# Almost coproducts of finite cyclic groups

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*Abstract.* A new class of $p$-primary abelian groups that are Hausdorff in the $p$-adic topology and that generalize direct sums of cyclic groups are studied. We call this new class of groups almost coproducts of cyclic groups. These groups are defined in terms of a modified axiom 3 system, and it is observed that such groups appear naturally. For example, $V(G)/G$ is almost a coproduct of finite cyclic groups whenever $G$ is a Hausdorff $p$-primary group and $V(G)$ is the group of normalized units of the modular group algebra over $Z/pZ$.

Several results are obtained concerning almost coproducts of cyclic groups including conditions on an ascending chain that implies that the union of the chain is almost a coproduct of cyclic groups.

*Keywords:* primary groups, coproduct of cyclic groups, almost coproducts, third axiom of countability

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## Introduction

In this paper we deal exclusively with torsion abelian groups, which are written additively. As is well known by an ancient theorem that rests on elementary number theory, any torsion abelian group is a coproduct (= direct sum) of primary groups. Thus we may assume that all groups considered here are $p$-primary for a fixed prime $p$.

If $G$ is a $p$-primary group, as usual we define $pG = \{px : x \in G\}$ and inductively we define $p^{n+1}G = p(p^nG)$. Recall that a subgroup $H$ of $G$ is said to be *pure* if $H \cap p^nG = p^nH$ for all $n < \omega$. The *$p$-adic topology* on $G$ has as a base for the neighborhoods of zero the subgroups $p^nG$ with $n < \omega$. All topological references herein are to the $p$-adic topology. Clearly, $G$ is Hausdorff (alternatively, $G$ is *without elements of infinite height* or, as in [F], $G$ is *separable*) if and only if $\cap p^nG = 0$, and a subgroup $H$ of $G$ is closed (in the $p$-adic topology) if and only if $G/H$ is Hausdorff. (In an effort to be less provincial, we are using terminology more commonly recognized by nonspecialists.)

As the title indicates, this paper is about groups that are almost coproducts of cyclic groups; this class of groups is precisely defined below. One thing that is required for a group to be almost a coproduct of cyclic groups is that the group be Hausdorff, but this condition alone is in no way sufficient except for countable groups.

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Henceforth, it is to be understood throughout that we are dealing exclusively with *p-primary abelian groups that are Hausdorff in the p-adic topology*, although we may repeat these conditions in our hypotheses from time to time for clarity and for emphasis.

**Preliminary results**

By a well-known theorem of Prüfer, if $G$ is countable then $G$ is a coproduct of cyclic groups [F]. Therefore, our interest will focus here on uncountable groups. Different criteria exist for (an uncountable group) $G$ to be a coproduct of cyclic groups, including a classical criterion established by Kulikov [K] in 1941. A more recent criterion (due to the author [H1]) is the existence of enough closed subgroups as stipulated by the *third axiom of countability*.

**Definition.** The group $G$ satisfies the third axiom of countability with respect to closed subgroups if there exists a collection $\mathcal{C}$ of closed subgroups that satisfy the following conditions:

(1) $0 \in \mathcal{C}$.

(2) The subgroup of $G$ generated by an arbitrary number of subgroups belonging to $\mathcal{C}$ again belongs to $\mathcal{C}$.

(3) If $B$ is a countable subgroup of $G$, there exists a countable subgroup $C$ belonging to $\mathcal{C}$ that contains $B$.

A collection $\mathcal{C}$ of closed subgroups of $G$ that satisfy conditions (1)–(3) is called an axiom 3 system (of closed subgroups) for $G$. As we indicated earlier, the following result is well known for a Hausdorff $p$-primary group $G$.

**Theorem 1** [H1]**.** *If $G$ has an axiom 3 system of closed subgroups, then $G$ is a coproduct of cyclic groups.*

Motivated by the preceding theorem, we say that $G$ is *almost a coproduct of cyclic groups* if $G$ possesses a collection $\mathcal{C}$ of closed subgroups that satisfy conditions (1) and (3) as well as the following weaker version of condition (2).

($2'$) The union of an ascending chain of subgroups belonging to $\mathcal{C}$ again belongs to $\mathcal{C}$.

A collection $\mathcal{C}$ of closed subgroups of $G$ that satisfy conditions (1), ($2'$), and (3) is called a *weak axiom 3 system* for $G$. In other words, a Hausdorff, $p$-primary group $G$ is almost a coproduct of cyclic groups, by definition, if and only if $G$ has a weak axiom 3 system of closed subgroups. Groups that are not Hausdorff having a weak axiom 3 system are studied in [HU2], but the results there do not substantially overlap those here. Indeed, the main results in [HU2] pertain to isotype subgroups of direct sums of countable groups, which is an important class of groups that have been carefully studied; see, for example [HM]. In the Hausdorff case, since a countable group is a direct sum of cyclics, (isotype) subgroups of direct sums of countable groups are in fact themselves direct sums of cyclic groups. Thus the theory of isotype subgroups, as a distinct class, evaporates in the Hausdorff case.

## Examples

We now give some broad classes of groups that are almost coproducts of cyclic groups.

(1) First, we have the trivial class, coproducts of cyclic groups. If $G = \oplus_{i \in I} G_i$ where $G_i$ is cyclic, then we can take $\mathcal{C}$ to be the collection of subgroups $G(J) = \oplus_{j \in J} G_i$ where $J$ denotes a subset of $I$. Clearly, $\mathcal{C}$ is a (weak) axiom 3 system of closed subgroups of $G$.

(2) The group of normalized units of modular group algebras. Suppose that $G$ is a Hausdorff $p$-group. Let $F(G)$ denote the group algebra over the field $F = \mathbb{Z}/p\,\mathbb{Z}$ or more generally over any countable perfect field $F$ of characteristic $p$. Let $U(G)$ denote the group of units of $F(G)$, and let $V(G)$ denote the subgroup of $U(G)$ consisting of those elements that have augmentation 1. The group of normalized units $V(G) \mod G$ is almost a coproduct of cyclic groups. Here, $\mathcal{C} = \{GV(H)/G : H \subseteq G\}$ is a weak axiom 3 system of closed subgroups of $V(G)/G$ according to [HU1].

(3) Hausdorff groups $G$ that have $v$-bases. The Hausdorff $p$-group $G$ is said to have a $v$-basis if there exists, for each nonnegative integer $n$, a set $B_n$ of representatives of the cosets of $p^{n+1}G$ in $p^n G$ for which each element $g$ in $G$ can be written as

$$g = b_{k(1)} + b_{k(2)} + \cdots + b_{k(m)}$$

where $k(1) < k(2) < \cdots < k(m)$ and where $b_{k(i)} \in B_{k(i)}$. If $G$ has a $v$-basis, a subgroup $H$ of $G$ is closed if $b_{k(1)}$, in the above representation, belongs to $H$ whenever $g$, itself, belongs to $H$; for a proof see [H2]. The collection $\mathcal{C}$ of all these (special) closed subgroups of $G$ is a weak axiom 3 system for $G$.

(4) Unions of a smooth ascending chain of coproducts of cyclic groups. One of the main results of this paper establishes the fact that the $p$-group $G$ is almost a coproduct of cyclic groups if $G$ is the union of a smooth ascending chain

$$G_0 \subseteq G_1 \subseteq \cdots \subseteq G_\alpha \subseteq \ldots$$

of pure and closed subgroups $G_\alpha$ having cardinality not exceeding $\aleph_1$ that are coproducts of cyclic groups. We also show that such chains exist for which $G$ is not a coproduct of cyclic groups.

## Main results

We begin with the following theorem, which is actually a consequence of well-known results, although not stated in the following specific form. This version is required for subsequent results herein, and we include it (with a short proof) for reference purposes.

**Theorem 2.** *A group $G$ that has a weak axiom 3 system of closed subgroups is a coproduct of cyclic groups provided that the cardinality of $G$ does not exceed $\aleph_1$.*

Proof: If $G$ has a weak axiom 3 system $\mathcal{C}$, then it is easy to see that $G$ has an axiom 3 system $\mathcal{C}'$ consisting of subgroups belonging to $\mathcal{C}$. Indeed, we can write $G$ as the union of a smooth ascending chain

$$0 = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_\alpha \subseteq \cdots \subseteq G = \bigcup G_\alpha$$

of countable subgroups $G_\alpha$ with $G_\alpha$ in $\mathcal{C}$. But $\mathcal{C}' = \{G\} \cup \{G_\alpha\}$, $\alpha < \omega_1$, satisfies axiom 3 for $G$. Since $\mathcal{C}'$ satisfies axiom 3, the fact that $G$ is a coproduct of cyclic groups is established by Theorem 1. $\square$

Obviously the concept of an axiom 3 system of closed subgroups of $G$, or that of a weak axiom 3 system, can be applied to an arbitrary subgroup property (such as purity). The following proposition is useful: its analogue for an axiom 3 system is known.

**Proposition 1.** *Let $\mathcal{C}_n$, $n < \omega$, be a weak axiom 3 system of subgroups of $G$ that satisfy some subgroup property $\mathcal{P}_n$ (where the properties $\mathcal{P}_n$ can be distinct or equal). Then $\mathcal{C} = \cap \mathcal{C}_n$ is a weak axiom 3 system of subgroups of $G$. Hence $G$ has a weak axiom 3 systems of subgroups that satisfy all the properties $\mathcal{P}_n$ simultaneously.*

Proof: Clearly, $\mathcal{C} = \cap \mathcal{C}_n$ satisfies conditions (1) and (2′). Thus we need only demonstrate that $\mathcal{C}$ also satisfies condition (3), which states that any countable subgroup $A$ of $G$ can be captured by a countable subgroup $C$ of $G$ that belongs to $\mathcal{C}$. Let $A_0 = A$. By hypothesis, there exists, for each $n$, a countable subgroup $B_{0,n}$ in $\mathcal{C}_n$ such that $A_0 \subseteq B_{0,n}$. Set $A_1 = \langle B_{0,n} \rangle_{n<\omega}$. Then $A_1$ is countable, and there exists a countable subgroup $B_{1,n}$ in $\mathcal{C}_n$ such that $A_1 \subseteq B_{1,n}$. Proceed inductively by choosing subgroups $B_{k,n}$ in $\mathcal{C}_n$ so that $A_k \subseteq B_{k,n}$, and then letting $A_{k+1} = \langle B_{k,n} \rangle_{n<\omega}$. Since $C = \bigcup_{k<\omega} A_k$ is equal to $B = \bigcup_{k<\omega} B_{k,n}$ for each $n$, we conclude that $C$ belongs to $\mathcal{C}_n$ for each $n$. Since $C$ is countable and belongs to $\mathcal{C}$, condition (3) is verified for the collection $\mathcal{C}$. $\square$

**Corollary 1.** *If $G$ has a weak axiom 3 system of closed subgroups, then $G$ has a weak axiom 3 system of subgroups that are both pure and closed in $G$.*

Proof: Every abelian group $G$ has a weak axiom 3 system of pure subgroups. Hence the corollary is an immediate consequence of the proposition. $\square$

Next, we shall establish for groups of cardinality $\aleph_1$ a new criterion for being a coproduct of cyclic groups. In the proof of this result we shall need to use the concept of compatibility.

**Definition.** Two subgroups $A$ and $B$ of $G$ are said to be *compatible*, and we write $A||B$, if for each $a \in A$ and $b \in B$, there exists $c$ in $A \cap B$ such that $|a + c| \geq |a + b|$.

Observe that compatibility is inductive in either variable (or both). Finally, in connection with the following theorem and subsequent results, recall that an ascending chain

$$A_0 \subseteq A_1 \subseteq \cdots \subseteq A_\lambda \subseteq \dots, \quad \lambda < \mu,$$

of subgroups indexed by an ordinal $\mu$ is *smooth* if $A_\beta = \bigcup_{\alpha<\beta} A_\alpha$ whenever $\beta$ is a limit ordinal less than $\mu$.

**Theorem 3.** *Let $G$ be a Hausdorff $p$-group of cardinality not exceeding $\aleph_1$. Then $G$ is a coproduct of cyclic groups if and only if $G$ is the union of a smooth ascending chain*

$$0 = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_\alpha \subseteq \dots, \quad \alpha < \mu,$$

*of pure and closed subgroups $G_\alpha$ that are themselves coproducts of cyclic groups.*

Proof: If $G$ is a coproduct of cyclic groups, the result is trivial. Hence, suppose that $G$ is the union of the smooth chain exhibited above of pure and closed subgroups $G_\alpha$. By Theorem 3 in [H2], $G$ is a coproduct of cyclic groups if $\mu$ is cofinal with $\omega_0$. Thus we may assume without loss of generality that $\operatorname{cof}(\mu) = \omega_1$; indeed, we may assume that $\mu = \omega_1$ since we can replace the original chain by a smooth cofinal subchain. For each $\alpha < \omega_1$, let $G_\alpha = \oplus_{\lambda<\mu(\alpha)} \langle g_{\alpha,\lambda} \rangle$ where $\mu(a)$ is a limit ordinal not exceeding $\omega_1$.

We plan to construct inductively a smooth chain of countable closed subgroups of $G$ leading up to $G$. Suppose that we have already chosen a smooth ascending chain

$$0 = C_0 \subseteq C_1 \subseteq \cdots \subseteq C_\alpha \subseteq \dots, \quad \alpha < \beta,$$

of countable closed subgroups $C_\alpha$ of $G$ that satisfy the following conditions.

(1) If $\alpha$ is a limit ordinal, $C_\alpha = \bigcup_{\gamma<\alpha} C_\gamma$.
(2) If $\alpha$ is isolated, $C_\alpha = \oplus_{\lambda<p(\alpha)} \langle g_{\alpha,\lambda} \rangle$ for some countable ordinal $p(\alpha)$.
(3) $C_\alpha || \, G_\gamma$ if $\gamma < \alpha$.
(4) If $\gamma < \alpha$, then $C_\alpha \cap G_\gamma = \oplus_{\lambda<\tau(\gamma,\alpha)} \langle g_{\gamma,\lambda} \rangle$ for some countable ordinal $\tau(\gamma, \alpha)$.
(5) $g_{\gamma,\delta} \in C_\alpha$ whenever $\delta < \mu(\gamma)$ and $\gamma$ and $\delta$ are less than $\alpha$.

In order to extend the chain, as usual we need to consider two cases.

**Case 1.** $\beta$ is a limit ordinal. In this case, we set $C_\beta = \bigcup_{\alpha<\beta} C_\alpha$. Clearly, $C_\beta$ is a countable subgroup of $G_\beta$, (2) is not relevant to limit ordinals $\beta$, and conditions (3)–(5) are inductive. It remains, however, to show that $C_\beta = \bigcup_{\alpha<\beta} C_\alpha$ is closed in $G_\beta$. Suppose that $C_\beta$ is not closed. If we choose $g$ in $\overline{C_\beta} \setminus C_\beta$, then

$$\sup\{|g - x| : x \in C_\beta\} = \infty;$$

more precisely, this sup is $\omega_0$. Since $G_\beta$ is closed, $\overline{C_\beta} \subseteq G_\beta$. Since $\beta$ is a limit and since $G_\beta = \bigcup_{\alpha<\beta} G_\alpha$, we know that $g \in G_\alpha$ for some $\alpha < \beta$. But $C_\beta || \, G_\alpha$, since

$C_\gamma || G_\alpha$ whenever $\alpha < \gamma < \beta$ by virtue of condition (3). Hence, for each $x \in C_\beta$, there exists $y \in C_\beta \cap G_\alpha$ for which $|g - y| \geq |g - x|$. This leads to the conclusion that

$$g \in \overline{C_\beta \cap G_\alpha}.$$

However, $C_\beta \cap G_\alpha = \oplus_{\lambda < \tau(\alpha,\beta)} \langle g_{\alpha,\lambda} \rangle$ is closed, and therefore $g \in C_\beta \cap G_\alpha$. But this is absurd since $g$ was chosen in $\overline{C_\beta} \setminus C_\beta$. This proves that $C_\beta$ is closed in $G$, and this case is finished.

**Case 2.** $\beta$ is isolated. Let $\beta = \delta + 1$. Again, taking advantage of the inductive properties of conditions (3)–(5), we can find a countable extension $C_\beta$ of $C_\delta$ in $G$ that satisfies all the conditions (1)–(5); the details of the back-and-forth procedure involved here are omitted. Observe that condition (2) implies that $C_\beta$ is closed in $G_\beta$.

In either of the two cases, we have completed the inductive construction of the desired chain of subgroups, and thereby we have shown that $G$ is the union of a smooth ascending chain of countable closed subgroups of $G$. Hence $G$ has an axiom 3 system of closed subgroups (consisting of the $C_\alpha$'s in the chain and $G$ itself). Therefore, $G$ is a coproduct of cyclic groups according to Theorem 1. □

**Corollary 2.** *Let $H_\alpha$ be an arbitrary Hausdorff $p$-group of cardinality $\aleph_1$, for $\alpha < \omega_2$. Then there exists a smooth ascending chain*

$$G_0 \subseteq G_1 \subseteq \cdots \subseteq G_\alpha \subseteq \ldots, \quad \alpha < \omega_2,$$

*of groups $G_\alpha$ that satisfy the following conditions.*

(i) *$G_\alpha$ is a coproduct of cyclic groups.*
(ii) *$G_\alpha$ is pure and closed in $G_{\alpha+1}$ for each $\alpha$; hence $G_\alpha$ is pure and closed in $G = \bigcup G_\alpha$.*
(iii) *$G_{\alpha+1}/G_\alpha = H_\alpha$.*

Proof: Let $G_0 = \oplus_{n<\omega} \oplus_{\aleph_1} C(p^n)$ be the coproduct of $\aleph_1$ cyclic groups of order $p^n$ for each $n < \omega$. Proceeding from $G_\alpha$ to $G_{\alpha+1}$, for any $\alpha < \omega_2$, under the guidelines of the specified conditions, is a simple application of the pure resolution of $H_\alpha$,

$$G_\alpha \rightarrowtail G_{\alpha+1} \twoheadrightarrow H_\alpha$$

(with a suitable coproduct of cyclic groups added, if necessary, to both $B$ and $C$ in the initial pure resolution $B \rightarrowtail C \twoheadrightarrow H_\alpha$). In the limit case, we take $G_\alpha = \bigcup_{\gamma<\alpha} G_\gamma$, and then apply Theorem 3. □

Motivated by the preceding corollary we define a class of groups as follows.

**Definition.** Define $\mathcal{A}$ to be the class of groups $G$ that can be expressed as the union of a smooth ascending chain of subgroups

$$G_0 \subseteq G_1 \subseteq \cdots \subseteq G_\alpha \subseteq \ldots\ , \quad \alpha < \mu,$$

indexed by an ordinal $\mu$, that satisfy the following conditions.

(1) $|G_\alpha| \leq \aleph_1$.
(2) $G_\alpha$ is a coproduct of cyclic groups for each $\alpha$.
(3) $G_\alpha$ is pure and closed in $G_{\alpha+1}$.

**Theorem 4.** *If $G \in \mathcal{A}$, then $G$ is almost a coproduct of cyclic groups.*

Proof: Condition (1) obviously implies that $|G| \leq \aleph_2$, and the index ordinal $\mu$ can be chosen to be less than or equal to $\omega_2$. If $|\mu| < \omega_2$, then $G$ has cardinality $\aleph_1$. In this case, $G$ is a coproduct of cyclic groups according to Theorem 3. Thus we may assume that $\mu = \omega_2$ and that $|G| = \aleph_2$.

For each $\alpha < \omega_2$, write $G_\alpha = \oplus_{i \in I(\alpha)} C_{\alpha,i}$ where $C_{\alpha,i}$ is cyclic and where $I(\alpha)$ is an index set of cardinality not exceeding $\aleph_1$. If $J \subseteq I(\alpha)$, we denote $\oplus_{j\in J} C_{\alpha,j}$ simply by $G_\alpha(J)$. Given the chain

$$G_0 \subseteq G_1 \subseteq \cdots \subseteq G_\alpha \subseteq \dots \quad (\alpha < \mu)$$

whose union is $G$ that satisfies the conditions (1)–(3), we define a function $\imath : G \to \mu$ by the following rule. If $g \in G$, then $\imath(g) = \alpha$ where $\alpha$ is the first ordinal for which $g \in G_\alpha$. Next, we define a class $\mathcal{C}$ of subgroups $H$ of $G$ as follows.

$$\mathcal{C} = \big\{ H \subseteq G : H || G_\alpha \ \text{ and } \ H \cap G_\alpha = G_\alpha(J) \ \text{ for some} \\ J \subseteq I(\alpha), \ \text{ whenever } \ \alpha = i(h) \ \text{ with } h \in H \big\}.$$

Clearly, for a fixed $\alpha$, $H || G_\alpha$ and $H \cap G_\alpha = G_\alpha(J)$ for some variable $J$ are both inductive. It follows that $\mathcal{C}$ is closed with respect to ascending unions. Trivially, $H = 0$ belongs to $\mathcal{C}$. Moreover, it can quickly be verified that if $B$ is any countable subgroup of $G$, there exists a countable subgroup $H$ belonging to $\mathcal{C}$ that contains $B$; the important point here is that $\imath(H)$ is countable (whenever $H$ is countable), and we only have a countable number of ordinals $\alpha = i(h)$ to consider in qualifying $H$ for membership in $\mathcal{C}$.

In order to demonstrate that $\mathcal{C}$ is a weak axiom 3 system and that $G$ is therefore an almost coproduct of cyclic groups, it remains only to prove that the members of $\mathcal{C}$ are in fact closed subgroups of $G$. Assume that $H \in \mathcal{C}$ is not closed in $G$. Choose $g \in \overline{H} \setminus H$. Since $g \in \overline{H}$, there exists $h_n \in H$ such that $|g - h_n| \geq n$ for each $n < \omega$. By replacing the sequence $h_n$ by a subsequence, if necessary, we can assume that $\imath(h_1) \leq \imath(h_2) \leq \cdots \leq \imath(h_n) \leq \dots$ . Indeed, this can be accomplished by choosing $k(1)$ so that $\imath(h_{k(1)}) = \min\{\imath(h_n)_{n<\omega}\}$, then by choosing $k(2) > k(1)$ so that $\imath(h_{k(2)}) = \min\{\imath(h_n) : n > k(1)\}$, and inductively choosing $k(j+1) > k(1)$ so that $\imath(h_{k(j+1)}) = \min\{\imath(h_n) : n > k(j)\}$. If the sequence $\imath(h_n)$ becomes constant, obviously we have that $g \in \overline{G}_\alpha = G_\alpha$. Indeed, we have that $g \in \overline{G_\alpha \cap H}$. But $G_\alpha \cap H = G_\alpha(J)$ is a direct summand of $G_\alpha$ and is therefore closed. Thus, we can conclude $g \in H$, which is a contradiction on the choice of $g$ in $\overline{H} \setminus H$. If the nondecreasing sequence $\imath(h_n)$ does not become constant, we may assume that $\imath(h_n)$ is strictly increasing (since repetitions can be deleted). Under this assumption, let $\beta = \sup \imath(h_n)$. We know that $g \in G_\beta = \bigcup G_\alpha$ where $\alpha < \beta$. Choose $\alpha$ so that $\alpha = \imath(h_n)$ for some $n$ and so that $g \in G_\alpha$. Now, since $\alpha = \imath(h_n)$, we know that $G_\alpha || H$ and $G_\alpha \cap H = G_\alpha(J)$ for some subset $J$ of $I(\alpha)$. But, as in the preceding case, this again leads to the conclusion that $g \in \overline{G_\alpha \cap H} = G_\alpha \cap H$. In any event, $H$ is closed, and we have finished the proof that $G$ is almost a coproduct of cyclic groups. □

The next corollary shows that there are many members of the class of groups $\mathcal{A}$ that are not coproducts of cyclic groups; such groups are nontrivial examples of almost coproducts of cyclic groups.

**Corollary 3.** *Let $G \in \mathcal{A}$ be the union of a smooth ascending chain*

$$G_0 \subseteq G_1 \subseteq \cdots \subseteq G_\alpha \subseteq \dots , \quad \alpha < \omega_2,$$

*where $G_\alpha$ is pure and closed in $G_{\alpha+1}$ and where each $G_\alpha$ is a coproduct of cyclic groups of cardinality not exceeding $\aleph_1$. If the set*

$$S = \{\alpha < \omega_2 : G_{\alpha+1}/G_\alpha = H_\alpha \ \text{ is not a coproduct of cyclic groups}\}$$

*is a stationary subset of $\omega_2$, then $G$ is almost a coproduct of cyclic groups but is not a coproduct of cyclic groups.* (*The existence of such a chain follows from Corollary* 2.)

By Theorem 4, $G$ is almost a coproduct of cyclic groups. Suppose that $S$ is stationary. A standard argument shows that $G$ is not a coproduct of cyclic groups. Indeed, assuming that $G = \oplus_{i \in I} C_i$ is a coproduct of cyclic groups $C_i$, we can show that the subset

$$\Lambda = \{\alpha < \omega_2 : G_\alpha = \oplus_{j \in J} C_j \ \text{ for some subset } J \text{ of } I\}$$

is closed and unbounded (*cub*). Hence $S \cap \Lambda \neq \emptyset$ since $S$ is stationary. But the existence of $\alpha$ in $S \cap \Lambda$ leads quickly to a contradiction because, on the one hand, $G_{\alpha+1}/G_\alpha = H_\alpha$ is not a coproduct of cyclic groups whenever $\alpha$ is in $S$. On the other hand, $G/G_\alpha = \oplus_{i \in I \setminus J} C_i$ is a coproduct of cyclic groups whenever $\alpha \in \Lambda$. Hence, $G_{\alpha+1}/G_\alpha$ is a coproduct of cyclic groups whenever $\alpha \in \Lambda$ because a subgroup of a coproduct of cyclic groups is again a coproduct of cyclic groups. Thus, we have a contradiction, and we have shown that $G$ is not a coproduct of cyclic groups after all.

We finish with some observations about closure properties of the class of almost coproducts of cyclic groups.

**Proposition 2.** *If $G$ is a Hausdorff $p$-group which is the countable extension of an almost coproduct of cyclic groups, then $G$ itself is almost a coproduct of cyclic groups.*

Proof: Let $G = H + C$ where $H$ is almost a coproduct of cyclic groups and $C$ is countable. Let $\mathcal{D}$ denote a weak axiom 3 system of closed subgroups of $H$. Set

$$\mathcal{D}' = \{D \in \mathcal{D} : D \supseteq C \cap H\}.$$

Finally, let $\mathcal{C} = \{0\} \cup \{C + D : D \in \mathcal{D}'\}$. We claim that the set $\mathcal{C}$ is closed with respect to ascending unions. To see that, notice that if $C + D_1 \subseteq C + D_2$ where $D_1$ and $D_2$ are in $\mathcal{D}'$ then $D_1 \subseteq D_2$ because $C \cap (D_1 + D_2) \subseteq C \cap H \subseteq D_2$. Therefore, an ascending chain

$$C + D_1 \subseteq C + D_2 \subseteq \cdots \subseteq C + D_\alpha \subseteq \dots$$

in $\mathcal{C}$ implies that the $D_\alpha$'s actually themselves ascend. Letting $D = \bigcup D_\alpha$, we conclude that $D$ belongs to $\mathcal{D}'$ since $D_\alpha \in \mathcal{D}'$ for each $\alpha$. Hence the union of the preceding ascending chain is $C + D$, which belongs to $C$.

Clearly, $\mathcal{C}$ has the property that every countable subgroup of $G$ is contained in a countable subgroup $C + D$ in $\mathcal{C}$. We need to show that each member of $\mathcal{C}$ is closed in $G$. Since $G$ is Hausdorff, we know that 0 is closed. Hence it remains only to show that $C + D$ is closed whenever $D \in \mathcal{D}'$. However, for such a subgroup $D$, we have that

$$G/(C{+}D) = (C{+}H)/(C{+}D) = (C{+}D{+}H)/(C{+}D) \simeq H/[(C{+}D){\cap}H] = H/D,$$

which is Hausdorff. Therefore, $C{+}D$ is indeed closed in $G$. We have demonstrated that $\mathcal{C}$ is a weak axiom 3 system of closed subgroups of $G$, and consequently $G$ is almost a coproduct of cyclic groups. □

The following proposition is dual to the preceding one, and its proof involves only the simple exercise of showing that $G$ has a weak axiom 3 system of closed groups based on the existence of such a system for the quotient group. The argument, however, relies heavily on the fact that the subgroup being extended is countable.

**Proposition 3.** *If $G$ is Hausdorff and is the extension of a countable group by an almost coproduct of cyclic groups, then $G$ itself must be almost a coproduct of cyclic groups.*

A special case of the above proposition of interest in the next corollary.

**Corollary 4.** *If $G$ is a pure extension of a countable Hausdorff group by an almost coproduct of cyclic groups, then $G$ is itself an almost coproduct of cyclic groups.*

Finally, we observe

**Proposition 4.** *A coproduct of groups that are almost coproducts of cyclic groups is again almost a coproduct of cyclic groups.*

We conclude with the following open problem.

**Problem.** If $G$ is almost a coproduct of cyclic groups, does a summand of $G$ have to be almost a coproduct of cyclic groups?

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