Nonnegative solutions of nonlinear integral equations

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Abstract. Existence results of nonnegative solutions of asymptotically linear, nonlinear integral equations are studied.

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1. Introduction

The purpose of this paper is to study the existence of nonnegative solutions of two integral equations given by

$$p\bigl(x,u(x)\bigr)=\int_0^1 q\bigl(x,t,u(t)\bigr)\,dt,\quad x\in[0,1], \tag{1.1}$$

$$p\bigl(x,u(x)\bigr)=m\left(x,\int_0^1 k(x,t)u(t)\,dt\right),\quad x\in[0,1], \tag{1.2}$$

where $k\in L_2([0,1]\times[0,1],\mathbb{R}_+)$ and p, $m\colon [0,1]\times\mathbb{R}\to\mathbb{R}$, $q\colon [0,1]\times[0,1]\times\mathbb{R}\to\mathbb{R}$ are nonnegative for $u\in\mathbb{R}_+=[0,\infty)$ and all remaining variables. Furthermore, p, m, q satisfy the Carathéodory continuity conditions (see [10]), they are bounded on bounded sets and they have at most linear growth in $u\in\mathbb{R}_+$. Finally, we suppose that p is nondecreasing in $u\in\mathbb{R}_+$ for any $x\in[0,1]$ and $p(x,0)=0\,\forall\,x\in[0,1]$.

The papers [3], [11], [12] have motivated us for the study of (1.1–2). We are interested in the existence of nonnegative solutions for (1.1–2), when p, q, m are asymptotically linear as $u\to+\infty$ uniformly in the remaining variables. We do not investigate uniqueness and properties of possible solutions like in the papers [3], [11], [12], where only the convolution case for $p(x,u)=u^\alpha$, $\alpha>1$ and $q(x,t,u)=h(x-t)u+f(x)\colon t\le x$; $q(x,t,u)=f(x)\colon t>x$ is studied. Hence the equation (1.1) had the following special form in [3], [11], [12]

$$u^\alpha(x)=\int_0^x h(x-t)u(t)\,dt+f(x).$$

Our method is based on abstract existence results derived by the theory of pseudomonotone operators like in [4], [6], [8], [10]. These results are extensions of fixed point theorems in [1], [9] of asymptotically linear maps leaving invariant cones in Banach spaces.

Finally, we note that problems like (1.2) naturally occur in the study of nonlinear boundary value problems of ordinary differential equations. As an example, let us consider the equation

$$\begin{aligned} p(x,-u'') &= m(x,u) \\ u(0) = u(1) &= 0, \end{aligned} \tag{1.3}$$

where p, m have the above properties. By putting $z=-u''$, i.e.

$$u(x)=\int_0^1 G(x,t)z(t)\,dt,$$

where G is the Green function of $-u''=z$, $u(0)=u(1)=0$, we have

$$p\big(x,z(x)\big)=m\left(x,\int_0^1 G(x,t)z(t)\,dt\right), \quad x\in[0,1]. \tag{1.4}$$

Now, if $z\geq 0$ then u is concave. Since $u(0)=u(1)=0$, we obtain $u\geq 0$. Hence nonnegative solutions of (1.4) generate nonnegative solutions of (1.3). In [13], there are studied similar problems of differential equations which are not solvable for highest-order derivatives. In the end of this paper, we also study a discontinuous version of (1.3) by using some ideas of the papers [2], [7]. The paper is completed by several remarks devoting to another possible applications.

2. Abstract existence theorems

In this section, we shall derive existence results of solutions for certain operator equations. Let H be a real separable Hilbert space with the inner product $(\cdot,\cdot)$ and norm $|\cdot|$, and let $K\subset H$ be a wedge, i.e. K is a closed, nonempty, convex subset of H such that $\lambda K\subset K\,\forall\,\lambda\geq 0$. We know (see [5, p. 71]) that there is a continuous metric retraction $\eta\colon H\to K$ such that $\eta(\lambda x)=\lambda\eta(x)\,\forall\lambda\geq 0$, $\forall x\in H$ and $|\eta(x)|\leq|x|\,\forall x\in H$. The following definitions will be needed in the sequel (see [4, p. 946]).

A mapping $f\colon H\to H$ is:

– *monotone* (denote $f\in MON$), if $(f(u)-f(v),u-v)\geq 0$ for all u, $v\in H$;

– *pseudomonotone* ($f\in PM$), if for any sequence $\{u_n\}$ in H with $u_n\rightharpoonup u$ (weak convergence) and $\overline{\lim}(f(u_n),u_n-u)\leq 0$, it follows that $f(u_n)\rightharpoonup f(u)$ and $(f(u_n),u_n)\to(f(u),u)$;

– *of class* S_+ ($f \in S_+$), if for any sequence $\{u_n\}$ in H with $u_n \rightharpoonup u$ and $\overline{\lim}(f(u_n), u_n - u) \le 0$, it follows that $u_n \to u$;

– *compact* ($f \in COMP$), if it is continuous and for any bounded sequence $\{u_n\}$ in H the sequence $\{f(u_n)\}$ has a convergent subsequence;

– *completely continuous* ($f \in CC$), if for any sequence $\{u_n\}$ in H with $u_n \rightharpoonup u$, it follows that $f(u_n) \to f(u)$;

– *bounded*, if it takes any bounded set of H into a bounded set.

We note that the following relations hold between the above definitions

$$\begin{aligned} &CC \subset COMP, \quad S_+ \subset PM, \quad MON \subset PM \\ &f_1 \in S_+,\ f_2 \in COMP \Rightarrow f_1 - f_2 \in S_+ \\ &f_1 \in PM,\ f_2 \in CC \Rightarrow f_1 - f_2 \in PM. \end{aligned} \tag{2.0}$$

In what follows, we shall assume that the mappings are bounded and continuous.

In the rest of this section, we solve the equation

$$L(x) = N(x) \quad x \in K, \tag{2.1}$$

where $N \in COMP$ satisfies $N(K) \subset K$ and $L \in MON$ is such that $(L+\varepsilon\mathbb{I})(K) = K$ for any $\varepsilon > 0$ sufficiently small. Here $\mathbb{I}$ is the identity map.

We suppose

($H1$) There is a linear map $L_\infty \in S_+$ such that $|L(x) - L_\infty x|/|x| \to 0$ as $|x| \to \infty$, $L_\infty x = 0 \Rightarrow x = 0$ and $L_\infty(K) = K$.

($H2$) There is a linear map $N_\infty \in COMP$ such that $|N(x) - N_\infty x|/|x| \to 0$ as $|x| \to \infty$ and $N_\infty(K) \subset K$.

Theorem 2.1. *Assume that* ($H1$), ($H2$) *hold and* $L - N \in PM$. *If the following condition holds*

$$L_\infty x = \lambda N_\infty x,\ x \in K,\ 0 < \lambda \le 1 \ \textit{ implies } \ x = 0, \tag{C}$$

then (2.1) *has a solution.*

Proof: We solve

$$L(x) + \varepsilon x = N(\eta(x)) \tag{2.2}$$

for $\varepsilon > 0$ small. We know that $L + \varepsilon I$ is strongly monotone (i.e. $\big((L(x_1) + \varepsilon x_1) - (L(x_2) + \varepsilon x_2), x_1 - x_2\big) \ge \varepsilon |x_1 - x_2|^2 \,\forall x_1, x_2 \in H$), so it is invertible (see [5, p. 100]). Hence, by using $(L + \varepsilon\mathbb{I})(K) = K$, we see that any solution of (2.2) belongs to K. Let $\overline{\cup_{n=1}^{\infty} H_n} = H$ and H_n be finite dimensional subspaces such that $H_n \subset H_{n+1} \, \forall\, n \in \mathbb{N}$. Let $P_n \colon H \to H_n$ be the orthogonal projections. We rewrite (2.2) in the form

$$L_\infty x + \varepsilon x - N_\infty(\eta(x)) = \big(L_\infty x - L(x)\big) + \big(N(\eta(x)) - N_\infty(\eta(x))\big). \tag{2.3}$$

By $(H2)$ for any $\omega > 0$ there is a constant $c(\omega) > 0$ such that

$$|N(x) - N_\infty(x)| \le \omega |x| + c(\omega) \quad \forall x \in H.$$

This implies

$$|N(\eta(x)) - N_\infty(\eta(x))| \le \omega |\eta(x)| + c(\omega) \le \omega |x| + c(\omega) \quad \forall x \in H.$$

So we obtain

$$|N(\eta(x)) - N_\infty(\eta(x))|/|x| \to 0 \ \text{ as } \ |x| \to \infty.$$

First we solve

$$\begin{aligned} &P_n\Big(L_\infty x + \varepsilon x - N_\infty(\eta(x))\Big) \\ &= P_n\Big(\big(L_\infty x - L(x)\big) + \big(N(\eta(x)) - N_\infty(\eta(x))\big)\Big) \quad x \in H_n. \end{aligned} \tag{2.4}$$

We claim that there are constants $c_1 > 0$, $\varepsilon_0 > 0$, $n_0 > 0$ such that

$$\begin{aligned} &|P_n\big(L_\infty x + \varepsilon x - \lambda N_\infty(\eta(x))\big)| \ge c_1 |x| \\ &\forall (x, \varepsilon, n, \lambda) \in H_n \times [0, \varepsilon_0] \times [n_0, \infty) \times [0, 1]. \end{aligned} \tag{2.5}$$

Indeed, if it is not true then there is a sequence

$$\Big\{(x_{n_i}, \varepsilon_i, n_i, \lambda_i)\Big\}_{i=1}^\infty \subset H \times \mathbb{R}_+ \times \mathbb{N} \times [0, 1]$$

such that $x_{n_i} \in H_{n_i}$, $|x_{n_i}| = 1$, $\varepsilon_i \to 0_+$, $n_i \to \infty$, $\lambda_i \to \lambda_0$ and

$$|P_{n_i}\big(L_\infty x_{n_i} + \varepsilon_i x_{n_i} - \lambda_i N_\infty(\eta(x_{n_i}))\big)| \to 0. \tag{2.6}$$

Here we have used $\eta(\lambda x) = \lambda \eta(x) \, \forall \lambda \ge 0$, $\forall x \in H$. We can assume the existence of $z \in H$ such that $x_{n_i} \rightharpoonup z$. (As a matter of fact, a subsequence of $\{x_{n_i}\}_{i=1}^\infty$ has this property, but for simplicity we can consider in this way. Similar arguments are used later on.) Let $\{z_i\}_{i=1}^\infty$ be a sequence satisfying $z_i \to z$ and $z_i \in H_i$, $\forall\, i \in \mathbb{N}$. Then by using the boundedness of $\{x_{n_i}\}_{i=1}^\infty$, we have

$$\big(L_\infty x_{n_i} + \varepsilon_i x_{n_i} - \lambda_i N_\infty(\eta(x_{n_i})), z_{n_i} - z\big) \to 0.$$

Moreover, the condition (2.6) and the boundedness of $\{x_{n_i}\}_{i=1}^\infty$ and $\{z_i\}_{i=1}^\infty$ as well as $z_i \in H_i$, $\forall\, i \in \mathbb{N}$ imply

$$\big(L_\infty x_{n_i} + \varepsilon_i x_{n_i} - \lambda_i N_\infty(\eta(x_{n_i})), x_{n_i} - z_{n_i}\big) \to 0.$$

So

$$(L_\infty x_{n_i} + \varepsilon_i x_{n_i} - \lambda_i N_\infty(\eta(x_{n_i})), x_{n_i} - z) \to 0.$$

Hence we obtain

$$(2.7) \qquad (L_\infty x_{n_i} - \lambda_0 N_\infty(\eta(x_{n_i})), x_{n_i} - z) \to 0.$$

Since $L_\infty - \lambda_0 N_\infty(\eta) \in S_+$ (see (2.0)), we can assume $x_{n_i} \to z$. Finally, we arrive at the equation

$$L_\infty z - \lambda_0 N_\infty(\eta(z)) = 0.$$

Now, if $\lambda_0 = 0$ then $L_\infty z = 0$, so $z = 0$; if $0 < \lambda_0 \leq 1$ then, by using both $L_\infty(K) = K$ and the fact that the assumptions $L_\infty \in S_+$ and $L_\infty x = 0 \Rightarrow x = 0$ give the invertibility of L_∞, we obtain $z \in K$ and $L_\infty z = \lambda_0 N_\infty(z)$, so $z = 0$. We have in both the cases $z = 0$. But $|z| = 1$, this contradiction proves the validity of (2.5).
(We prove, for the reader convenience, the claim that $L_\infty \in S_+$ and $L_\infty x = 0 \Rightarrow x = 0$ give the invertibility of L_∞: The equation $P_n L_\infty x = P_n h$ has a unique solution $x_n \in H_n$, by (2.5), for any $n \in \mathbb{N}$ sufficiently large and $h \in H$ such that $|x_n| \leq |h|/c_1$. Hence we can assume $x_n \rightharpoonup x$. Similarly as for (2.7), we have $(L_\infty x_n - h, x_n - x) \to 0$. This gives $(L_\infty x_n, x_n - x) \to 0$. Since $L_\infty \in S_+$, we can assume $x_n \to x$, and so x is a unique solution of $L_\infty x = h$ satisfying $|x| \leq |h|/c_1$. The claim is proved.)

Now (2.5) implies for n sufficiently large

$$(2.8) \qquad \deg\Big(P_n(L_\infty + \varepsilon\mathbb{I} - N_\infty(\eta)), B_{1n}, 0\Big) = \deg\Big(P_n L_\infty, B_{1n}, 0\Big) \neq 0,$$

where $B_{1n} = \{x \in H_n \mid \quad |x| < 1\}$. Here we have used the invertibility of $P_n L_\infty$ for n sufficiently large. This follows from (2.5).

Let $\mathcal{L}_n \colon H_n \to H_n$, respectively $\mathcal{N}_n \colon H_n \to H_n$, be the semi-linear, respectively nonlinear, operator defined by the left-hand, respectively right-hand, side of (2.4). Hence (2.4) has the form $\mathcal{L}_n(x) = \mathcal{N}_n(x)$. The property (2.5) gives $|\mathcal{L}_n(x)| \geq c_1|x|$ for any $x \in H_n$ and $n \in \mathbb{N}$ sufficiently large. Since the right-hand side of (2.4) is asymptotically sublinear, for any $r > 0$ there is a constant $c = c(r) > 0$ satisfying $|\mathcal{N}_n(x)| \leq r|x| + c$ for any $n \in \mathbb{N}$ and $x \in H_n$. Hence there is a constant $M > 0$ such that $\mathcal{L}_n(x) \neq \lambda\mathcal{N}_n(x)$ for any n sufficiently large, $\lambda \in [0, 1]$ and $x \in S_{Mn} = \{x \in H_n \mid \quad |x| = M\}$. So by (2.8) we have

$$\deg\big(\mathcal{L}_n - \mathcal{N}_n, B_{Mn}, 0\big) = \deg(\mathcal{L}_n, B_{Mn}, 0) \neq 0,$$

where $B_{Mn} = \{x \in H_n \mid \quad |x| < M\}$. Now it is clear that for any $n \in \mathbb{N}$ sufficiently large, the equation (2.4) has a solution $x_n \in H_n$ satisfying $|x_n| \leq M$. Since $L + \varepsilon\mathbb{I} - N(\eta)$ is pseudomonotone (see (2.0)) (we note that a strongly monotone operator is of class S_+ and $L+\varepsilon\mathbb{I}$ is strongly monotone for any $\varepsilon > 0$), by using the standard arguments (see [4], [8] and [10, pp. 54–55]), (2.2) has a solution

x_ε, $|x_\varepsilon| \le M$. We already know $x_\varepsilon \in K$. So the sequence $\{x_\varepsilon\}$ possesses a weakly convergent subsequence as $\varepsilon \to 0_+$ and again by using the pseudomonotony of $L - N$ as well as the weak closeness of K, we obtain the desired solution. The proof is finished. □

Remark 2.2. 1. It is clear that $L(K) \subset K$.

2. Since $L_\infty \in S_+$ and $L_\infty x = 0 \Rightarrow x = 0$, we know from the above proof that L_∞ is invertible. So the equation $L_\infty x = \lambda N_\infty x$ is equivalent to $x = \lambda L_\infty^{-1} N_\infty x$. Assume that the interior $\mathring{K}$ of K is nonempty and if $u \in K \setminus \{0\}$, then $-u \notin K$. If $N_\infty(K \setminus \{0\}) \subset \mathring{K}$ then the condition (C) of Theorem 2.1 is equivalent to $1/\lambda \ne r(L_\infty^{-1} N_\infty)$, where r denotes the spectral radius (see [5, the Krein-Rutman theorem]). Since $0 < \lambda \le 1$ this means $r(L_\infty^{-1} N_\infty) < 1$. On the other hand, we note that the condition $r(L_\infty^{-1} N_\infty) < 1$ always implies the validity of (C).

3. Theorem 2.1 is an extension of [9, Theorem 4.10] and [1, Theorem 1].

By using Remark 2.2 we have

Corollary 2.3. *Assume that* $(H1)$, $(H2)$ *hold and moreover, suppose*

$$r(L_\infty^{-1} N_\infty) < 1,$$

then (2.1) *has a solution.*

Theorem 2.4. *If all assumptions of Theorem* 2.1 *hold except* $L - N \in PM$, *then* (2.1) *is almost solvable, i.e.* $0 \in \overline{(L - N)(K)}$.

Proof: We follow the proof of Theorem 2.1. So there is a constant $M > 0$ such that for any $\varepsilon > 0$ sufficiently small, there is a solution x_ε, $L(x_\varepsilon) + \varepsilon x_\varepsilon - N(x_\varepsilon) = 0$, $x_\varepsilon \in K$, $|x_\varepsilon| \le M$. The proof is finished. □

Now we replace the assumption $N \in COMP$ by $N \in CC$. Then, of course, $N \in COMP$ and $L - N \in PM$ (see (2.0)). So we obtain the following

Theorem 2.5. *If the assumption* $N \in COMP$ *is strengthened to* $N \in CC$ *in* (2.1) *and all assumptions of Theorem* 2.4 *hold. Then* (2.1) *has a solution.*

Remark 2.6. Problems at resonances of (2.1), i.e. if $L_\infty x - N_\infty x = 0$ has a solution $x \in K \setminus \{0\}$, can be investigated as well by using both an approach suggested in [14] and the method of the proof of Theorem 2.1.

3. Nonnegative solutions

In this section, we study the existence of nonnegative solutions of (1.1–2) by using the results from the previous section. We assume that there are constants $\alpha > 0$, $\beta \ge 0$ and $\gamma \in L_2([0,1] \times [0,1], \mathbb{R}_+)$ satisfying (see Remark 3.9 below)

$$\lim_{|u| \to \infty} |p(x,u) - \alpha u| / |u| = 0 \text{ uniformly in } x \in [0,1] \tag{3.1}$$

$$\lim_{|u| \to \infty} |m(x,u) - \beta u| / |u| = 0 \text{ uniformly in } x \in [0,1] \tag{3.2}$$

$$\lim_{|u| \to \infty} |q(x,t,u) - \gamma(x,t) u| / |u| = 0 \text{ uniformly in } x,\, t \in [0,1]. \tag{3.3}$$

Theorem 3.1. *Assume that* (3.1), (3.3) *hold and moreover, suppose*

$$\big(p(x,u_1)-p(x,u_2)\big)(u_1-u_2)>0 \quad \forall\, x\in[0,1],\ u_1\neq u_2. \tag{3.4}$$

If the equation

$$\alpha u(x)=\lambda\int_0^1 \gamma(x,t)u(t)\,dt \tag{3.5}$$

has no nonzero nonnegative solution for any $0<\lambda\le 1$, *then* (1.1) *has a nonnegative solution.*

Proof: We apply Theorem 2.1 by putting

$$H=L_2([0,1],\mathbb{R}),\ K=\Big\{u\in L_2([0,1],\mathbb{R})\mid u\ge 0 \text{ almost everywhere on } [0,1]\Big\}$$

$$L(u)=p(\cdot,u),\quad N(u)=\int_0^1 q\big(\cdot,t,u(t)\big)\,dt$$

$$L_\infty u=\alpha u,\quad N_\infty u=\int_0^1 \gamma(\cdot,t)u(t)\,dt.$$

It is clear that $L\in MON$, $N\in COMP$, $N(K)\subset K$, $L_\infty\in S_+$, $N_\infty\in COMP$. Since the function $p(x,y)+\varepsilon y$ is strictly increasing in y and it tends asymptotically linearly to $\pm\infty$ as $y\to\pm\infty$ uniformly in $x\in[0,1]$, we see that this function has the continuous inverse function in y for any $x\in[0,1]$ with at most asymptotically linear growth in y uniformly in $x\in[0,1]$. Now we easily verify that $(L+\varepsilon\mathbb{I})(K)=K$.

By [10, p. 61], we know that (3.4) implies $L\in S_+$. So $L-N\in PM$ (see (2.0)). The assumptions $(H1-2)$ are proved as usually by using (3.1) and (3.3) (see [6], [10]). For instance, let us prove $(H2)$. By (3.3) and the boundedness of q, for any $\omega>0$, there is a constant $c(\omega)>0$ such that

$$|q(x,t,u)-\gamma(x,t)u|\le\omega|u|+c(\omega)\quad\forall\,(x,t,u)\in[0,1]\times[0,1]\times\mathbb{R}\,.$$

This gives

$$\begin{aligned}
|N(u)-N_\infty u|_{L_2}&\le\sqrt{\int_0^1\int_0^1|q(x,t,u(t))-\gamma(x,t)u(t)|^2\,dt\,dx}\\
&\le\sqrt{\int_0^1\big(\omega|u(t)|+c(\omega)\big)^2\,dt}\le\sqrt{2}\sqrt{\omega^2\int_0^1 u^2(t)\,dt+c^2(\omega)}\\
&\le\sqrt{2}\omega|u|_{L_2}+\sqrt{2}c(\omega),
\end{aligned}$$

where we have used the inequality $(a+b)^2 \leq 2(a^2+b^2)$. So $(H2)$ is proved.

The last assumption of Theorem 2.1: $L_\infty u = \lambda N_\infty u$, $u \in K$, $0 < \lambda \leq 1$ implies $u = 0$, is guaranteed by (3.5). Hence the proof is finished. □

Concerning the equation (3.5), we have the following easy result from Corollary 2.3.

Theorem 3.2. *In addition to* (3.1), (3.3), (3.4), *assume that either*

$$\alpha > \sqrt{\int_0^1 \int_0^1 \gamma^2(x,t)\, dx\, dt}$$

or γ *is, moreover, bounded on* $[0,1] \times [0,1]$ *satisfying*

$$\alpha > \min\left\{ \sup_{x\in[0,1]} \int_0^1 \gamma(x,t)\, dt,\ \sup_{t\in[0,1]} \int_0^1 \gamma(x,t)\, dx \right\}.$$

Then (1.1) *has a nonnegative solution.*

Proof: To prove this theorem, we use the inequality $r(A) \leq ||A||_X$, where $A\colon X \to X$ is a bounded linear operator with the norm $||A||_X$ in a Banach space X, to the operator

$$Au = \frac{1}{\alpha} \int_0^1 \gamma(\cdot,t) u(t)\, dt$$

considered gradually on the Banach spaces $L_\infty([0,1],\mathbb{R})$, $L([0,1],\mathbb{R})$, $L_2([0,1],\mathbb{R})$. The inequalities of this theorem ensure that $||A||_X < 1$ holds at least for one of these three cases, so the proof is finished by Corollary 2.3. □

Since the set $\{x \in C([0,1],\mathbb{R}) \mid x(\cdot) \geq 0\}$ has a nonempty interior in $C([0,1],\mathbb{R})$, by applying Remark 2.2, we can strengthen Theorem 3.2 as follows.

Theorem 3.3. *In addition to* (3.1), (3.3), (3.4), *assume that*

$$\gamma \in C\big([0,1] \times [0,1], (0,\infty)\big).$$

Then there is a unique $\alpha_0 > 0$ *such that*

$$\alpha_0 u(x) = \int_0^1 \gamma(x,t) u(t)\, dt$$

has a positive solution. Moreover, if $\alpha \notin [0, \alpha_0]$ *then* (1.1) *has a nonnegative solution.*

Now we consider that γ in (3.5) has a convolution form, i.e. we assume

$$\begin{aligned} \gamma(x,t) &= 0 && \text{for } t > x \\ \gamma(x,t) &= h(x-t) && \text{for } t \le x, \end{aligned}$$

where $h \in L_2([0,1], \mathbb{R}_+)$. Then (3.5) has the form

$$\alpha u(x) = \lambda \int_0^x h(x-t) u(t)\, dt. \tag{3.6}$$

It is well-known that (3.6) has the only zero solution for any $0 < \lambda \le 1$. Indeed, we have

$$\alpha^2 u^2(x) \le \int_0^x h^2(x-t)\, dt \int_0^x u^2(t)\, dt \le \int_0^1 h^2(t)\, dt \int_0^x u^2(t)\, dt.$$

The Gronwall lemma gives $u = 0$.

So we have

Theorem 3.4. *In addition to* (3.1), (3.3), (3.4), *assume that*

$$\begin{aligned} \gamma(x,t) &= 0 && \textit{for } t > x \\ \gamma(x,t) &= h(x-t) && \textit{for } t \le x, \end{aligned}$$

where $h \in L_2([0,1], \mathbb{R}_+)$. *Then* (1.1) *has a nonnegative solution.*

By applying Theorem 2.4, we obtain the following result.

Theorem 3.5. *If the assumption* (3.4) *in Theorems* 3.1–4 *is dropped* (*so* p *is only nondecreasing in* u) *and the remaining ones are valid, then* (1.1) *has almost a nonnegative solution.*

Now we apply Theorem 2.5 to study (1.2) by assuming (3.1), (3.2) and setting

$$H = L_2([0,1], \mathbb{R}),\ K = \Big\{ u \in L_2([0,1], \mathbb{R}) \mid u \ge 0 \text{ almost everywhere on } [0,1] \Big\}$$

$$L(u) = p(\cdot, u), \quad N(u) = m\left(\cdot, \int_0^1 k(\cdot, t) u(t)\, dt \right)$$

$$L_\infty u = \alpha u, \quad N_\infty u = \beta \int_0^1 k(\cdot, t) u(t)\, dt.$$

Since a compact linear mapping is completely continuous and a composition of a continuous mapping and a completely continuous one is also completely continuous, we have $N \in CC$. Hence Theorem 2.5 is applicable. Moreover, (1.2) is very similar to (1.1). So, by using the above procedure, we obtain

Theorem 3.6. *Assume that* (3.1), (3.2) *hold. Then replacing* γ *by* k *and the equation* (3.5) *by*

$$\alpha u(x) = \lambda\beta \int_0^1 k(x,t)u(t)\,dt,$$

Theorems 3.1–4 *can be straightforwardly rewritten to obtain the existence results of a nonnegative solution of* (1.2).

For instance, a modified version of Theorem 3.4 has the following form.

Theorem 3.7. *Assume that* (3.1), (3.2) *hold and* (1.2) *has the form*

$$p\big(x,u(x)\big) = m\left(x, \int_0^x h(x-t)u(t)\,dt\right), \quad x \in [0,1], \tag{3.7}$$

where $h \in L_2([0,1],\mathbb{R}_+)$. *Then* (3.7) *has a nonnegative solution.*

Remark 3.8. We see that the nonlinearity on the right-hand side of (1.2) is stronger than the corresponding one of (1.1), because the assumption (3.4) can be dropped in (1.2) for obtaining its nonnegative solution.

Remark 3.9. The asymptotic behaviors in (3.1–3) can be considered only for $u \to +\infty$, because we are only interested in nonnegative solutions.

Finally, we apply Theorem 3.6 to solve (1.3) and its discontinuous version.

Theorem 3.10. *Consider* (1.3) *and assume* (3.1), (3.2) *hold. If* $\beta < \pi^2\alpha$ *then* (1.3) *has a nonnegative concave solution.*

Proof: In this case, the linear equation in Theorem 3.6 has the form (see (1.4))

$$u(x) = \frac{\lambda\beta}{\alpha} \int_0^1 G(x,t)u(t)\,dt,$$

which is equivalent to $-u''(x) = \frac{\lambda\beta}{\alpha}u(x)$, $u(0) = u(1) = 0$. It is well known (see [5]) that this equation has a nonzero nonnegative concave solution only if $\frac{\lambda\beta}{\alpha} = \pi^2$. Since $0 < \lambda \le 1$ and $\beta < \pi^2\alpha$, the proof is finished. □

In the end of this paper, we consider the following discontinuous version of (1.3)

$$\begin{aligned} &p(x,-u''(x)) = g(u(x)) + f(x) \\ &u(0) = u(1) = 0, \end{aligned} \tag{3.8}$$

where we assume

1. p is continuous possessing the properties of Introduction;

2. $g\colon \mathbb{R} \to \mathbb{R}$ is increasing and continuous on $\mathbb{R} \setminus \mathcal{A}$ for a subset $\mathcal{A} \subset \mathbb{R}$ with no finite accumulation points such that

$$\begin{gathered}\inf g = -\infty, \quad \sup g = +\infty \\ g(0+) = \lim_{s \to 0_+} g(s) \geq 0, \quad g(0-) = \lim_{s \to 0_-} g(s) \leq 0\,;\end{gathered}$$

3. $f \in C([0,1], \mathbb{R}_+)$.

By a solution of (3.8) we mean u such that $u'' \in L_2([0,1], \mathbb{R})$ and the following relation holds almost everywhere on $[0,1]$

$$\begin{aligned}&p(x, -u''(x)) - f(x) \in [g(u(x)-), g(u(x)+)] \\ &u(0) = u(1) = 0.\end{aligned} \tag{3.9}$$

Now we rewrite (3.9) by using a method from [2], [7]. So we are able to find a continuous nondecreasing function $e\colon \mathbb{R} \to \mathbb{R}$ such that

$$e(t) = a, \quad \text{if} \quad t \in [g(a-), g(a+)],$$

where $g(a\pm) = \lim\limits_{s \to a_\pm} g(s)$.

Hence the equation (3.9) is equivalent to

$$e\big(p(x, -u''(x)) - f(x)\big) = u(x).$$

Finally, we obtain

$$e\big(p(x, z(x)) - f(x)\big) - e(-f(x)) = \int_0^1 G(x,t) z(t)\,dt - e(-f(x)), \tag{3.10}$$

where G is the above Green function. We see that (3.10) has the form of (1.4). Moreover, if (3.1), respectively (3.2), holds for p, respectively $m(x,u) = g(u) + f(x)$ with $\beta > 0$, then the function

$$p_1(x,z) = e\big(p(x,z) - f(x)\big) - e(-f(x))$$

has the linear asymptote $\alpha z/\beta$ as $z \to \pm\infty$ uniformly in $x \in [0,1]$. It is also clear that p_1 is nondecreasing in z, $p_1(\cdot, 0) = 0$ and it is nonnegative for $z \geq 0$. Lastly, the function $-e(-f(x))$ is nonnegative, since $e(0) = 0$ and e is nondecreasing. So, by applying Theorem 3.6 like in the proof of Theorem 3.10, we obtain

Theorem 3.11. *Consider* (3.8) *and assume* (3.1), (3.2) *hold for* p, $m(x,u) = g(u) + f(x)$, *respectively. If* $\alpha\pi^2 > \beta > 0$ *then* (3.8) *has a nonnegative concave solution* (*see* (3.9)).

4. Concluding remarks

Remark 4.1. The proof of Theorem 2.1 can be considered also for the equation

$$L(x) = N_1(x) + N_2(x), \quad x \in K, \tag{4.1}$$

where L has the above properties, $N_1 \in COMP$, $N_2 \in CC$ are such that

1. $(N_1 + N_2)(K) \subset K$ and $L - N_1 \in PM$;
2. There are linear $N_{1,\infty}$, $N_{2,\infty} \in COMP$ such that $|N_1(x)+N_2(x)-N_{1,\infty}x-N_{2,\infty}x|/|x| \to 0$ as $|x| \to \infty$ and $(N_{1,\infty} + N_{2,\infty})(K) \subset K$;
3. $L_\infty x = \lambda(N_{1,\infty}x + N_{2,\infty}x)$, $x \in K$, $0 < \lambda \leq 1$ implies $x = 0$.

Then (4.1) has a solution.

Remark 4.2. Remark 4.1 implies the existence of a nonnegative solution of the equation

$$p\big(x, u(x)\big) = \int_0^1 q\big(x, t, u(t)\big)\, dt + m\left(x, \int_0^1 k(x,t)u(t)\, dt\right), \quad x \in [0,1], \tag{4.2}$$

where p, q, m, k possessing the properties from Introduction and satisfying (3.1–4) are such that the linear equation

$$\alpha u(x) = \lambda \int_0^1 \Big(\gamma(x,t) + \beta k(x,t)\Big) u(t)\, dt$$

has no nonzero nonnegative solution for any $0 < \lambda \leq 1$.

Remark 4.3. Theorem 2.5 can be applied to certain nonlinear boundary value problems of integrodifferential equations similarly as for (1.3) and (3.8). For instance, it is applicable to the equation

$$\begin{aligned} &p\big(x, -u''(x)\big) = \int_0^1 q\big(x, t, u(t)\big)\, dt + m(x, u(x)), \quad x \in [0,1] \\ &u(0) = u(1) = 0, \end{aligned} \tag{4.3}$$

where p, q, m possessing the properties from Introduction and satisfying (3.1–3) are such that the linear equation

$$\begin{aligned} -\,u''(x) &= \frac{\lambda}{\alpha}\left(\int_0^1 \gamma(x,t)u(t)\, dt + \beta u(x)\right) \\ u(0) &= u(1) = 0 \end{aligned}$$

has no nonzero nonnegative concave solution for any $0 < \lambda \leq 1$. Then the equation (4.3) has a nonnegative solution. Indeed, the equation (4.3) is equivalent to

$$p\big(x, z(x)\big) = \int_0^1 q\left(x, t, \int_0^1 G(t,s)z(s)\,ds\right) dt + m\left(x, \int_0^1 G(x,t)z(t)\,dt\right) \tag{4.4}$$
$$x \in [0,1],$$

where G is the above Green function. It is clear that the right-hand side of (4.4) is completely continuous (see the arguments over Theorem 3.6). So Theorem 2.5 can be used similarly as for (1.2) in Theorem 3.6.

References

[1] Amann H., *Fixed points of asymptotically linear maps in ordered Banach spaces*, J. Functional Analysis **14** (1973), 162–171.

[2] Ambrosetti A., Badiale M., *The Dual Variational Principle and elliptic problems with discontinuous nonlinearities*, J. Math. Anal. Appl. **140** (1989), 363–373.

[3] Askhabov S.N., *Integral equations of convolution type with power nonlinearity*, Coll. Math. **62** (1991), 49–65.

[4] Berkovits J., Mustonen V., *An extension of Leray-Schauder degree and applications to nonlinear wave equations*, Diff. Int. Equations **3** (1990), 945–963.

[5] Deimling K., *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, 1985.

[6] Fečkan M., *Critical points of asymptotically quadratic functions*, Annales Polon. Math. **LXI.1** (1995), 63–76.

[7] Fečkan M., *Ordinary differential equations with discontinuous nonlinearities*, Atti Sem. Mat. Fis. Univ. Modena **XLI** (1993), 431–444.

[8] Kittilä A., *On the topological degree for a class of mappings of monotone type and applications to strongly nonlinear elliptic problems*, Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes **91** (1994).

[9] Krasnoselskii M.A., *Positive Solutions of Operator Equations*, Noordhoff, Groningen, 1964.

[10] Nečas J., *Introduction to the Theory of Nonlinear Elliptic Equations*, Teubner, Leipzig, 1983.

[11] Okrasiński W., *On a non-linear convolution equation occurring in the theory of water percolation*, Annal. Polon. Math. **37** (1980), 223–229.

[12] Okrasiński W., *On the existence and uniqueness of nonnegative solutions of certain nonlinear convolution equation*, Annal. Polon. Math. **36** (1979), 61–72.

[13] Petryshyn W.V., *Solvability of various boundary value problems for the equation $x'' = f(t,x,x',x'') - y$*, Pacific J. Math. **122** (1986), 169–195.

[14] Santanulla J., *Existence of nonnegative solutions of a semilinear equation at resonance with linear growth*, Proc. Amer. Math. Soc. **105** (1989), 963–971.

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