On the equivalence of algebraic and geometric local cohomology

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Abstract. In this paper, we will present several necessary and sufficient conditions on a commutative ring such that the algebraic and geometric local cohomologies are equivalent.

Keywords: local cohomology, commutative ring, quasi-flabby, sheaf, localization

Classification: 13D45, 13C10, 13B30

1. Introduction

As is well known, one of the main features of local cohomology, permitting to calculate it effectively in practice, is the equivalence of its geometric and algebraic definitions (see Grothendieck [2], or Hartshorne [1, p. 217, 3.3 (b)]).

More precisely, let R be a commutative noetherian ring and Z a closed subset of $\operatorname{Spec} R$, say $Z = \operatorname{Spec} R - D(K)$ with K an ideal. Then for each R-module M, there are the following isomorphisms:

$$(*) \qquad H^n_Z(\tilde{M}) \cong \widetilde{H^n_K M},$$

and

$$(**) \qquad H^n_Z(X, \tilde{M}) \cong H^n_K M$$

where $H^n_Z = R^n\Gamma_Z$ and $H^n_Z(X, -) = R^n\Gamma_Z(X, -)$ are the right derived functors of the support functors Γ_Z and $\Gamma_Z(X, -)$ with respect to Z, respectively, and H^n_K is the right derived functor of the torsion functor τ_K determined by K (i.e. $\tau_K M = \{m \in M \mid K^n m = 0$ for some natural numbers $n\}$).

In this paper, we will consider general commutative (not necessarily noetherian) rings, and give several necessary and sufficient conditions on a commutative ring R such that $(*)$ or $(**)$ holds (see Theorem 8 below), which shows that the noetherian assumption in the above classical results is not necessary.

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2. Preliminaries

In this paper, a ring means a commutative ring with an identity. We denote by $\operatorname{Spec} R$ the set of all prime ideals of R, endowed with the Zariski topology whose typical open subsets are of the form $D(I) = \{P \in \operatorname{Spec} R \,|\, I \nsubseteq P\}$ with an ideal I of R.

For each ideal I, the intersection of all prime ideals containing I is called *the radical* of I, and will be denoted $\sqrt{I}$.

For each ideal I of R, define $T(I)$ *to be the filter of ideals consisting of those ideals* J *whose radical* $\sqrt{I} \supseteq I$. For each R-module M, define

$$\tau_I M = \{m \in M \,|\, Jm = 0 \ \text{ for some } \ J \in T(I)\}.$$

Then τ_I is a left exact subfunctor of the identity functor on R-Mod. Note that if I is finitely generated, then $T(I)$ has a cofinal base consisting of $\{I^n \,|\, n \leq \omega\}$, and hence τ_I coincides with the usual one (e.g., the one mentioned in the introduction).

For any subset Z of $\operatorname{Spec} R$, each sheaf F of $\tilde{R}$-modules and each open subset U of $\operatorname{Spec} R$, let

$$\Gamma_Z F(U) = \{s \in F(U) \,|\, s_P = 0 \ \text{ for all } \ P \in U \setminus Z\}$$

(the support of $F(U)$). Then Γ_Z defines a left exact endofunctor on $\widetilde{R}$-Mod, the category of all sheaves of $\tilde{R}$-modules.

Let $H^n_Z = R^n\Gamma_Z$ *and* $H^n_Z(\operatorname{Spec} R, -) = R^n\Gamma_Z(\operatorname{Spec} R, -)$ *be the right derived functors of* Γ_Z *and* $\Gamma_Z(\operatorname{Spec} R, -)$ *respectively.*

For any $Y \in \operatorname{Spec} R$, let $F(U \cap Y) = \operatorname{colim}_{V \supseteq U \cap Y} F(V)$ and for each $s \in F(U)$ let $s\,|\,U \cap Y$ denote the image of s under the canonical morphism.

Lemma 0. *For any* $F \in \widetilde{R}$-Mod, *we have*

$$(\Gamma_Z F)(U) = \{s \in F(U) \,|\, s\,|\,U \cap Y = 0\}.$$

Proof: If $s \in F(U)$ with $s\,|\,U \cap Y = 0$, then there is an open set $V \supseteq U \cap Y$ such that $V \subseteq U$ and $s\,|\,V = 0$. Thus $s_x = 0$ for each $x \in V$, in particular, for each $x \in U \cap Y$.

On the other hand, if $s \in F(U)$ with $s_x = 0$ for each $x \in U \cap Y$, then there are open subsets $V_x \subseteq U$ with $x \in V_x$ and $F^U_{V_X}(s) = 0$, and hence $F^U_V(s) = 0$, where $V = \bigcup_{x \in U \cap Y} V_x$, since F is separated. Now $s\,|\,U \cap Y = 0$ follows from the fact that $V \supseteq U \cap Y$. $\square$

For any subset Y of $\operatorname{Spec} R$, we may also define a left exact endofunctor τ_Y on R-Mod by letting $\tau_Y M = \{m \in M \,|\, (\exists \text{ an ideal } J)(Y \subseteq D(J))(Jm = 0)\}$.

Let H^n_Y be the n-th right derived functor of τ_Y.

3. Main results

We shall first give the following lemma, from which we will derive that in the corresponding result in [1, Ex. 5.6 (d), p. 124] the noetherian hypothesis can be omitted.

Lemma 1. *Let R be any commutative ring and M an R-module, and $Z = \operatorname{Spec} R - Y$ for an arbitrary subset Y. Then there is a natural monomorphism*

$$\phi_M : \tau_Y M \longrightarrow \Gamma_Z \tilde{M}.$$

Proof: It suffices to show that it is true for those components at $D(a)$ with $a \in R$. First note that there is a canonical morphism $\phi_M : \widetilde{\tau_Y M} \to \tilde{M}$, whose component at $D(a)$ sends $r/^n \in (\tau_Y M)_a$ to the element in $r/a^n \in M_a$.
This morphism is natural in $M \in R$-Mod: If $f : M \to N$ is an R-linear morphism, then $\tilde{f} : \tilde{M} \to \tilde{N}$ is defined by $\tilde{f}(D(a)) : M_a \to N_a$, which sends m/a^n to $f(m)/a^n$. On the other hand, the restriction $f\,|\,\tau_Y M$ factorizes through $\tau_Y N$. Write $\tau_Y(f)$ for the morphism $f\,|\,\tau_Y M$, it is an R-linear morphism from $\tau_Y M$ to $\tau_Y N$. Thus for each $D(a)$, we have a morphism $\widetilde{\tau(f)}(D(a)) : (\tau_Y M)_a \to (\tau_Y N)_a$ which sends $m/a^n \in (\tau_Y M)_a$ to $f(m)/a^n \in (\tau_Y N)_a$, and hence we obtain a morphism $\widetilde{\tau(f)} : \widetilde{\tau_Y M} \to \widetilde{\tau_Y N}$ such that the following diagram commutes

$$\begin{array}{ccc} \widetilde{\tau_Y M} & \xrightarrow{\phi_M} & \tilde{M} \\ {\scriptstyle \widetilde{\tau_Y f}}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{f}} \\ \widetilde{\tau_Y N} & \xrightarrow[\phi_N\,,]{} & \tilde{N} \end{array}$$

since for each $a \in R$, the following diagram commutes:

$$\begin{array}{ccc} (\tau_Y M)_a & \xrightarrow{(\phi_M)(D(a))} & M_a \\ {\scriptstyle (\widetilde{\tau_Y f})D(A)}\big\downarrow & & \big\downarrow{\scriptstyle (\tilde{f})D(A)} \\ (\tau_Y N)_a & \xrightarrow[(\phi_N)(D(A))]{} & N_a\,. \end{array}$$

Next we want further to show that ϕ_M factorizes through $\Gamma_Z \tilde{M}$, or equivalently, $\phi_M(D(a))((\tau_Y M)_a) \subseteq \Gamma_Z(D(a), \tilde{M})$.

To prove this, note that $m/a^n \in \Gamma_Z(D(a), \tilde{M})$ if and only if $m/a^n = 0$ in M_P for each $P \in D(a) \cap Y$.

Suppose $m/a^n \in (\tau_Y M)_a$ with $m \in \tau_Y M$, then there exists an ideal J with $Y \subseteq D(J)$ such that $Jm = 0$. Now for each $P \in Y$, we have $J \not\subseteq P$, and hence may find $y \in J \setminus P$ such that $ym \in Jm = \{0\}$. That is, m/a^n is zero in M_P. Since it holds for each $P \in D(a) \cap Y$, we have $m/a^n \in \Gamma_Z(D(a), \tilde{M})$. That is to say, $\phi_M D(a)$ factorizes through $\Gamma_Z \tilde{M}(D(a))$. It is clear that each $\phi_M(D(a))$ is injective, and hence so is ϕ_M. $\square$

Corollary 1. *If Y is an open subset $D(I)$ with a finitely generated ideal I of R, then ϕ_M is an isomorphism.*

PROOF: It remains to show that each $\phi_M(D(a))$ is surjective: Let $m/a^n \in \Gamma(D(A), \tilde{M})$ such that $m/a^n = 0$ in M_P for each $P \in D(a) \cap D(I)$. Then there exists an $s_P \notin P$ such that $s_P m = 0$. Let $J = \sum_{P \in Y} Rs_P$. Then $D(aI) = D(a) \cap D(I) \subseteq D(J)$ and $Jm = 0$. Thus, $aI \subseteq \sqrt{J}$ and hence there is a natural k such that $I^k a^k = (aI)^k \subseteq J$ which implies that $I^k a^k m = 0$ and hence $a^k m \in \tau_I M$. The conclusion follows from the fact that $m/a^n = a^k m/a^{n+k}$ in M_a. □

Corollary 2. *If R is noetherian, then each ϕ_M is an isomorphism.* □

Now we would like to introduce the following notion: Let $X = \operatorname{Spec} R$.

Definition 3. A sheaf F is called *quasi-flabby* if for each quasi-compact open subset U of X, the restriction map F_U^X is surjective.

It is clear that each flabby sheaf is quasi-flabby, in particular, each injective sheaf is quasi-flabby.

Lemma 4. *If F is quasi-flabby, then $H^1(U, F) = 0$ for each quasi-compact open subset U of X.*

PROOF: To show $H^1(U, F) = 0$, it suffices to show that for any exact sequence of sheaves

$$0 \longrightarrow F \longrightarrow E \xrightarrow{\beta} G \longrightarrow 0,$$

the following sequence

$$\Gamma(U, E) \xrightarrow{\beta_U} \Gamma(U, G) \longrightarrow 0,$$

is exact. Let $s \in \Gamma(U, G)$ and $\mathcal{S} = \{(V, t) \,|\, t \in E(V), V \subseteq U, \beta_V(t) = s\,|\,V\}$. Then $\bigcup\{V \,|\, (V, t) \in \mathcal{S}\} = U$ since β is epic.

Since U is quasi-compact, it suffices to show that if (V_1, t_1) and (V_2, t_2) are two members of $\mathcal{S}$, then there is a $t \in E(V_1 \cup V_2)$ such that $(V1 \cup V_2, t) \in \mathcal{S}$. In fact, $t_1\,|\,V_1 \cap V_2 - t_2\,|\,V_1 \cap V_2 \in F(V_1 \cap V_2)$ since $\beta_{V_1 \cap V_2}(t_1\,|\,V_1 \cap V_2 - t_2\,|\,V_1 \cap V_2) = 0$. Since F is quasi-flabby and $V_1 \cap V_2$ is quasi-compact open, there exists $t' \in F(V_1) \subseteq E(V_1)$ such that $F^{V_1}_{V_1 \cap V_2}(t') = t_1\,|\,V_1 \cap V_2 - t_2\,|\,V_1 \cap V_2$. Now let $t_1' = t_1 - t'$. Then $t_1'\,|\,V_1 \cap V_2 = t_2\,|\,V_1 \cap V_2$ and $\beta_{V_1}(t_1') = s\,|\,V_1$. Since E is a sheaf, we may patch t_1' and t_2 together to get a section $t \in E(V_1 \cup V_2)$, whose image under $\beta_{V_1 \cup V_2}$ is $s\,|\,V_1 \cup V_2$ since G is also a sheaf. □

Lemma 5. *If $0 \to F \to E \to G \to 0$ is an exact sequence of sheaves and F and E are quasi-flabby, then so is G.*

PROOF: By Lemma 4, for each quasi-compact open subset U of $\operatorname{Spec} R$, we have the following commutative exact diagrams:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma(X,F) & \longrightarrow & \Gamma(X,E) & \longrightarrow & \Gamma(X,G) & \longrightarrow & 0 \\
 & & {\scriptstyle F_U^X}\downarrow & & {\scriptstyle E_U^X}\downarrow & & {\scriptstyle G_U^X}\downarrow & & \\
0 & \longrightarrow & \Gamma(U,F) & \longrightarrow & \Gamma(U,E) & \longrightarrow & \Gamma(U,G) & \longrightarrow & 0.
\end{array}$$

Now the surjectivity of G_U^X follows from the fact that both morphisms E_U^X and $\Gamma(U,E) \to \Gamma(U,G)$ are surjective. $\square$

Lemma 6. *Let $Z = \operatorname{Spec} R - U$ with U quasi-compact open, and let F be quasi-flabby. Then $H^1_Z(X,F) = 0 = H^1_Z F$.*

PROOF: Consider the short exact sequence $0 \to F \to E \to G \to 0$ of sheaves, where E is an injective sheaf. To show $H^1_Z(F) = 0$ is to show the induced map $\Gamma_Z(E) \to \Gamma_Z(G)$ is epic. It suffices to show that each induced morphism $\Gamma_Z(V,E) \to \Gamma_Z(V,G)$ is surjective, for each quasi-compact open subset V. In fact, it follows from the following diagram,

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma_Z(V,F) & \longrightarrow & \Gamma_Z(V,E) & \longrightarrow & \Gamma_Z(V,G) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma(V,F) & \longrightarrow & \Gamma(V,E) & \longrightarrow & \Gamma(V,G) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma(U\cap V,F) & \longrightarrow & \Gamma(U\cap V,E) & \longrightarrow & \Gamma(U\cap V,G) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

since the vertical sequences are exact by Lemma 5 and the row sequences are exact by Lemma 4. $\square$

Lemma 7. *Let Z be a complement of a quasi-compact open subset of* $\operatorname{Spec} R$, *and F quasi-flabby. Then $H^n_Z(V,F) = 0 = H^n_Z F$ for any $n \geq 1$ and any open V.*

PROOF: Consider the short exact sequence $0 \to F \to E \to G \to 0$ of sheaves, where E is an injective sheaf. By the long exact sequence of cohomology of the short sequence above, we have $H^{n+1}_Z(V,F) \cong H^n_Z(V,G)$ for $n \geq 1$. The conclusion follows, by induction, from Lemma 5 and Lemma 6. $\square$

Theorem 8. *For a commutative ring R, the following are equivalent:*

(1) *The associated sheaf $\tilde{E}$ is quasi-flabby for each injective R-module E;*

(2) *$H^n_Z(\tilde{E}) = 0$, for all injective E, $n \geq 1$ and for all complement Z of quasi-compact open subsets;*

(3) *$H^n_Z(\tilde{M}) \cong \widetilde{H^n_K M}$, for all R-modules M, $n \geq 1$ and $Z = \operatorname{Spec} R - D(K)$ with K finitely generated;*

(4) *$H^n_Z(\operatorname{Spec} R, \tilde{M}) \cong H^n_K M$, for all M, $n \geq 1$ and $Z = \operatorname{Spec} R - D(K)$ with K finitely generated;*

(5) *$H^1_Z(\operatorname{Spec} R, \tilde{E}) = 0$, for all injective E and all complements Z of quasi-compact open subsets of $\operatorname{Spec} R$;*

Proof: (1) $\Rightarrow$ (2) follows from Lemma 7.

(2) $\Rightarrow$ (3) Consider the following exact sequence

$$0 \longrightarrow M \longrightarrow E \longrightarrow E/M \longrightarrow 0,$$

where E is an injective hull of M; which induces the following exact sequence

$$0 \longrightarrow \tilde{M} \longrightarrow \tilde{E} \longrightarrow \widetilde{E/M} \longrightarrow 0,$$

since the structure sheaf functor is exact.

Therefore we have another exact sequence:

$$0 \longrightarrow \Gamma_Z \tilde{M} \longrightarrow \Gamma_Z \tilde{E} \longrightarrow \Gamma_Z \widetilde{E/M} \longrightarrow H^1_Z \tilde{M} \longrightarrow H^1_Z \tilde{E} = 0,$$

and $H^{n+1}_Z \tilde{M} \cong H^n_Z \widetilde{E/M}$ for $n \geq 1$ by (2). In particular, $H^1_Z \tilde{M}$ is the cokernel of $\Gamma_Z \tilde{E} \to \Gamma_Z \widetilde{E/M}$.

On the other hand,

$$0 \longrightarrow M \longrightarrow E \longrightarrow E/M \longrightarrow 0$$

also induces another exact sequence

$$0 \longrightarrow \tau_K M \longrightarrow \tau_K E \longrightarrow \tau_K E/M \longrightarrow H^1_K M \longrightarrow H^1_K E = 0$$

and hence induces the following exact sequence

$$0 \longrightarrow \widetilde{\tau_K M} \longrightarrow \widetilde{\tau_K E} \longrightarrow \widetilde{\tau_K E/M} \longrightarrow \widetilde{H^1_K M} \longrightarrow \widetilde{H^1_K E} = 0,$$
$$H^{n+1}_K \tilde{M} \simeq H^n_K \widetilde{E/M}$$

for all $n \geq 1$.

By Corollary 1, we have the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \Gamma_Z \tilde{M} \arrow[r] & \Gamma_Z \tilde{E} \arrow[r] & \Gamma_Z \widetilde{E/M} \arrow[r] & H^1_Z \tilde{M} \arrow[r] & 0 \\ 0 \arrow[r] & \widetilde{\tau_K M} \arrow[r] \arrow[u, "\phi_M"] & \widetilde{\tau_K E} \arrow[r] \arrow[u, "\phi_E"] & \widetilde{\tau_K E/M} \arrow[r] \arrow[u, "\phi_{E/M}"'] & \widetilde{H^1_K M} \arrow[r] & 0 \end{tikzcd}$$

so that $H^1_Z \tilde{M} \simeq \widetilde{H^1_K M}$.

Now the conclusion follows from the fact that

$$H^{n+1}_Z \tilde{M} \simeq H^n_Z \widetilde{E/M}$$
$$H^{n+1}_K \tilde{M} \simeq H^n_K \widetilde{E/M}$$

for all $n \geq 1$.

(3) $\Rightarrow$ (2) is obvious since the right hand side is zero when E is injective.

The proof of (2) $\Rightarrow$ (4) is similar to that (2) $\Rightarrow$ (3) (or by using the result of Grothendieck that $H^n_Z(\tilde{M}) \cong H^n_Z(\widetilde{\operatorname{Spec} R}, \tilde{M})$, see [2]).

(4) $\Rightarrow$ (5) is trivial.

(5) $\Rightarrow$ (1) Consider the exact sequence of functors, where K is a finitely generated ideal:

$$0 \longrightarrow \Gamma_Z(X, -) \longrightarrow \Gamma(X, -) \longrightarrow \Gamma(D(K), -),$$

which is exact on injective sheaves. It induces the following exact sequence

$$\begin{aligned} 0 &\longrightarrow \Gamma_Z(X, \tilde{E}) \longrightarrow \Gamma(X, \tilde{E}) \longrightarrow \Gamma(D(K), \tilde{E}) \\ &\longrightarrow H^1_Z(X, \tilde{E}) \longrightarrow H^1(X, \tilde{E}) \longrightarrow H^1(D(K), \tilde{E}). \end{aligned}$$

By (5), $H^1_Z(X, \tilde{E}) = 0$, so we have the surjective map

$$\Gamma(X, \tilde{E}) \longrightarrow \Gamma(D(K), \tilde{E}).$$

That is to say, E is quasi-flabby. $\square$

Example 9. The following example is due to Hartshorne ([1, p. 218, Example 3.8]), which shows that not all commutative ring satisfies (1) in Theorem 8.

Let $R = k[x_0, x_1, x_2 \dots]$ with the relations $x_0^n x_n = 0$ for $n = 1, 2 \dots$. Let E be an injective R-module containing R. Then $E \to E_{x_0}$ is not surjective.

Example 10. If R is a commutative von Neumann regular ring, then for each injective R-module E, $\tilde{E}$ is quasi-flabby.

Note that there are many examples which are von Neumann regular rings but not noetherian (e.g., a product of infinite copies of a field).

It is almost immediate that X in equivalence $(**)$ may be replaced by any basic open subset $D(a)$ with $a \in R$, by the fact that $H^n_Z(D(a), \tilde{E}) \cong H^n_Z(X, \tilde{E}_a)$ since the functor $(-)_a$ is exact.

Proposition 11. *For a commutative ring R, R satisfies* (1) *in Theorem* 8 *iff* $H^n_{Z\cap D(a)}(D(a), \tilde{M}) \cong H^n_K M_a$ *for each basic open subset $D(a)$ of* $\operatorname{Spec} R$ *and for each R-module M and for any $f \cdot g \cdot K$ with $Z = \operatorname{Spec} R \setminus K$.*

Lastly, we would like to ask one question: Is the quasi-flabbiness equivalent to the weaker one that all restriction maps to basic open subset are surjective? We do not know the answer, but we will give some information instead.

Lemma 12. *Let R be a commutative ring and I, J be two finitely generated ideals of R. Then for each R-module M, $\tilde{M}(D(IJ)) \cong \widetilde{\tilde{M}(D(I))}(D(J))$, in particular, $\tilde{M}(D(Ia)) \cong (\tilde{M}(D(I)))_a$, where $a \in R$.*

Proof: Let $I = \sum_i^n Rb_i$. Note that $(Mb_i)_a \cong M_{b_i a}$ and consider the equalizer diagram

$$\tilde{M}(D(I)) \to \Pi_{i\le n} M_{b_i} \rightrightarrows \Pi_{i\le n}\Pi_{j\le n} M_{b_i b_j}\,.$$

We have the following equalizer diagram

$$(\tilde{M}(D(I)))_a \to \Pi_{i\le n} M_{b_i a} \rightrightarrows \Pi_{i\le n}\Pi_{j\le n} M_{b_i b_j a}\,.$$

On the other hand, we also have the following equalizer diagram

$$\tilde{M}(D(Ia)) \to \Pi_{i\le n} M_{b_i a} \rightrightarrows \Pi_{i\le n}\Pi_{j\le n} M_{b_i b_j a}\,.$$

Thus $\tilde{M}(D(Ia)) \cong (\tilde{M}(D(I)))_a$. The conclusion follows by using a similar proof once more. $\square$

Theorem 13. *If a quasi-coherent sheaf F satisfies that all restriction maps to basic open subsets are surjective, then $H^n(U, F) = 0$ for each quasi-compact open subset U, $n \ge 1$; and $H^n_Z(X, F) = 0$ for all $n \ge 2$, where Z is a complement of a quasi-compact open subset of* $\operatorname{Spec} R$.

Proof: If $0 \to F \to E \to G \to 0$ is an exact sequence of sheaves, and U is a quasi-compact open subset of $\operatorname{Spec} R$, we want to prove that $\Gamma(U, E) \to \Gamma(U, G)$ is surjective.

For each $s \in \Gamma(U, G)$, since $E \to G \to 0$ is exact, there exists a basic open cover $D(a_i)$ of U with the property that there is a $t_i \in E(D(a_i))$ such that the image of t_i is $s\,|\,D(a_i)$ for all i. Since U is quasi-compact, we may assume

$U = \bigcup_{i \le n} D(a_i)$. Let $I_1 = Ra_1$ and $I_l = \sum_i^l Ra_i$ for each $l \le n$. We claim that for each I_l there exists $u_l \in E(D(I_l))$ such that the image of u_l is $s\,|\,D(I_l)$. For $l = 1$ it is true. Assume that it is true for l, we want to show that it is true for $l+1$. Now the image of $u_l\,|\,D(I_l a_{l+1}) - t_{l+1}\,|\,D(I_l a_{l+1})$ is $s\,|\,D(I_l a_{l+1}) - s\,|\,D(I_l a_{l+1}) = 0$, and hence it is in $F(D(I_l a_{l+1}))$. However, by Lemma 12, we see $F(D(I_l a_{l+1})) \cong (F(D(I_l)))_{a_{l+1}}$. Now by assumption, there exists $w \in FD(I_l)$ such that $w\,|\,D(I_l a_{l+1}) = u_l\,|\,D(I_l a_{l+1}) - t_{l+1}\,|\,D(I_l a_{l+1})$. Let $u_l' = u_l - w \in E(D(I_l))$. Then $u_l'\,|\,D(I_l a_{l+1}) = t_{l+1}\,|\,D(I_l a_{l+1})$. Thus there exists an extension $u_{l+1} \in E(D(I_{l+1}))$ of u_l' and t_{l+1}. Note that the image of u_l' is also $s\,|\,D(I_l)$, so that the image of u_{l+1} is $s\,|\,D(I_{l+1})$. This completes the induction.

Thus we have shown that $H^1(U, F) = 0$. Now observe that if E is flabby, then G is quasi-flabby and $H_Z^n(X, G) \cong H_Z^{n+1}(X, F)$ for all $n \ge 1$. We finally have $H^n(U, F) = 0$ for all $n \ge 1$ and $H_Z^n(X, F) = 0$ for all $n \ge 2$ and all Z, where Z's are complements of quasi-compact open subsets of $\operatorname{Spec} R$.

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