Hausdorff topology and uniform convergence topology in spaces of continuous functions

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Abstract. The local coincidence of the Hausdorff topology and the uniform convergence topology on the hyperspace consisting of closed graphs of multivalued (or continuous) functions is related to the existence of continuous functions which fail to be uniformly continuous. The problem of the local coincidence of these topologies on $C(X,Y)$ is investigated for some classes of spaces: topological groups, zero-dimensional spaces, metric manifolds.

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Let P be a completely regular Hausdorff space and $H(P)$ be the hyperspace of all non-empty closed subsets of P. For any pseudometric d compatible with the topology of P, we recall that the Hausdorff pseudometric $\tilde{d}$ on $H(P)$ is defined as follows:

$$\tilde{d}(A,B) = (\sup\{d(a,B) :\ a \in A\}) \vee (\sup\{d(b,A) :\ b \in B\}),$$

for every $A, B \in H(P)$. We denote by A_ε the ε-neighborhood of A, that is the set $\{x \in P :\ d(x,A) < \varepsilon\}$. Then we have:

$$\tilde{d}(A,B) = \inf\{\varepsilon > 0 :\ A \subseteq B_\varepsilon, B \subseteq A_\varepsilon\}.$$

If P is a uniform space, the Hausdorff uniformity on $H(P)$ can be defined as the weak uniformity of the family of the pseudometrics $\tilde{d}$, where d ranges over the set of all compatible pseudometrics of P.

Let X and Y be uniform spaces. The space $C(X,Y)$ of continuous functions of X into Y is a subset of $H(X \times Y)$ if we identify every function f with its graph $G(f)$. In case of no ambiguity, we simply write f instead of $G(f)$.

The uniform product $X \times Y$ is equipped with the weak uniformity of the "box" pseudometrics $d_X \times d_Y$, where d_X and d_Y range over all the pseudometrics compatible with the uniformities of X and Y respectively.

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We will denote by $M(X,Y)$ the subspace of $H(X\times Y)$ consisting of the (closed) graphs of upper semi-continuous multivalued functions of X into Y.

Recall that a multivalued function F is upper semi-continuous if $F(x)$ is a closed set for every $x\in X$ and for every open subset V of Y the set $\{x\in X:\ F(x)\subseteq V\}$ is an open subset of X (it is easy to check that the graph of such a function is a closed subset of $X\times Y$). Obviously, $C(X,Y)$ is contained in $M(X,Y)$.

On $M(X,Y)$, as well as on $C(X,Y)$, one may consider the uniform convergence uniformity, which is finer than the Hausdorff uniformity. It may be described as the weak uniformity of the pseudometrics $\hat{d}_Y$, for all the uniformly compatible pseudometrics d_Y on Y, where

$$\hat{d}_Y(F,G)=\sup\{\tilde{d}_Y\big(F(x),G(x)\big):\ x\in X\},$$

for every F, G belonging to $M(X,Y)$. Of course, on $C(X,Y)$ this uniformity coincides with the usual uniform convergence uniformity.

We may therefore consider on $M(X,Y)$ and $C(X,Y)$ the topologies induced by the two above uniformities and investigate the problem of their local or global coincidence. For example, it is easy to prove that the neighborhood systems of a uniformly continuous function coincide in $C(X,Y)$ [4, III, Exercise 10].

In [1] G. Beer introduced the problem of the global coincidence on $C(X,Y)$ for metric spaces, and proved that if the metric space Y contains a non trivial arc, than the above two topologies coincide on $C(X,Y)$ if and only if every continuous function of X into Y is uniformly continuous. Moreover, he proved that metric spaces with such a property are fine, that is, the metric uniformity is the finest one compatible with the topology. The proof of [1, Theorem 1] requires the global coincidence of the two topologies.

It is interesting to tackle the question from a local point of view, by formulating the following problem:

Local problem. *Let X and Y be uniform spaces, and let f be a function of $C(X,Y)$. Is it true that f is uniformly continuous, under the assumption that the neighborhood systems of f in the above two topologies coincide?*

The following example shows that, even in simple metric spaces, the answer is negative.

1. Example. Let $X=[-1,1]\setminus\{0\}$. The real-valued function f defined by $f(x)=\dfrac{x}{|x|}$ is clearly not uniformly continuous, while the ε-neighborhood of f in the Hausdorff topology coincides with the ε-neighborhood of f in the uniform convergence topology, as we now turn to prove.

Assume that $\tilde{\varrho}(g,f)<\varepsilon<\frac{1}{2}$, so that $g\subseteq f_\varepsilon$. If $(x,y)\in f_\varepsilon$, then y cannot be 0 and it is positive whenever $x>\varepsilon$. By the continuity of g, it follows that $g(x)>0$ for every $x>0$. Then we have $1-\varepsilon<g(x)<1+\varepsilon$, for every $x>0$.

Similarly, $-1-\varepsilon<g(x)<-1+\varepsilon$ for every $x<0$, which concludes our proof.

A similar example can be produced for a connected space, by identifying the points -1 and 1 in the above space X, and by modifying the function f near -1 and 1 so as to make it continuous.

Remark. Let X be a subset of $\mathbb{R}$ with a finite number of connected components and let $f : X \mapsto \mathbb{R}$ be a continuous function. It is not hard to prove that the neighborhood systems of f in the two topologies coincide if and only if the restriction of f to every component of X is uniformly continuous.

An answer to the local problem is provided by the next theorem.

2. Theorem. *Let f belong to $C(X,Y)$. The following conditions are equivalent:*

(1) *f is uniformly continuous;*
(2) *in $M(X,Y)$ the neighborhood systems of f in the uniform convergence topology and in the Hausdorff topology coincide.*

Proof: $1 \Rightarrow 2$. Since the uniform convergence topology is stronger than the Hausdorff topology, it is enough to prove that every neighborhood of f in the former topology is a neighborhood in the latter. Let d_Y be a pseudometric and ε be a positive number. An element C of $M(X,Y)$ belongs to the ε-neighborhood of f in the pseudometric $\hat{d}_Y$ if and only if $f(x) \in (C(x))_\varepsilon$ and $C(x) \subseteq \{f(x)\}_\varepsilon$, for every $x \in X$.

Since f is uniformly continuous, there exist a pseudometric d_X and a real number $\delta > 0$ such that $d_Y(f(x), f(z)) < \frac{\varepsilon}{2}$ if $d_X(x,z) < \delta$. Let $\lambda = \min\left\{\frac{\varepsilon}{2}, \delta\right\}$ and $\varrho = d_X \times d_Y$.

We now turn to prove that the λ-neighborhood of f in $\tilde{\varrho}$ is contained in the ε-neighborhood of f in $\hat{d}_Y$. Let C be an element of $M(X,Y)$ such that $\tilde{\varrho}(C,f) < \lambda$. Then, for every $(x, c_x) \in C$, there exists $(z, f(z))$ such that $d_X(x,z) < \lambda \leq \delta$ and $d_Y(c_x, f(z)) < \lambda \leq \frac{\varepsilon}{2}$. Then $d_Y(f(x), f(z)) < \frac{\varepsilon}{2}$ and consequently

$$d_Y(c_x, f(x)) \leq d_Y(c_x, f(z)) + d_Y(f(z), f(x)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.$$

Accordingly, we have $C(x) \subseteq \{f(x)\}_\varepsilon$ and $f(x) \in (C(x))_\varepsilon$, and the proof is complete.

$2 \Rightarrow 1$. Let $\{d_\alpha : \alpha \in A\}$ be the family of all pseudometrics compatible with X (this family is a directed set in the direction of finer pseudometrics). By way of contradiction, assume that f is not uniformly continuous. Then there exists a pseudometric d_Y such that for every pseudometric d_α we can find two points $x_\alpha, y_\alpha \in X$ such that $d_\alpha(x_\alpha, y_\alpha) < 1$ and $d_Y(f(x_\alpha), f(y_\alpha)) > 1$. For every α, let $C_\alpha = \{(x_\alpha, f(y_\alpha))\} \cup G(f)$. It is easy to prove that the sets C_α belong to $M(X,Y)$. The net $(C_\alpha)_{\alpha\in A}$ cannot converge to f in the uniform convergence uniformity, as the following inequality shows:

$$\hat{d}_Y(f, C_\alpha) \geq \tilde{d}_Y\big(\{f(x_\alpha)\}, C_\alpha(x_\alpha)\big) \geq d_Y(f(x_\alpha), f(y_\alpha)) > 1.$$

We now obtain a contradiction by showing that the net $(C_\alpha)_{\alpha\in A}$ converges to f in the Hausdorff topology. A neighborhood base of f in this topology is given by the sets of the form $\{C \in M(X,Y) : \tilde{\varrho}(C,f) < 1\}$, where ϱ ranges over the box pseudometrics of $X \times Y$. Given such a neighborhood, ϱ is of the form $d_\alpha \times d$. Since $C_\beta \setminus G(f) = \{(x_\beta, f(y_\beta))\}$, we have:

$$\begin{aligned}\tilde{\varrho}(C_\beta, f) = \varrho((x_\beta, f(y_\beta), G(f)) \leq \\ \leq \varrho((x_\beta, f(y_\beta), (y_\beta, f(y_\beta)) = d_\alpha(x_\beta, y_\beta) \leq d_\beta(x_\beta, y_\beta) < 1,\end{aligned}$$

for every $\beta \geq \alpha$. Therefore $\tilde{\varrho}(C_\beta, f) < 1$ for every $\beta \geq \alpha$, and the proof is complete. □

Of course, the implication $1 \Rightarrow 2$ of Theorem 2 holds if we consider $C(X,Y)$ instead of $M(X,Y)$. If X is a uniform fine space, we get, as a consequence, that on $C(X,Y)$ the Hausdorff topology coincides with the uniform convergence topology [5], [1].

Remark. An element $F \in M(X,Y)$ is said to be uniformly continuous if for every pseudometric d_Y there exists a pseudometric d_X such that $\tilde{d}_Y(F(x_1), F(x_2)) < 1$ whenever $d_X(x_1, x_2) < 1$ (this means that $F : X \longrightarrow H(Y)$ is uniformly continuous, where $H(Y)$ is equipped with the Hausdorff uniformity).

If the neighborhood systems of an element $F \in M(X,Y)$ in the two topologies coincide, then F is uniformly continuous, as one can easily show by imitating the proof of $2 \Rightarrow 1$ in Theorem 2. Nevertheless, the neighborhood systems of a uniformly continuous element $F \in M(X,Y)$ may differ from each other. For example, consider:

$$F : [0,1] \longrightarrow \mathbb{R}, \quad F(x) = \{0,1\}.$$

The sequence F_n of elements of $M(X,Y)$ defined by:

$$F_n(x) = \begin{cases} \{0,1\} & \text{if } x \geq \frac{1}{n}, \\ \{0\} & \text{if } x < \frac{1}{n}, \end{cases}$$

converges to F in the Hausdorff topology and does not converge in the uniform convergence topology.

The question of finding conditions on X or on Y under which the implication $2 \Rightarrow 1$ of Theorem 2 holds in $C(X,Y)$ is still open. In the following proposition we exhibit two classes of spaces X for which an answer can be given.

3. Proposition. *If $M(X,Y)$ is replaced by $C(X,Y)$ in condition* 2 *of Theorem* 2*, then implication $2 \Rightarrow 1$ holds in the following cases:*

(a) *the space X is a topological group (equipped, for example, with the left uniformity);*
(b) *the topology of X has a base of clopen sets.*

PROOF: We argue by contradiction. As in Theorem 2 take a suitable pseudometric d_Y, and for every α select two points x_α and y_α such that $d_\alpha(x_\alpha, y_\alpha) < 1$ and $d_Y(f(x_\alpha), f(y_\alpha)) > 1$.

In case (a) we can assume that the pseudometrics d_α are left-invariant. Let us use the additive notation. If $\delta_\alpha = -y_\alpha + x_\alpha$, consider the continuous functions defined as follows: $g_\alpha(x) = f(x + \delta_\alpha)$ for every $x \in X$.

We have $g_\alpha(y_\alpha) = f(y_\alpha - y_\alpha + x_\alpha) = f(x_\alpha)$, so that

$$\hat{d}_Y(g_\alpha, f) \geq d_Y\big(g_\alpha(y_\alpha), f(y_\alpha)\big) = d_Y\big(f(x_\alpha), f(y_\alpha)\big) > 1,$$

for every α; hence the net $(g_\alpha)_{\alpha\in A}$ does not converge uniformly to f.

On the other hand, $(x, y) \in G(f) \iff (x - \delta_\alpha, y) \in G(g_\alpha)$. For every box pseudometric $\varrho_\alpha = d_\alpha \times d$ and for every $\beta \geq \alpha$, we have

$$\begin{aligned}\varrho_\alpha\big((x, y), (x - \delta_\beta, y)\big) &= d_\alpha(x, x - \delta_\beta) \leq d_\beta(x, x - x_\beta + y_\beta) = \\ &= d_\beta(x_\beta - x + x, x_\beta - x + x - x_\beta + y_\beta) = d_\beta(x_\beta, y_\beta) < 1.\end{aligned}$$

Then for all box pseudometric ϱ_α we have $\tilde{\varrho}_\alpha(g_\beta, f) < 1$, if $\beta \geq \alpha$, so that the net $(g_\alpha)_{\alpha\in A}$ converges to f in the Hausdorff topology.

In case (b), let $\{d_\lambda : \lambda \in \Lambda\}$ be the set of all compatible pseudometrics on Y. For every $(\alpha, \lambda) \in A \times \Lambda$, take two disjoint clopen neighborhoods $U_{\alpha,\lambda}$, $V_{\alpha,\lambda}$ of x_α, y_α such that the d_α-diameter of $U_{\alpha,\lambda}$ and $V_{\alpha,\lambda}$ and the d_λ-diameter of $f(U_{\alpha,\lambda})$ and $f(V_{\alpha,\lambda})$ respectively are less than 1.

Now, for every $(\alpha, \lambda) \in A \times \Lambda$ define the continuous function $g_{\alpha,\lambda}$ as follows:

$$g_{\alpha,\lambda}(x) = \begin{cases} f(y_\alpha) & \text{if } x \in U_{\alpha,\lambda}\ , \\ f(x_\alpha) & \text{if } x \in V_{\alpha,\lambda}\ , \\ f(x) & \text{otherwise.} \end{cases}$$

Of course,

$$\tilde{d}_Y(g_{\alpha,\lambda}\ , f) \geq d_Y(g_{\alpha,\lambda}(x_\alpha), f(x_\alpha)) = d_Y(f(y_\alpha), f(x_\alpha)) \geq 1.$$

Let $\varrho_{\alpha,\lambda} = d_\alpha \times d_\lambda$, for every $(\alpha, \lambda) \in A \times \Lambda$. In order to verify the Hausdorff convergence, it is enough to prove that for every $(\beta, \mu) \geq (\alpha, \lambda)$ we have $\tilde{\varrho}_{\alpha,\lambda}(g_{\beta,\mu}\ , f) \leq 2$.

Let (x, y) be an arbitrary point of $G(g_{\beta,\mu}) \setminus G(f)$.
Clearly, $x \in U_{\beta,\mu} \cup V_{\beta,\mu}$. Assume that $x \in U_{\beta,\mu}$. Then $y = f(y_\beta)$ and

$$\varrho_{\alpha,\lambda}\big((x, y), (y_\beta, f(y_\beta))\big) = d_\alpha(x, y_\beta) < 2.$$

Indeed

$$d_\alpha(x, y_\beta) \leq d_\alpha(x, x_\beta) + d_\alpha(x_\beta, y_\beta) \leq d_\beta(x, x_\beta) + d_\beta(x_\beta, y_\beta) < 2.$$

Similarly, if $x \in V_{\beta,\mu}$, we have $\varrho_{\alpha,\lambda}\big((x, y), (x_\beta, f(x_\beta))\big) < 2$.
Therefore, the $\varrho_{\alpha,\lambda}$-distance of the point (x, y) from $G(f)$ is less than 2.
Now, let $(x, f(x))$ be a point of $G(f) \setminus G(g_{\beta,\mu})$. If $x \in U_{\beta,\mu}$, then the d_μ-diameter

of $f(U_{\beta,\mu})$ is less than 1 and we have
$d_\mu\left(f(x), f(x_\beta)\right) < 1$. Then by equality $g_{\beta,\mu}(y_\beta) = f(x_\beta)$, we have

$$\varrho_{\alpha,\lambda}\big((x, f(x)), (y_\beta, g_{\beta,\mu}(y_\beta))\big) \leq \max\{d_\alpha(x, y_\beta), d_\lambda(f(x), f(x_\beta))\} \leq \\ \leq \max\{d_\beta(x, y_\beta), d_\mu(f(x), f(x_\beta))\} < 2.$$

Similarly, if $x \in V_{\beta,\mu}$, we have $\varrho_{\alpha,\lambda}\big((x, f(x)), (x_\beta, g_{\beta,\mu}(x_\beta))\big) < 2$. This means that the $\varrho_{\alpha,\lambda}$-distance of the point $(x, f(x))$ from $G(g_{\beta,\mu})$ is less than 2. □

In the first part of the previous proposition, translations play an important role. This suggests the way of defining a class of metric spaces for which the same result holds.

4. Definition. A metric space (X, d) is said to be transitive if there exist positive numbers γ and L such that, whenever the distance of two points a, b is less than γ, then there exists a continuous function φ of X onto X such that $\varphi(a) = b$ and $d(x, \varphi(x)) \leq Ld(a, b)$, $\forall x \in X$.

A function φ as above is called transition function from a to b.

5. Theorem. *Let (X, d) be a transitive metric space and let f belong to $C(X, Y)$, where Y is a uniform space. If the neighborhood systems of f in the two topologies coincide in $C(X, Y)$, then f is uniformly continuous.*

PROOF: Assume on the contrary that f is not uniformly continuous. Then, as in Theorem 1 we may take d_Y and, for every $n > \frac{1}{\gamma}$, two points x_n, y_n such that $d(x_n, y_n) < \frac{1}{n}$ and $d_Y(f(x_n), f(y_n)) > 1$. For every n, we define $g_n = f \circ \varphi_n$, where φ_n is a transition function from y_n to x_n.
Since $g_n(y_n) = f(x_n)$, then $d(g_n(y_n), f(y_n)) = d(f(x_n), f(y_n)) > 1$ and consequently the sequence g_n does not converge uniformly to f.

Let $\varrho = d \times d'$ be a compatible box pseudometric on $X \times Y$. Since (x, y) belongs to $G(g_n)$ if and only if $(\varphi_n(x), y)$ belongs to $G(f)$, we have:

$$\varrho\big((x, y), (\varphi_n(x), y)\big) = d(x, \varphi_n(x)) < \frac{L}{n},$$

and therefore $G(g_n) \subseteq G(f)_{\frac{L}{n}}$.

On the other hand, if $(z, y) \in G(f)$, then there exists x_z such that $\varphi_n(x_z) = z$, so that $(x_z, y) \in G(g_n)$. Since $\varrho\big((z, y), (x_z, y)\big) = d(\varphi_n(x_z), x_z) < \frac{L}{n}$ we have that $G(f) \subseteq G(g_n)_{\frac{L}{n}}$. Consequently, $\tilde{\varrho}(f, g_n) < \frac{L}{n}$, so that the sequence g_n converges to f in the Hausdorff pseudometric $\tilde{\varrho}$. □

It is clear that the above result holds by more simply assuming that the images of transition maps are dense in X.

There exist very elementary metric spaces which are not transitive. For example, see the space described in Example 1 or the subset of $\mathbb{R}^2$ consisting of the complement of a closed half line. While for these two spaces the local problem has a negative answer, there exist compact metric spaces that are not transitive, as the subset of $\mathbb{R}^2$ defined by $\{(x,y) : |y| = x^2, 0 \leq x \leq 1\}$.

Anyway, there exists a large class of metric manifolds that are transitive metric spaces. Recall that a metric manifold X is a metric space equipped with an atlas of compatible charts (a chart is a homeomorphism from an open subset of X onto an open ball of a Banach space) cf. *e.g.* [2].

Lemma 6 and Theorem 7 which follow below contain some ideas of the so called "Isotopy Lemma" (cf. *e.g.* [2] [3]).

6. Lemma. *Let B be the closed unit ball of a Banach space. For every positive number $r < 1$ there exists $\varepsilon > 0$ such that for every $a \in B$ with $\|a\| \leq \varepsilon$ there exists a homeomorphism h of B onto B such that $h(0) = a$ and $h(x) = x$ whenever $\|x\| \geq r$.*

Proof: For every $x \in B$, let $\varrho(x) = \frac{1}{r} d(x, B \setminus rB)$. The number $\frac{1}{r}$ is a Lipschitz constant for the $[0,1]$-valued function ϱ, and ϱ is 1 at 0, while it vanishes on $B \setminus rB$. Let $a \in B$ with $\|a\| \leq \varepsilon$, where $\varepsilon < r \wedge (1-r)$. The required homeomorphism is defined by $h(x) = x + \varrho(x)a$. Obviously $h(x)$ is a Lipschitz continuous function, $h(0) = a$, and $h(x) = x$ for every $x \in B \setminus rB$.

If $\|x\| \leq r$ we have $\|h(x)\| \leq \|x\| + \varrho(x)\|a\| \leq r + \varepsilon < 1$ and therefore $h(x) \in B$ for every $x \in B$.

Moreover

$$\begin{aligned}\|h(x_1) - h(x_2)\| &= \\ &= \|x_1 - x_2 + (\varrho(x_1) - \varrho(x_2))a\| \geq \|x_1 - x_2\| - \frac{1}{r}\|x_1 - x_2\| \cdot \|a\| = \\ &= \left(1 - \frac{\|a\|}{r}\right) \|x_1 - x_2\| \geq (1 - \frac{\varepsilon}{r})\|x_1 - x_2\|.\end{aligned}$$

Then the function h is injective and its inverse is a Lipschitz continuous function. To show that h is onto B, we prove that for every $w \in rB$ the equation $x + \varrho(x)a = w$ has a solution in B. The solvability of such equation in B is equivalent to existence of a fixed point in B for the function $\gamma(x) = w - \varrho(x)a$. Since $\|\gamma(x)\| \leq \|w\| + \|a\| \leq r + \|a\| \leq 1$, we have $\gamma(x) \in B$ for all $x \in B$.

Furthermore, $\|\gamma(x_1) - \gamma(x_2)\| \leq \frac{\|a\|}{r}\|x_1 - x_2\| \leq \frac{\varepsilon}{r}\|x_1 - x_2\|$.
Since $\varepsilon < r$ the function γ is a contraction of the unit ball and therefore it has a fixed point. $\square$

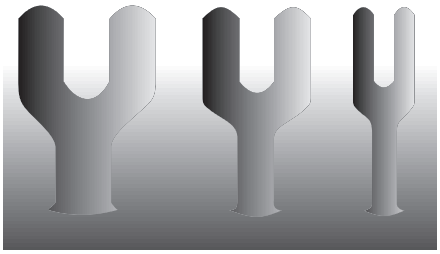
7. Theorem. *A metric manifold (X, d) is transitive provided that there exist positive constants γ, L such that one of the following equivalent conditions is satisfied whenever $d(a,b) < \gamma$:*

(1) *there exists a path σ joining a and b such that the diameter of the image of σ is less than $Ld(a,b)$;*
(2) *there exists a connected open set U containing a and b, and the diameter of U is less than $Ld(a,b)$.*

Proof: It is easy to prove the equivalence of (1) and (2), if we use the fact that a manifold is locally arcwise connected.

If the distance of the points a and b is less than γ, we consider a set U as in condition (2) above and we define an equivalence relation on U as follows. We say that two points $y, z \in U$ are equivalent if there exists a homeomorphism $\varphi : X \longrightarrow X$ and a closed subset K of X, $K \subseteq U$, such that $\varphi(y) = z$ and $\varphi(x) = x$ for every $x \notin K$. If we prove that each equivalence class is an open set, the connectivity of U implies that a is equivalent to b. Then the corresponding homeomorphism φ is the required transition map from a to b. Indeed $d(x, \varphi(x))$ does not exceed the diameter of U, which is at most $Ld(a, b)$.

Let $p \in U$. We must show that there exists a neighborhood of p consisting of points equivalent to p. Let A be a closed neighborhood of p, contained in U. There is no loss of generality in assuming that there exists a homeomorphism ψ of A onto the closed unit ball of a Banach space, such that $\psi(p) = 0$. Let $r < 1$ and ε as in Lemma 6. We prove that every point $q \in W = \psi^{-1}(\varepsilon B)$ is equivalent to p. Indeed, $\|\psi(q)\| \leq \varepsilon$. If h denotes the homeomorphism of Lemma 6, with $a = \psi(q)$, then the homeomorphism $\psi^{-1} \circ h \circ \psi : A \longrightarrow A$ maps p into q, induces the identity on the complement of the closed set $\psi^{-1}(rB)$ and accordingly it can be extended to all of X by means of the identity map. $\square$



A connected metric manifold can be transitive (with homeomorphic transition maps) although the conditions of Theorem 7 are not fulfilled. Consider a discrete sequence of bounded open subsets of $\mathbb{R}^3$ which have the form of a cactus with two parallel branches of the same height. We require that all the cactuses have the same height and that the thickness and the distance between the two branches of each cactus both tend to zero as the index of the cactus-like sets tends to infinity (see the picture). It is not difficult to realize that the connected manifold obtained by attaching all the trunks of the cactuses to an open semispace is transitive, but does not satisfy condition 1.

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