Type III_0 cocycles without unbounded gaps

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Abstract. An example of type III_0 cocycle without unbounded gaps of an ergodic probability measure preserving transformation will be shown.

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1. Introduction

In this note, we give an answer to M. Lemańczyk's question about type III_0 cocycles ([3]). Let T be an ergodic probability measure preserving transformation of a Lebesgue space $(X, \mathcal{B}, m)$. A measurable function $f : X \to \mathbf{R}$ is called a cocycle with unbounded gaps if there exists a sequence of open intervals P_n such that $|P_n| \to \infty$ and

$$\{f^{(k)}(x) : x \in X, k \in \mathbf{Z}\} \cap P_n = \emptyset$$

for all $n \geq 1$. Here $f^{(k)}(x) = \sum_{i=0}^{k-1} f(T^i x)$, if $k > 0$, $f^{(0)}(x) = 0$, $f^{(k)}(x) = -\sum_{i=k}^{-1} f(T^i x)$ if $k < 0$. In [4] M. Lemańczyk considers cocycles whose restriction to a measurable subset has unbounded gaps. This property is invariant up to cohomology. His question is whether it is a generic property among all type III_0 recurrent cocycles or not. We will show that there exists an example of type III_0 recurrent cocycle of an ergodic probability measure preserving transformation whose no restriction has unbounded gaps.

2. Construction

Here let us recall the notion of orbit cocycle. Let T be an ergodic probability measure preserving transformation of a Lebesgue space $(X, \mathcal{B}, m)$. Each measurable function $f : X \to \mathbf{R}$ is called a cocycle. Denote by

$$\mathcal{R} = \mathcal{R}(T) = \{(x, T^k x) : \ x \in X, \ k \in \mathbf{Z}\}$$

and call it the relation generated by T. An orbit cocycle is any measurable function $\psi : \mathcal{R} \to \mathbf{R}$ satisfying

$$\psi(x, y) + \psi(y, z) = \psi(x, z)$$

for all (x,y), $(y,z) \in \mathcal{R}$. Since T acts freely, the set of all cocycles is bijectively mapped to the set of all orbit cocycles by the map

$$f \to \psi$$

where

$$\psi(x,y) = f^{(k)}(x) \ \text{ if } \ y = T^k x.$$

If $B \in \mathcal{B}$ then put

$$\mathcal{R}_B = \mathcal{R} \cap (B \times B).$$

The corresponding restricted orbit cocycle ψ_B is defined as

$$\psi_B(x,y) = \psi(x,y),\ (x,y) \in \mathcal{R}_B.$$

Let $\{N_s\}_{s \geq 0}$ be a sequence of positive integers satisfying that

$$\sum_{s=1}^{\infty} \frac{1}{\sqrt{N_s}} < \infty, \quad N_0 = 0$$

and set $M_s = N_1 + N_2 + \cdots + N_s$ and $I_s = \{2M_{s-1}+1, 2M_{s-1}+2, \cdots, 2M_s\}$. Define the infinite product probability measure space

$$(X,m) = \prod_{s=1}^{\infty} \prod_{i \in I_s} (\{0,1\}, \{1/2, 1/2\})$$

and let $\mathcal{B}$ be the smallest sigma algebra which makes each coordinate variable of X measurable. The transformation of X which we consider is the adding machine transformation T defined for $x = (x_n) \in X$ by

$$Tx = (x_1, x_2, \dots) + (1, 0, 0, \dots)$$

where the addition is the coordinatewise addition with right carry. Then,

$$\mathcal{R} = \{(x,y) \in X \times X : \ x_n = y_n \ \text{ for all but a finite number of } \ n\}.$$

Define an orbit cocycle $\psi(x,y)$ by setting for $(x,y) \in \mathcal{R}$

$$\psi(x,y) = \sum_{s=1}^{\infty} 2^s \Big(\sum_{i \in I_s} x_i - \sum_{i \in I_s} y_i\Big).$$

Notice that the sum is a finite sum.

Theorem 2.1. *The above cocycle ψ of $\mathcal{R}$ is of type* III_0, *recurrent and does not admit any restricted cocycle with unbounded gaps.*

In a series of lemmas and propositions, we will complete the proof of Theorem 2.1.

Let $n \geq 1$ and define the probability space $(X_n, m_n) = \prod_1^{2n}(\{0,1\}, \{\frac{1}{2}, \frac{1}{2}\})$.

Definition 2.1. *Let $A, B \subset X_n$ and $\phi : A \to B$ be an bijection. Suppose ϕ satisfies the two conditions:*

1. $\sum_1^{2n} \phi(x)_i - x_i = 1, \quad \forall\, x \in A.$
2. *The subset A is maximal in the sense that if $\phi' : A' \to B'$ is another bijection satisfying the condition* (1) *and if $A' \supset A$, then $A = A'$.*

We call such a map ϕ a lacunary map and write $A = \mathrm{Dom}(\phi)$, $B = \mathrm{Im}(\phi)$.

Lemma 2.1. *Any lacunary map ϕ satisfies $m_n((\mathrm{Dom}(\phi))^c) = O(\frac{1}{\sqrt{n}})$ as $n \to \infty$.*

Proof: Set $S_{2n}(x) = \sum_1^{2n} x_i$, and $E_k = \{S_{2n}(x) = k\}$, $0 \leq k \leq 2n$. If $k < n$ then $\sharp E_k < \sharp E_{k+1}$. This means $\cup_{k=0}^{n-1} E_k \subset \mathrm{Dom}\,\phi$. On the other hand, $\sharp E_k > \sharp E_{k+1}$, if $k \geq n$. Therefore, $\sharp((\mathrm{Dom}\,\phi)^c \cap E_k) = \sharp E_k - \sharp E_{k+1}$, $k \geq n$. Hence,

$$\begin{aligned} m((\mathrm{Dom}\,\phi)^c) &= \frac{1}{2^n} \sum_{k=n}^{2n} (\sharp E_k - \sharp E_{k+1}) \\ &= (\sharp E_n - \sharp E_{2n})/2^{2n} \\ &< \sharp E_n / 2^{2n} \\ &= \frac{(2n)!}{2^{2n} n! n!}\,. \end{aligned}$$

Apply Stirling's formula, the right hand $\sim \frac{1}{\sqrt{n\pi}}$. □

Definition 2.2. *By $[\mathcal{R}]_*$ we denote the set of all measurable injective maps $g : A \to B = g(A)$, where A and B are measurable subsets of X, such that*

$$gx \in \{y \,|\, (y, x) \in \mathcal{R}\}, \quad \text{a.e. } x \in A.$$

Such maps are called $\mathcal{R}$-partial transformations.

Note that $\mathcal{R}$-partial transformations preserve the restricted measures.

Proposition 2.1. *For any measurable subset $E \subset X$ of positive measure, the restricted cocycle ψ_E of $\mathcal{R}_E$ does not have unbounded gaps.*

Proof: Let $E \subset X$. Notice that for a.e. $x \in E$,

$$\lim_{n\to\infty} \frac{m(E \cap [x_1, \cdots, x_n]_1^n)}{m([x_1, \cdots, x_n]_1^n)} = 1.$$

For each $s \geq 1$, we let ϕ_s be a lacunary map for $\prod_{i \in I_s}(\{0,1\},\{1/2,1/2\})$. Let ϕ_s act on X by setting

$$\phi_s(x)_i = \begin{cases} (\phi_s([x]_{I_s}))_{i-2M_{s-1}} & \text{if } i \in I_s, \\ x_i & \text{otherwise.} \end{cases}$$

Then $\phi_s \in [\mathcal{R}]_*$ and $\psi(\phi_s x, x) = 2^s$, $x \in \mathrm{Dom}(\phi_s)$.

For a.e. $x \in E$, there exists an integer $T \geq 1$ such that

$$\frac{m(E \cap [x_1, \cdots, x_n]_1^n)}{m([x_1, \cdots, x_n]_1^n)} > \frac{3}{4}, \quad \forall n \geq 2N_T.$$

By $\epsilon_1 \epsilon_1 \cdots \epsilon_{2M_T}$ we denote the word $x_1 x_2 \cdots x_{2M_T}$. It follows from Lemma 2.1 and the assumption on $\{N_s\}_{s \geq 1}$ that we may assume that

$$\prod_{s=T+1}^{\infty} m(\mathrm{Dom}\ \phi_s) > \frac{3}{4}.$$

We are going to show that

$$m(x \in E \,|\, \exists\, y \in E \text{ such that } (y,x) \in \mathcal{R},\ \psi(y,x) = l2^{T+1}) > 0, \quad \forall\, l \geq 1.$$

Then this means ψ_E does not admit unbounded gaps.

Notice that

$$\begin{aligned} \frac{m([\epsilon_1 \cdots \epsilon_{2M_T}]_1^{2M_T} \cap \bigcap_{s=T+1}^{\infty} \mathrm{Dom}\ \phi_s)}{m([\epsilon_1 \cdots \epsilon_{2M_T}]_1^{2M_T})} &= m(\bigcap_{s=T+1}^{\infty} \mathrm{Dom}\ \phi_s) \\ &> \frac{3}{4}. \end{aligned}$$

Hence,

$$\frac{m([\epsilon_1 \cdots \epsilon_{2M_T}]_1^{2M_T} \cap E \cap \bigcap_{s=T+1}^{\infty} \mathrm{Dom}\ \phi_s)}{m([\epsilon_1 \cdots \epsilon_{2M_T}]_1^{2M_T})} > \frac{1}{2}.$$

Set

$$E' = [\epsilon_1 \cdots \epsilon_{2M_T}]_1^{2M_T} \cap E \cap \bigcap_{s=T+1}^{\infty} \mathrm{Dom}\ \phi_s,$$

and for each $l \geq 1$ let $l \cdot 2^{T+1} = \sum_{s=1}^{L} 2^s l_s$ be a dyadic expansion. Set $I = \{s \in [T+1, L] \,|\, l_s = 1\}$, then $I \neq \emptyset$. Let us define an $\mathcal{R}$-partial transformation f by setting for $x \in E'$, $s \geq 0$, $j \in I_s$,

$$f(x)_j = \begin{cases} (\phi_s(x))_j, & \text{if } j \in I_s, \quad \text{for some } s \in I, \\ x_j & \text{otherwise.} \end{cases}$$

Since f is measure preserving, it follows that $f \in [\mathcal{R}]_*$ with $\mathrm{Dom}\, f = E'$ and that

$$\frac{m(f(E') \cap [\epsilon_1 \cdots \epsilon_{2M_T}]_1^{2M_T} \cap E)}{m([\epsilon_1 \cdots \epsilon_{2M_T}]_1^{2M_T})} > \frac{1}{4} > 0.$$

If we set $F = \{x \in E' \,|\, f(x) \in E\}$. Then, $m(F) > 0$, $f(F) \subset E$, $\psi(f(x), x) = l2^{T+1}$, $\forall\, x \in F$. $\square$

Proposition 2.2. *The orbit cocycle ψ is recurrent.*

Proof: For each $s \geq 0$ let θ_s be a transformation of $\prod_{i \in I_s}(\{0,1\},\{1/2,1/2\})$ which transitively acts on each set $\{x \mid S_{2N_s} = j\}$, $0 \leq j \leq 2N_s$. θ_s can naturally act on the infinite product probability measure space (X, m). Then, $\theta_s \in [\mathcal{R}]_*$ and $\psi(\theta_s(x), x) = 0$, $x \in X$. Now for any measurable subset E of positive measure, we have an integer T and a word $\epsilon_1 \cdots \epsilon_{2M_T}$ such that

$$\frac{m(E \cap [\epsilon_1 \cdots \epsilon_{2M_T}]_1^{2M_T})}{m([\epsilon_1 \cdots \epsilon_{2M_T}]_1^{2M_T})} > \frac{1}{2}.$$

Since θ_{T+1} is measure preserving, we have

$$m(E \cap [\epsilon_1 \cdots \epsilon_{2M_T}]_1^{2M_T} \cap \theta_{T+1}(E \cap [\epsilon_1 \cdots \epsilon_{2M_T}]_1^{2M_T})) \;>\; 0.$$

Thus, we see that

$$\begin{aligned} &m(x \in E \mid \exists\, y \in E \;\text{ such that }\; y \neq x,\; (y,x) \in \mathcal{R} \;\text{ and }\; \psi(y,x) = 0) \\ &\geq m(x \in E \mid \theta_{T+1}(x) \in E) \\ &> 0. \end{aligned}$$

This means ψ is recurrent. □

Remark 2.1. *The cocycle defined by*

$$f(x) = \psi(x,y), \quad \textit{where} \;\; y = x + (1,0,0,\cdots)$$

is integrable with mean 0 *and hence recurrent. This was kindly told the author by M. Lemańczyk.*

Next we will show that ψ is of type III_0. For this, let us recall a "T-set" $T(\psi)$ of a cocycle ψ ([1]) which is the set of all real numbers t such that there exists a real measurable function $\xi(x)$ satisfying

$$e^{it\psi(y,x)} = \frac{e^{i\xi(y)}}{e^{i\xi(x)}}, \quad \text{a.e. } x.$$

Lemma 2.2 ([1]). *Let $t \in \mathbf{R}$. Then, $t \in T(\psi)$ if and only if there exists a sequence of real numbers $\{a_{n,t}\}_{n \geq 1}$ such that*

$$\lim_{n \to \infty} e^{it \sum_{j=1}^{n} \{X_j(x) - a_{j,t}\}} \quad \textit{exists a.e. } x$$

where for $x \in X$, $X_j(x) = X_j(x_j) = 2^s x_j$, if $j \in I_s$, $s \geq 1$.

Proof: $(\leftarrow)$ Set

$$e^{i\xi_t(x)} = \lim_{n \to \infty} e^{it \sum_{j=1}^{n} \{X_j(x) - a_{j,t}\}}.$$

Then

$$\frac{e^{i\xi_t(y)}}{e^{i\xi_t(x)}} = \lim_{n\to\infty} e^{it\sum_{j=1}^{n}\{X_j(y)-X_j(x)\}}$$
$$= e^{it\psi(y,x)}.$$

($\rightarrow$) Let for each $i \geq 1$ $g_i(k) = k+1 \pmod 2$. For each $n \geq 1$, $1 \leq i \leq n$ and $\epsilon_i = 0$ or 1, we have

$$\exp(i\{\xi_t(g_1^{\epsilon_1}x_1, \cdots, g_n^{\epsilon_n}x_n, x_{n+1}, \cdots) + t\{X_1(g_1^{\epsilon_1}x_1) + \cdots + X_n(g_n^{\epsilon_n}x_n)\}\})$$
$$= \exp(i\{\xi_t(x) + t\{X_1(x_1) + \cdots + X_n(x_n)\}\}).$$

The orbit of $(x_1 \cdots, x_n)$ by the group generated by the transformations $g_1^{\epsilon_1} \times g_2^{\epsilon_2} \times \cdots \times g_n^{\epsilon_n}$ is the whole set $\prod_{i=1}^{n}\{0,1\}$. Hence,

$$e^{i\xi_t(x)+t\sum_{i=1}^{n}X_i(x)} = e^{i\xi_{n+1,t}(x)},$$

where $\xi_{n+1,t}(x)$ is a function of $x_{n+1}, x_{n+2}, \cdots$. If we put

$$e^{iC_{n+1,t}} = \frac{E(e^{i\xi_{n+1,t}})}{|E(e^{i\xi_{n+1,t}})|}$$

then, by the martingale convergence theorem,

$$\lim_{n\to\infty} e^{-it\sum_{j=1}^{n}X_j + iC_{n+1,t}} = \lim_{n\to\infty} \frac{E\{e^{i\xi_t} \mid \prod_{j=1}^{n}\mathcal{F}_j\}}{|E\{e^{i\xi_t} \mid \prod_{j=1}^{n}\mathcal{F}_j\}|}$$
$$= \frac{e^{i\xi_t}}{|e^{i\xi_t}|}$$
$$= e^{i\xi_t}.$$

Here, $\mathcal{F}_j$ denotes the sub σ-algebra generated by the cylinder sets $[0]_j$ and $[1]_j$. If we set

$$a_{n,t} = C_{n+1,t} - C_{n,t}, \quad C_{0,t} = 0,$$

then

$$\lim_{n\to\infty} e^{it\sum_{j=1}^{n}\{X_j(x)-a_{j,t}\}} \text{ exists a.e. } x.$$

$\square$

Lemma 2.3 ([1]). *Let* $t \in \mathbf{R}$. *Then,* $t \in \{\frac{k}{2^s} \mid s \geq 0,\ k \in \mathbf{Z}\}$ *if and only if* $\lim_{s\to\infty} e^{2\pi i 2^s t} = 1$.

Proof: ($\rightarrow$) Obvious.

($\leftarrow$) Let $t = \sum_{s=0}^{\infty} \frac{t_s}{2^s}$, where $t_s \in \{0,1\}$. Suppose $t \notin \{\frac{k}{2^s} \mid s \geq 0,\ k \in \mathbf{Z}\}$. Then, there are infinitely many s such that

$$t_s = 1 \ \text{ and } \ t_{s+1} = 0.$$

On the other hand, there exists an integer $L \geq 1$ such that

$$|e^{2\pi i 2^s t} - 1| < \sqrt{2}, \quad \forall\, s \geq L.$$

So, one can get an s such that $t_s = 1$, $t_{s+1} = 0$, $s \geq L + 1$.

Then,

$$e^{2\pi i 2^{s-1} t} = e^{2\pi i \{\frac{1}{2} + \frac{t_{s+2}}{2^3} + \frac{t_{s+3}}{2^4} + \cdots\}}.$$

Hence

$$|1 - e^{2\pi i 2^{s-1} t}| > \sqrt{2}.$$

This is a contradiction. □

Lemma 2.4 ([1])**.** $T(\psi) = 2\pi\{\frac{k}{2^s} \mid k \in \mathbf{Z},\ s \geq 0\}$.

PROOF: Let $t = 2\pi \cdot \frac{k}{2^s}$. Then,

$$e^{it \sum_{j=1}^{n} X_j(x)} = e^{2\pi i \frac{k}{2^s} \sum_{j=1}^{M_{s-1}} X_j(x)}, \quad \forall\, n > M_{s-1}.$$

By Lemma 2.2, we see that $t \in T(\psi)$.

Conversely, let $t \in T(\psi)$. Then, again by Lemma 2.2, there exists a sequence of real numbers $\{a_{n,t}\}$ such that

$$\lim_{n \to \infty} e^{it \sum_{j=1}^{n} \{X_j(x) - a_{j,t}\}} \text{ exists a.e. } x.$$

Since $m(X_n(x) = 0) = \frac{1}{2}$, this implies

$$e^{ita_{n,t}} \to 1, \quad \text{as} \quad n \to \infty.$$

Hence,

$$e^{itX_n} \to 1 \quad \text{a.e.}$$

On the other hand

$$m(e^{itX_n} = e^{it2^s}) = \frac{1}{2}, \quad \text{if} \quad n \in I_s.$$

Therefore, $e^{it2^s} \to 1$ as $s \to \infty$. Thus by Lemma 2.3, we see that $t \in 2\pi\{\frac{k}{2^s} \mid k \in \mathbf{Z},\ s \geq 0\}$. □

Proposition 2.3. *The cocycle ψ is of type* III_0.

PROOF: It is known ([2]) that $T(\psi)$ coincides with the L^∞ spectrum of the associated flow of the cocycle ψ and hence by Lemma 2.4 we see that L^∞-spectrum of the associated flow is the set $2\pi\{\frac{k}{2^n} \mid n \geq 1,\ k \in \mathbf{Z}\}$. This implies that the flow is neither the translation of the real line nor periodic flow, that is, ψ is of type III_0 ([2]). □

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