma characterises normality. Indeed, if the functions $f_{A,U}$ exist then we can construct a monotone normality operator G by defining $G(A,U) = (f_{A,U})^{-1}[0, 1/2)$. It is clear that $G(A,U)$ is an open set, $A \subseteq G(A,U) \subseteq \overline{G(A,U)} \subseteq U$ and if (B,V) is a pair of subsets of X such that B is closed, V is open and $B \subseteq V$ and $A \subseteq B$ and $U \subseteq V$ then $f_{B,V}(x) \le f_{A,U}(x)$ for all $x \in X$ and hence $G(A,U) \subseteq G(B,V)$. We shall call a set of functions satisfying the conditions of Lemma 2.1, a collection of *monotone Urysohn functions* and say that they *witness* the monotone normality of X.

We note that, without loss of generality, we can actually have a collection of monotone Urysohn functions which are symmetric in a certain sense, as detailed in the following lemma.

Lemma 2.2. *If X is monotonically normal then X has a collection of monotone Urysohn functions, $\{g_{A,U} : A$ closed, U open, $A \subseteq U\}$, such that for all pairs, (A,U), $g_{A,U} = 1 - g_{X \setminus U, X \setminus A}$.*

Proof: Given a set of monotone Urysohn functions, as in Lemma 2.1, define $g_{A,U}$ as follows,

$$g_{A,U}(x) = \frac{f_{A,U}(x) - f_{X\setminus U, X\setminus A}(x) + 1}{2}.$$

It is easily verified that the $g_{A,U}$ form a collection of monotone Urysohn functions for X with the desired property. □

So, now that we have an analogue of Urysohn's Lemma, the obvious question is, what about the Tietze-Urysohn theorem? In [6], it was shown that monotonically normal spaces satisfy the monotone extension property and it was asked whether the converse was true. The converse, however, is not true, as van Douwen showed in his thesis [2]. The question still remains as to whether we can find a monotonically normal analogue of the Tietze-Urysohn theorem. Van Douwen in fact showed that various extension properties fail in this task. Van Douwen's example is a countable retractifiable space which is not monotonically normal. It has a wide range of extension properties, among them the monotone extension property. The reason this example can be constructed is because all of the extension properties van Douwen considered fail to link functions defined on different closed subspaces. If we remember this crucial fact, then it is possible to characterise monotone normality in the desired fashion. Indeed, we have the following theorem.

Theorem 2.3. *A space X is monotonically normal if and only if for each closed subspace F of X there exists an extender $\Phi_F : C(F,[0,1]) \to C(X,[0,1])$ such that*

(1) *if $F_1 \subseteq F_0$ are closed subspaces and $f_i \in C(F_i,[0,1])$ such that $f_0 \restriction_{F_1} \geq f_1$ and $f_0(x) = 1$ for all $x \in F_0 \setminus F_1$ then $\Phi_{F_0}(f_0) \geq \Phi_{F_1}(f_1)$,*
(2) *if $F_1 \subseteq F_0$ are closed subspaces and $f_i \in C(F_i,[0,1])$ such that $f_0 \restriction_{F_1} \leq f_1$ and $f_0(x) = 0$ for all $x \in F_0 \setminus F_1$ then $\Phi_{F_0}(f_0) \leq \Phi_{F_1}(f_1)$.*

At first glance conditions (1) and (2) look rather complicated. In condition (1), the function f_0 defined on the larger subspace F_0 dominates the function f_1 on the smaller subspace F_1. If the extension $\Phi_{F_0}(f_0)$ of f_0 is to dominate the extension $\Phi_{F_1}(f_1)$ of f_1, then f_0 must, at the very least, take "sensible" values on the subspace $F_0 \setminus F_1$, since the extenders we are constructing are, to a certain degree, range-preserving ($\Phi_F(f)(x) \in [\inf f, \sup f]$ for all x). We must have that $f_0(y) \geq \inf f_1$ for $y \in F_0 \setminus F_1$. Condition (1) goes further by insisting that for points in $F_0 \setminus F_1$, f_0 takes values as large as possible (namely 1).

Proof: First, assume that X is monotonically normal with monotone normality operator G. As opposed to the usual textbook proof of the Tietze-Urysohn theorem which follows from Urysohn's Lemma by applying Weierstrass's M-test and uniform convergence, we use here a 'more topological' proof based on a paper by Mandelkern [7]. The idea of the proof is reminiscent of the 'onion skin' method of Urysohn's Lemma. Mandelkern's construction is fully described below for completeness (although, here, in terms of monotone normality operators).

So, suppose F is a closed subspace of X and f is a continuous function, $f : F \to [0,1]$. Define for $r,s \in \mathbb{Q} \cap [0,1)$, $A_r = \{x \in F : f(x) \leq r\}$ and $U_s = X \setminus \{x \in F : f(x) \geq s\}$. Index the set $P = \{(r,s) : r,s \in \mathbb{Q}$ and $0 \leq r < s < 1\}$ so that $P = \{(r_n,s_n) : n \in \mathbb{N}\}$.

Suppose closed sets H_k have been constructed for all $k < n$ such that,

$$A_{r_k} \subseteq H_k^o \subseteq H_k \subseteq U_{s_k} \qquad \text{for} \quad k < n$$

$$H_j \subseteq H_k^o \qquad \text{when} \quad j,k < n, \quad r_j < r_k, \quad \text{and} \quad s_j < s_k.$$

Let $J = \{j : j < n, r_j < r_n$ and $s_j < s_n\}$ and let $K = \{k : k < n, r_n < r_k$ and $s_n < s_k\}$. We now define H_n as follows:

$$H_n = \overline{G(A_{r_n} \cup \bigcup_{j \in J} H_j,\ U_{s_n} \cap \bigcap_{k \in K} H_k^o)}.$$

Writing H_{rs} for H_n where $r_n = r$ and $s_n = s$ we have, by induction (the details are straightforward), a family of closed subsets of X, $\{H_{rs} : (r,s) \in P\}$ such that,

$$A_r \subseteq H_{rs}^o \subseteq H_{rs} \subseteq U_s \qquad (r,s) \in P$$
$$H_{rs} \subseteq H_{tu}^o \qquad \text{when} \quad r < t, \quad \text{and} \quad s < u.$$

Defining $X_r = \bigcap_{s>r} H_{rs}$ for $r \in \mathbb{Q} \cap [0,1)$ and letting $X_r = \emptyset$ for $r < 0$ and $X_r = X$ for $r \geq 1$ we have the following: if $0 \leq r < s < 1$, choose t such that $r < t < s$, then,

$$X_r \subseteq H_{rt} \subseteq H_{ts}^o \subseteq H_{ts} \subseteq \bigcap_{u>s} H_{su} = X_s$$

and

$$A_r \subseteq X_r \cap F = F \cap \bigcap_{s>r} H_{rs} \subseteq F \cap \bigcap_{s>r} U_s = A_r.$$

We therefore have a family of closed subsets of X, $\{X_r : r \in \mathbb{Q}\}$ such that, $\bigcup_{r\in\mathbb{Q}} X_r = X$, $\bigcap_{r\in\mathbb{Q}} X_r = \emptyset$ and $X_r \subseteq X_s^o$ whenever $r < s$ and

$$\text{(A)} \qquad X_r \cap F = A_r \qquad r \in \mathbb{Q}.$$

Defining $\Phi_F(f)(x) = \inf\{r : x \in X_r\}$ gives a continuous extension of f by Lemma 1.1 and Equation (A). It remains to check that conditions (1) and (2) are satisfied. We prove that condition (1) holds. The proof for (2) is similar. We use the notation A_r, U_s, H_{rs} and X_r as above for the extension of f_0 and A'_r, U'_s, H'_{rs} and X'_r for the extension of f_1.

If $x \in A_r$ (where $r < 1$), then $f_0(x) \leq r < 1$ and $x \in F_0$. By assumption $x \in F_1$ (else $f_0(x) = 1$) and therefore, again by assumption, $x \in A'_r$ (since $f_1 \leq f_0 \restriction_{F_1}$). We therefore have that $A_r \subseteq A'_r$ for all $r \in \mathbb{Q} \cap [0,1)$. Similarly $U_s \subseteq U'_s$ for all $s \in \mathbb{Q} \cap [0,1)$. By a simple induction, since G is a monotone operator, we have that $H_{rs} \subseteq H'_{rs}$ for all $(r,s) \in P$ and consequently, $X_r \subseteq X'_r$ for all $r \in \mathbb{Q} \cap [0,1)$. Hence $\Phi_{F_0}(f_0) \geq \Phi_{F_1}(f_1)$ as required.

Before we prove the converse we define some notation: If $x \notin E$ a closed subset of X, then define the continuous functions $\chi_{xE} : \{x\} \cup E \to [0,1]$ and $\hat{\chi}_{xE} : \{x\} \cup E \to [0,1]$ by, $\chi_{xE}(y) = 0$ if $y = x$ and $\chi_{xE}(y) = 1$ if $y \in E$ and $\hat{\chi}_{xE}(y) = 1$ if $y = x$ and $\hat{\chi}_{xE}(y) = 0$ if $y \in E$. Now for x in open U, let $F_{xU} = \Phi_{\{x\}\cup(X\setminus U)}(\chi_{xX\setminus U})$ and $K_{xU} = \Phi_{\{x\}\cup(X\setminus U)}(\hat{\chi}_{xX\setminus U})$.

If $x \in U$ where U is open in X, then define $G(x,U)$ as follows:

$$G(x,U) = F_{xU}^{-1}[0,1/2) \cap K_{xU}^{-1}(1/2,1].$$

It is clear that $G(x,U)$ is open, contains x and lies inside U. It is straightforward to check that $G(x, X\setminus\{y\}) \cap G(y, X\setminus\{x\}) = \emptyset$ for $x \neq y$. So assume $x \in U \subseteq V$. Then $G(x,V) = F_{xV}^{-1}[0,1/2) \cap K_{xV}^{-1}(1/2,1]$. Now, $F_1 = \{x\} \cup (X \setminus V) \subseteq \{x\} \cup (X \setminus U) = F_0$, $\chi_{xX\setminus V}(y) \leq \chi_{xX\setminus U}(y)$ for $y \in F_1$ and $\chi_{xX\setminus U}(y) = 1$ for $y \in F_0 \setminus F_1$ therefore, by condition (1), $F_{xU} \geq F_{xV}$. Similarly, using condition (2), $K_{xU} \leq K_{xV}$ hence, $G(x, U) \subseteq G(x, V)$. We have, therefore, shown that G is a monotone normality operator as in Theorem 1.3. □

Another 'more topological' proof of the Tietze-Urysohn theorem by Scott [10] may also be used to prove this result. It is, however, more complicated, using four applications of normality. We also note that conditions (1) and (2) both imply that the extender is monotone, and so, the above gives us an alternative (and, the author believes, simpler) proof to the one given in [6] showing that monotonically normal spaces satisfy the monotone extension property.

The reader may be interested to compare the above analogue of the Tietze-Urysohn theorem with the following, alternative, analogue given in [11]. Notation is as in the above proof.

Theorem 2.4. *X is monotonically normal iff the following three conditions hold,*

(1) *for each closed $E \subseteq X$ there is an extender $\Phi_E : C^*(E) \to C^*(X)$,*
(2) *if $x \notin E$ closed and $F \subseteq E$ then $\Phi_{\{x\}\cup F}(\chi_{xF}) \leq \Phi_{\{x\}\cup E}(\chi_{xE})$,*
(3) *if $f, g : E \to [0, 1]$ are such that $f = 1 - g$ then $\Phi_E(f) = 1 - \Phi_E(g)$.*

It can be seen that the extender constructed here is, in a weak sense, linear (condition (3)). To prove this result, one simply duplicates the usual proof of the Tietze-Urysohn theorem (using uniform convergence and Weierstrass's M-test) using monotone Urysohn functions. Condition (2) follows since, for the special case of the functions χ_{xE}, the construction of the extension collapses and condition (3) follows from repeated use of Lemma 2.2. The details are left to the reader.

3. Extending Urysohn functions

Assume that X is a monotonically normal space and Y is a closed subspace with a set, $\mathcal{F}_Y$, of monotone Urysohn functions witnessing the monotone normality of Y. In his survey paper [9], San-ou asked whether one could find a set of monotone Urysohn functions, $\mathcal{F}_X$, witnessing the monotone normality of X and such that every element of $\mathcal{F}_Y$ is the restriction of some element of $\mathcal{F}_X$. That is, can we 'extend' monotone Urysohn functions to monotone Urysohn functions? In this section we answer this question in the affirmative, as well as look at the alternative question for monotone normality operators.

If we consider this alternative problem first, the following theorem shows that we can 'extend' monotone normality operators from closed subspaces. More precisely we have:

Theorem 3.1. *If X is monotonically normal and Y is a closed subspace of X with a monotone normality operator H_Y, then X has a monotone normality operator H_X such that for all pairs (A, U) with A closed in X, U open in X, $A \subseteq U$, and $A \cap Y \neq \emptyset$,*

(1) $H_X(A, U) \cap Y = H_Y(A \cap Y, U \cap Y)$,
(2) $\overline{H_X(A, U)} \cap Y = \overline{H_Y(A \cap Y, U \cap Y)}$.

Before we begin the proof of this result, recall the following lemma due to Borges, which shows monotonically normal spaces have a monotone normality operator with stronger properties than the usual one.

Lemma 3.2 ([1])**.** *If X is monotonically normal, then there is a function G which assigns to each pair (A,U), where A is closed and U is open, an open set $G(A,U) \subseteq U$ such that (denoting $G(\{x\},U)$ by $G(x,U)$)*

(a) *if $A \subseteq B$ and $U \subseteq V$ then $G(A,U) \subseteq G(B,V)$,*
(b) *$A \cap U \subseteq G(A,U) \subseteq \overline{G(A,U)} \subseteq A \cup U$,*
(c) *if $G(x,U) \cap G(y,V) \neq \emptyset$, then $x \in V$ or $y \in U$,*
(d) *$G(A,U) = \bigcup_{x\in A} G(x,U)$ if $A \subseteq U$.*

Lemma 3.3. *Assume X is monotonically normal and G is an operator as in Lemma 3.2. If $V \subseteq Y \subseteq X$ and $A \cap Y \subseteq V$ with V open in Y, Y closed in X and A closed in X and U is open in X with $V \subseteq U$ and $A \subseteq U$, then for $H = G(\overline{V}, (U \setminus Y) \cup V)$ and $K = \bigcup_{x\in A\setminus Y} G(x, U \setminus Y)$ and $G = H \cup K$ we have $G \cap Y = V$ and $\overline{G} \cap Y = \overline{V}$.*

Proof: We first check that H is well-defined, i.e. that $S = (U \setminus Y) \cup V$ is indeed an open set. Since $V \subseteq U$, $(X \setminus S) = (X \setminus U) \cup (Y \setminus V)$ which, since Y is closed, is the union of two closed sets. Next we show that $G \cap Y = V$. Now, by Lemma 3.2, $H \subseteq (U \setminus Y) \cup V$ hence $H \cap Y \subseteq V$. By Lemma 3.2 (b), $V \subseteq \overline{V} \cap ((U \setminus Y) \cup V) \subseteq H$ hence $V \subseteq H \cap Y$. It is clear that $K \cap Y = \emptyset$. We have therefore proved that $G \cap Y = V$. We immediately have that $\overline{V} \subseteq \overline{G} \cap Y$. To complete the proof we show that $\overline{G} \cap Y \subseteq \overline{V}$. Note: $\overline{G} = \overline{H} \cup \overline{K}$. By Lemma 3.2 (b), $\overline{H} \subseteq (U \setminus Y) \cup \overline{V}$ hence $\overline{H} \cap Y \subseteq \overline{V}$. We now show that $\overline{K} \cap Y \subseteq \overline{V}$. If $z \in \overline{K} \cap Y$ but $z \notin \overline{V}$ then, in particular $z \in Y$ and hence $z \notin A \setminus Y$. Therefore, since $A \cap Y \subseteq \overline{V}$, $z \in T = X \setminus (A \cup \overline{V})$ which is open. By the assumption that $z \in \overline{K}$, $G(z,T) \cap G(x, U \setminus Y) \neq \emptyset$ for some $x \in A \setminus Y$. Consequently, by Lemma 3.2 (c), $z \in U \setminus Y$ or $x \in T \subseteq X \setminus A$, each of which is a contradiction. $\square$

Proof of Theorem 3.1: Assume X is monotonically normal with operator G as in Lemma 3.2. We define H_X as follows. For all ordered pairs (A,U) where A is closed in X, U is open in X and $A \subseteq U$, define:

$$\begin{aligned} H_X(A,U) &= G(A, U \setminus Y) && \text{if } A \cap Y = \emptyset \\ &= G\left(\overline{H_Y(A \cap Y, U \cap Y)},\ (U \setminus Y) \cup H_Y(A \cap Y, U \cap Y)\right) \\ &\qquad \cup \bigcup_{x\in A\setminus Y} G(x, U \setminus Y) && \text{otherwise.} \end{aligned}$$

As we noted in Lemma 3.3, this is well-defined. Clearly $A \subseteq H_X(A,U)$. It remains to check that $H_X(A,U) \subseteq H_X(B,V)$ whenever $A \subseteq B$ and $U \subseteq V$, that $\overline{H_X(A,U)} \subseteq U$ and that H_X 'extends' H_Y (that is that conditions (1) and (2) are satisfied). Assume $A \subseteq B$ and $U \subseteq V$.

Case 1. $B \cap Y = \emptyset$.
This implies $A \cap Y = \emptyset$. Hence, $H_X(A, U) = G(A, U \setminus Y) \subseteq G(B, V \setminus Y) = H_X(B, V)$.
Case 2. $A \cap Y = \emptyset$ and $B \cap Y \neq \emptyset$.
We immediately have that $A \subseteq B \setminus Y$ therefore $H_X(A, U) = G(A, U \setminus Y) = \bigcup_{a \in A} G(a, U \setminus Y) \subseteq \bigcup_{b \in B \setminus Y} G(b, V \setminus Y) \subseteq H_X(B, V)$.
Case 3. $B \cap Y \neq \emptyset \neq A \cap Y$.
We therefore have that $A \setminus Y \subseteq B \setminus Y$ and $A \cap Y \subseteq B \cap Y$. Hence, $\bigcup_{x \in A \setminus Y} G(x, U \setminus Y) \subseteq \bigcup_{x \in B \setminus Y} G(x, V \setminus Y)$ and $G(\overline{H_Y(A \cap Y, U \cap Y)}, (U \setminus Y) \cup H_Y(A \cap Y, U \cap Y)) \subseteq G(\overline{H_Y(B \cap Y, V \cap Y)}, (V \setminus Y) \cup H_Y(B \cap Y, V \cap Y))$, since G and H_Y are monotone. Thus, $H_X(A, U) \subseteq H_X(B, V)$, as required.

We now check that $\overline{H_X(A, U)} \subseteq U$.
Case 1. $A \cap Y = \emptyset$.
This follows directly from the analogous property of G.
Case 2. $A \cap Y \neq \emptyset$.
Since the closure operator respects finite unions (i.e. $\overline{A \cup B} = \overline{A} \cup \overline{B}$) the following is sufficient. First, $\overline{\bigcup_{x \in A \setminus Y} G(x, U \setminus Y)} \subseteq U$ since, if $y \notin U$ then $y \notin A$ so $G(y, X \setminus A) \cap G(x, U \setminus Y) = \emptyset$ for all $x \in A \setminus Y$ (else $x \in X \setminus A$). Second, by Lemma 3.2 (b)

$$\begin{aligned} &\overline{G\left(\overline{H_Y(A \cap Y, U \cap Y)},\ (U \setminus Y) \cup H_Y(A \cap Y, U \cap Y)\right)} \\ &\subseteq (U \setminus Y) \cup \overline{H_Y(A \cap Y, U \cap Y)} \subseteq U. \end{aligned}$$

So we have proved that H_X is, indeed, a monotone normality operator for X. Finally, conditions (1) and (2) follow immediately from Lemma 3.3. □

This result now suggests a 'naive' proof to answer San-ou's question in the affirmative. This naive proof, however does not work. More precisely, assume now that, if we have a set of monotone Urysohn functions $\mathcal{F}_Y$ which witness the monotone normality of Y, then they were constructed by the method of Lemma 2.1 from a monotone normality operator H_Y. Using the above theorem, we can construct a monotone normality operator H_X for X which extends H_Y. By repeating the construction in Lemma 2.1 with this operator, the reader can easily check that we acquire a set of Urysohn functions $\mathcal{F}_X$ for X which 'extends' $\mathcal{F}_Y$. Indeed, in the notation of Lemma 2.1, if $f_{A,U} \in \mathcal{F}_Y$ is a monotone Urysohn function for Y, then it is the restriction of the monotone Urysohn function $f_{A,V} \in \mathcal{F}_X$ for X, where V is open in X and $V \cap Y = U$. However, there is a limitation to this argument. The problem arises from our assumption that the given set of monotone Urysohn functions was acquired by the construction of Lemma 2.1. Given a monotone normality operator we can construct a set of monotone Urysohn functions and, given a set of monotone Urysohn functions, we can construct a monotone normality operator. However, it is not clear whether composing these two operations gets us back to where we started. The author does not know the answer to this question. We can, however, still answer San-ou's question in the affirmative but we have to construct our Urysohn functions directly to extend the given set. Precisely, we have:

Theorem 3.4. *If X is monotonically normal and Y is a closed subspace of X with a set of monotone Urysohn functions $\{f_{A,U} : A \subseteq U,\ A$ closed in Y and U open in $Y\}$, then X has a set of monotone Urysohn functions $\{g_{A,U} : A \subseteq U,\ A$ closed in X and U open in $X\}$ such that $g_{A,U}\restriction_Y = f_{A\cap Y, U\cap Y}$ for all pairs (A, U) with $A \cap Y \neq \emptyset$.*

Proof: The idea of the proof is very similar to that of the above theorem. Assume X has an operator G as in Lemma 3.2. We now construct the functions $g_{A,U}$. So, take A closed in X and U open in X and containing A. If $A \cap Y = \emptyset$ then let $g_{A,U}$ be the monotone Urysohn function for the pair $(A, U \setminus Y)$ constructed from G as in Lemma 2.1 (G is in particular a monotone normality operator). If $A \cap Y \neq \emptyset$ then proceed as follows:

Let $V(r) = f^{-1}_{A\cap Y, U\cap Y}[0, r)$, $U(0) = A$ and $U(1) = U$. Now define,

$$U(1/2) = G\left(\overline{V(1/2)}, (U \setminus Y) \cup V(1/2)\right) \cup \bigcup_{x \in A \setminus Y} G(x, U \setminus Y).$$

By Lemma 3.3, $U(1/2) \cap Y = V(1/2)$ and $\overline{U(1/2)} \cap Y = \overline{V(1/2)}$. Proceeding inductively as in Lemma 2.1, if $p/2^q$ is a dyadic rational in its lowest terms define:

$$U(p/2^q) = G\left(\overline{V(p/2^q)},\ (U((p+1)/2^q) \setminus Y) \cup V(p/2^q)\right) \cup \bigcup_{x \in \overline{U(\frac{p-1}{2^q})} \setminus Y} G\left(x, U((p+1)/2^q) \setminus Y\right).$$

Inductively, $\overline{U((p-1)/2^q)} \cap Y = \overline{V((p-1)/2^q)} \subseteq V(p/2^q) \subseteq \overline{V(p/2^q)} \subseteq V((p+1)/2^q) \subseteq U((p+1)/2^q)$ and $\overline{U((p-1)/2^q)} \subseteq U((p+1)/2^q)$. By Lemma 3.3, $U(p/2^q) \cap Y = V(p/2^q)$ and $\overline{U(p/2^q)} \cap Y = \overline{V(p/2^q)}$. It is easy to prove that $\overline{U((p-1)/2^q)} \subseteq U(p/2^q)$ and in the same way as in the proof of $\overline{H_X(A, U)} \subseteq U$ in Theorem 3.1 we can show that $\overline{U(p/2^q)} \subseteq U((p+1)/2^q)$. We therefore have a family of open sets $\{U(r) : r$ dyadic$\}$ such that $U(r) \subseteq \overline{U(r)} \subseteq U(s)$ whenever $r < s$. Defining $g_{A,U}(x) = \inf\{r : x \in U(r)\}$ gives us the required function. It only remains to check that the collection of functions we have constructed 'extends' the original collection and is monotone in the sense that, if $A \subseteq B$ and $U \subseteq V$, then $g_{A,U} \geq g_{B,V}$. For this it is sufficient to prove that $U(r) \subseteq W(r)$ for all r, where the $U(r)$ are as in the construction above of g_{AU} and the $W(r)$ are the corresponding sets in the construction of g_{BV}.

Case 1. $B \cap Y = \emptyset$.
This follows from Lemma 2.1.

Case 2. $B \cap Y \neq \emptyset$ and $A \cap Y = \emptyset$.
This implies $A \subseteq B \setminus Y$. Also, note that $U(r) \subseteq X \setminus Y$ for all $r \in [0,1]$.

$$U(1/2) = G(A, U \setminus Y) = \bigcup_{x \in A} G(x, U \setminus Y) \subseteq \bigcup_{x \in B \setminus Y} G(x, V \setminus Y) \subseteq W(1/2).$$

Inductively,

$$U(p/2^q) = G\left(\overline{U((p-1)/2^q)}, U((p+1)/2^q)\right) = \bigcup_{x \in \overline{U((p-1)/2^q)}} G(x, U((p+1)/2^q))$$

$$\subseteq \bigcup_{x \in \overline{W(\frac{p-1}{2^q})} \setminus Y} G(x, W((p+1)/2^q) \setminus Y) \subseteq W(p/2^q).$$

Case 3. $A \cap Y \neq \emptyset$.
This means that $A \setminus Y \subseteq B \setminus Y$, $A \cap Y \subseteq B \cap Y$, $U \setminus Y \subseteq V \setminus Y$ and $U \cap Y \subseteq V \cap Y$. This implies, $f_{A \cap Y, U \cap Y} \geq f_{B \cap Y, V \cap Y}$ and therefore $V(r) \subseteq V'(r)$ for all r where $V(r)$ are the sets as in the construction of the $U(r)$ and $V'(r)$ are the corresponding sets in the construction of the $W(r)$. By monotonicity of G, and a simple induction, it follows that $U(r) \subseteq W(r)$ for all r.

We finally check that $g_{A,U} \restriction_Y = f_{A \cap Y, U \cap Y}$. So take $x \in Y$. If $g_{A,U}(x) = s$, then $x \in U(r)$ for all dyadic $r > s$. Hence $x \in U(r) \cap Y = V(r)$ for all $r > s$ and therefore $f_{A \cap Y, U \cap Y}(x) \leq s$. If $f_{A \cap Y, U \cap Y}(x) = s$ then for all $r > s$, $x \in V(r) \subseteq U(r)$ and hence, $g_{A,U}(x) \leq s$. $\square$

4. Semi-continuous functions

In [2], van Douwen considered extension of functions satisfying conditions weaker than continuity. He showed that the class of K_1-spaces could be characterised in terms of extending semi-continuous functions. First recall, a function $f : X \to \mathbb{R}$ is said to be upper [lower] semi-continuous if for all $r \in \mathbb{R}$, $f^{-1}(-\infty, r)$ [$f^{-1}(r, \infty)$] is open. We denote the collection of all bounded, upper [lower] semi-continuous functions on X by $C^*_{usc}(X)$ [$C^*_{lsc}(X)$]. Clearly, a function $f : X \to \mathbb{R}$ is continuous if and only if it is both upper and lower semi-continuous. Van Douwen's Theorem is as follows.

Theorem 4.1 (van Douwen)**.** *A completely regular space X is a K_1-space if and only if for every non-empty closed subspace F of X there are extenders $\Phi : C^*_{usc}(F) \to C^*_{usc}(X)$ and $\Psi : C^*_{lsc}(F) \to C^*_{lsc}(X)$ such that,*
(a) *$\Phi(f) \leq \Phi(g)$ whenever $f \leq g$, for $f, g \in C^*_{usc}(F)$,*
(b) *$\Psi(f) \leq \Psi(g)$ whenever $f \leq g$, for $f, g \in C^*_{lsc}(F)$,*
(c) *$\Psi(f) \leq \Phi(f)$ for $f \in C^*(F)$.*

It seems natural to ask whether this theorem can be strengthened to characterise the class of K_0-spaces. In answer to this question we have the following.

Theorem 4.2. *A completely regular space X is a K_0-space if and only if for every non-empty closed subspace F of X there are extenders $\Phi : C^*_{usc}(F) \to C^*_{usc}(X)$ and $\Psi : C^*_{lsc}(F) \to C^*_{lsc}(X)$ such that,*

(a) $\Phi(\max(f,g)) = \max(\Phi(f),\Phi(g))$ *for* $f,g \in C^*_{usc}(F)$,
(b) $\Psi(\min(f,g)) = \min(\Psi(f),\Psi(g))$ *for* $f,g \in C^*_{lsc}(F)$,
(c) $\Psi(f) \le \Phi(f)$ *for* $f \in C^*(F)$.

Proof: First assume that X is a K_0-space. The extenders are constructed in the same way as in van Douwen's construction for K_1-spaces. We include the details for completeness. Let F be a non-empty closed subspace of X and let $\kappa : \tau F \to \tau X$ be a K_0-function.

For $f \in C^*_{usc}(F)$, define $\Phi(f) : X \to \mathbb{R}$ by,

$$\Phi(f)(x) = \inf\left\{t \in \mathbb{R} : x \in \kappa(f^{-1}(-\infty,t))\right\}.$$

Since $\kappa(f^{-1}(-\infty, 1+\sup f)) = \kappa(F) = X$ and $\kappa(f^{-1}(-\infty,\inf f)) = \kappa(\emptyset) = \emptyset$, $\Phi(f)$ is well-defined and bounded. Furthermore, if $f \in C^*_{usc}(F)$ then for each $t \in \mathbb{R}$,

$$\begin{aligned}\Phi(f)^{-1}(-\infty,t) &= \{x \in X : \exists r < t \text{ such that } x \in \kappa(f^{-1}(-\infty,r))\}\\ &= \bigcup\{\kappa(f^{-1}(-\infty,r)) : r < t\}\end{aligned}$$

and hence $\Phi(f)^{-1}(-\infty,t)$ is open and $\Phi(f)$ is upper semi-continuous. Also if $f \in C^*_{usc}(F)$ and $x \in F$, then, since $\kappa(U) \cap F = U$ by 1.4 (i), $\Phi(f)(x) = \inf\left\{t \in \mathbb{R} : x \in f^{-1}(-\infty,t)\right\} = f(x)$, so Φ is indeed an extender. It remains to check that Φ satisfies condition (a).

First, it is clear that Φ is monotone, since, if $f \le g$, then $f^{-1}(-\infty,t) \supseteq g^{-1}(-\infty,t)$ for all t which implies that $\kappa(f^{-1}(-\infty,t)) \supseteq \kappa(g^{-1}(-\infty,t))$ for all t (by 1.4 (ii)) and so, $\Phi(f) \le \Phi(g)$. By monotonicity, we have that $\Phi(\max(f,g)) \ge \max(\Phi(f),\Phi(g))$. As for the reverse inequality: If $\max(\Phi(f),\Phi(g))(x) = s$, then $\Phi(f)(x) \le s$ and $\Phi(g)(x) \le s$. So, for all $t > s$, by 1.4 (iii)

$$\begin{aligned}x \in \kappa(f^{-1}(-\infty,t)) \cap \kappa(g^{-1}(-\infty,t)) &= \kappa\left(f^{-1}(-\infty,t) \cap g^{-1}(-\infty,t)\right)\\ &= \kappa((\max(f,g))^{-1}(-\infty,t))\end{aligned}$$

and hence, $\Phi(\max(f,g))(x) \le s$.

For $f \in C^*_{lsc}(F)$ define, $\Psi(f) : X \to \mathbb{R}$ by, $\Psi(f)(x) = \sup\{t \in \mathbb{R} : x \in \kappa(f^{-1}(t,\infty))\}$. As above we can check that Ψ is an extender for the lower semi-continuous functions satisfying (b).

Finally, if $f \in C^*(F)$ then,

$$\begin{aligned}&\Phi(f)^{-1}(-\infty,t) \cap \Psi(f)^{-1}(t,\infty)\\ &\qquad = \bigcup\left\{\kappa(f^{-1}(-\infty,r)) \cap \kappa(f^{-1}(s,\infty)) : r < t < s\right\} = \emptyset\end{aligned}$$

by 1.4 (iii), and hence, $\Psi(f) \leq \Phi(f)$.

As for the converse, if F is a closed subspace of X and U is open in F, define,

$$\phi(U) = \bigcup \left\{ \Phi(f)^{-1}(-\infty, 0) : f \in C(F, [-1, 2]) \text{ and } f(F \setminus U) \subseteq \{2\} \right\}$$
$$\psi(U) = \bigcup \left\{ \Psi(f)^{-1}(0, \infty) : f \in C(F, [-2, 1]) \text{ and } f(F \setminus U) \subseteq \{-2\} \right\}.$$

Now define $\kappa : \tau F \to \tau X$ by $\kappa(U) = \phi(U) \cap \psi(U)$. It is straightforward to check that, $\kappa(U) \cap F = U$ (by complete regularity), that $\kappa(\emptyset) = \emptyset$ (by condition (c)) and that $\kappa(U) \subseteq \kappa(V)$ for $U \subseteq V$. It remains to check that $\kappa(U \cap V) = \kappa(U) \cap \kappa(V)$. It is sufficient to prove that $\kappa(U \cap V) \supseteq \kappa(U) \cap \kappa(V)$. Take $x \in \kappa(U) \cap \kappa(V)$. This implies that there exist continuous $f_U, f_V : F \to [-1, 2]$ and $g_U, g_V : F \to [-2, 1]$ such that,

$$\begin{array}{llll} f_U(F \setminus U) \subseteq \{2\} & f_V(F \setminus V) \subseteq \{2\} & g_U(F \setminus U) \subseteq \{-2\} & g_V(F \setminus V) \subseteq \{-2\} \\ \Phi(f_U)(x) < 0 & \Phi(f_V)(x) < 0 & \Psi(g_U)(x) > 0 & \Psi(g_V)(x) > 0. \end{array}$$

Let $f = \max(f_U, f_V)$ and $g = \min(g_U, g_V)$. Therefore $f : F \to [-1, 2]$ and $g : F \to [-2, 1]$ are both continuous, $f(F \setminus (U \cap V)) \subseteq \{2\}$ and $g(F \setminus (U \cap V)) \subseteq \{-2\}$. $\Phi(f)(x) = \Phi(\max(f_U, f_V))(x) = \max(\Phi(f_U)(x), \Phi(f_V)(x)) < 0$. Similarly $\Psi(g)(x) > 0$. Hence, by definition, $x \in \kappa(U \cap V)$. $\square$

In light of Theorem 1.5, it seems reasonable to expect to be able to strengthen van Douwen's Theorem to characterise monotonically normal spaces. We have the following result, the proof of which requires only slight modification of the proofs of Theorems 4.2 and 2.3.

Theorem 4.3. *A space X is monotonically normal if and only if for every non-empty closed subspace F of X there are extenders $\Phi_F : C^*_{usc}(F) \to C^*_{usc}(X)$ and $\Psi_F : C^*_{lsc}(F) \to C^*_{lsc}(X)$ such that*

(1) *Φ_F and Ψ_F are both monotone and for all $f \in C^*(F)$, $\Psi_F(f) \leq \Phi_F(f)$,*
(2) *if $Y_1 \subseteq Y_0$ are closed subpaces of X and $f_i \in C^*_{usc}(Y_i)$ $(i = 0, 1)$ are such that $f_0 \restriction_{Y_1} \geq f_1$ and $f_0(x) \geq \sup f_1$ for all $x \in Y_0 \setminus Y_1$ then $\Phi_{Y_0}(f_0) \geq \Phi_{Y_1}(f_1)$,*
(3) *if $Y_1 \subseteq Y_0$ are closed subpaces of X and $f_i \in C^*_{lsc}(Y_i)$ $(i = 0, 1)$ are such that $f_0 \restriction_{Y_1} \leq f_1$ and $f_0(x) \leq \inf f_1$ for all $x \in Y_0 \setminus Y_1$ then $\Psi_{Y_0}(f_0) \leq \Psi_{Y_1}(f_1)$.*

We end this section with an answer to another question of San-ou. In [9], San-ou asked whether one could replace continuity by semi-continuity in the analogue of Urysohn's Lemma for monotonically normal spaces. By a semi-continuous set of monotone Urysohn functions for X we will mean a collection $\{f_{A,U} : A$ closed, U open and $A \subseteq U\}$ of lower semi-continuous functions, $f_{A,U} : X \to \mathbb{R}$ such that $f_{A,U}(x) = 1$ if $x \in A$ and $f_{A,U}(x) = 0$ if $x \notin U$ and such that, if $A \subseteq B$ and $U \subseteq V$, then $f_{A,U} \leq f_{B,V}$. San-ou's question therefore becomes: if X has a semi-continuous set of monotone Urysohn functions, then is X monotonically normal? We shall answer this question firmly in the negative by showing that ALL spaces have a set of semi-continuous monotone Urysohn functions. We first prove the following:

Proposition 4.4. *If X is a space and, for each non-empty closed subspace F of X, there exists a function $\kappa_F : \tau F \to \tau X$ such that,*

(1) $\kappa_F(U) \cap F = U$ *for all* $U \in \tau F$,
(2) $\kappa_F(U) \subseteq \kappa_F(V)$ *whenever* $U \subseteq V$,
(3) $\kappa_F(\emptyset) = \emptyset$ *and* $\kappa_F(F) = X$,
(4) *if* $U_i \in \tau F_i$ *for* $i = 0, 1$ *are such that* $U_0 \subseteq U_1$ *and* $F_1 \setminus U_1 \subseteq F_0 \setminus U_0$, *then* $\kappa_{F_0}(U_0) \subseteq \kappa_{F_1}(U_1)$,

then X has a semi-continuous set of monotone Urysohn functions.

Proof: We note that κ_F satisfies every condition in Theorem 1.5 except the condition: (iii) $\kappa_F(U) \cap \kappa_F(V) = \emptyset$ whenever $U \cap V = \emptyset$.

Construct the extender $\Psi_F : C^*_{lsc}(F) \to C^*_{lsc}(X)$ as in Theorem 4.2. The proof that Ψ_F is an extender does not depend on condition (iii). We now define our semi-continuous set of monotone Urysohn functions.

For A closed, U open and $A \subseteq U$ define $g_{A,U} : A \cup (X \setminus U) \to \mathbb{R}$, by $g_{A,U}(x) = 1$ if $x \in A$ and $g_{A,U}(x) = 0$ if $x \notin U$. Let $f_{A,U} = \Psi_{A\cup(X\setminus U)}(g_{A,U})$. Clearly, each $f_{A,U}$ is lower semi-continuous and $f_{A,U}(x) = 1$ if $x \in A$ and $f_{A,U}(x) = 0$ if $x \notin U$.

Assume that $A \subseteq B$ and $U \subseteq V$. It is clear that,

$$g^{-1}_{A,U}(t,\infty) = \begin{cases} A \cup (X \setminus U) & \text{if } t < 0 \\ A & \text{if } 0 \le t < 1 \\ \emptyset & \text{if } t \ge 1. \end{cases}$$

Similar statements hold for $g_{B,V}$, with the inverse image being $B \cup (X \setminus V)$, B or $\emptyset$.

Since $A \subseteq B$ and $(B \cup (X \setminus V)) \setminus B \subseteq (A \cup (X \setminus U)) \setminus A$, then $\kappa_{A\cup(X\setminus U)}(A) \subseteq \kappa_{B\cup(X\setminus V)}(B)$ (by (4)) and hence, for all $t \in \mathbb{R}$,

$$\kappa_{A\cup(X\setminus U)}(g^{-1}_{A,U}(t,\infty)) \subseteq \kappa_{B\cup(X\setminus V)}(g^{-1}_{B,V}(t,\infty)).$$

Consequently, by definition of Ψ_F, $f_{A,U} \le f_{B,V}$. The proof is therefore complete. □

To complete our claim, we now show that EVERY space X has functions κ_F satisfying the conditions of the above proposition.

If F is a non-empty closed subspace of any space X, then define $\kappa_F : \tau F \to \tau X$ by,

$$\kappa_F(U) = \begin{cases} X \setminus (F \setminus U) & \text{if } U \ne \emptyset \\ \emptyset & \text{if } U = \emptyset. \end{cases}$$

The reader may easily verify that these κ_F do, indeed, satisfy the required conditions.

5. Acyclic monotone normality

In [8], a stronger concept than monotone normality, called acyclic monotone normality, was introduced. This property is defined as follows.

Definition 5.1. *A space X is said to be acyclic monotonically normal if there is a function H which assigns to each pair (x, U), where U is an open set containing x, an open set $H(x, U)$ such that $x \in H(x, U) \subseteq U$ and,*

(a) *if $x \in U \subseteq V$, with U and V open, then $H(x, U) \subseteq H(x, V)$,*
(b) *$H(x, X \setminus \{y\}) \cap H(y, X \setminus \{x\}) = \emptyset$ for $x \neq y$,*
(c) *if $x_0 \dots x_{n-1}$ are distinct points of X and $x_n = x_0$, then $\bigcap_{i=0}^{n-1} H(x_i, X \setminus \{x_{i+1}\}) = \emptyset$.*

It is clear from conditions (a) and (b) that acyclic monotonically normal spaces are monotonically normal. Also, condition (b) follows from condition (c).

We have the following analogue of Urysohn's Lemma for acyclic monotonically normal spaces. The proof is a straightforward alteration of the proof of Lemma 2.1 using Lemma 2.2 and can be found in [11].

Theorem 5.2. *A space X is acyclic monotonically normal if and only if for all pairs (x, U), where U is an open subset of X containing x, we can assign a continuous function $g_{x,U} : X \to [0, 1]$ such that $g_{x,U}(x) = 0$ and $g_{x,U}(y) = 1$ for $y \notin U$ and such that,*

(i) *if $x \in U \subseteq V$, then $g_{x,V} \leq g_{x,U}$,*
(ii) *$g_{x,X\setminus\{y\}} = 1 - g_{y,X\setminus\{x\}}$ for all $x \neq y$,*
(iii) *if $x_0 \dots x_{n-1}$ are distinct points of X and $x_n = x_0$ then, for all $y \in X$, there is an $i \leq n-1$ such that $g_{x_i,X\setminus\{x_{i+1}\}}(y) \geq \frac{1}{4}$.*

We end with a question. Theorem 4.3 followed from the fact that monotone normality can be characterised as in Theorem 1.5. Since acyclic monotonically normal spaces can be similarly characterised by adding the condition (MK) to K_0-spaces [8], can we also characterise acyclic monotone normality in terms of the extension of semi-continuous functions?

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