Normal integrands and related classes of functions

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Abstract. Let $D \subset T \times X$, where T is a measurable space, and X a topological space. We study inclusions between three classes of extended real-valued functions on D which are upper semicontinuous in x and satisfy some measurability conditions.

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1. Preliminaries

Throughout this paper $(T, \mathcal{T})$ is a measurable space, X a topological space, and D a subset of $T \times X$. For a set $A \subset T \times X$, $\operatorname{proj}_T A$ denotes the projection of A on T. We shall always assume that $\operatorname{proj}_T D = T$. We say that X is Souslin if it is a continuous image of a Polish space. By $\mathcal{B}(X)$ and $\mathcal{T} \otimes \mathcal{B}(X)$ we mean, respectively, the Borel σ-field on X and the product σ-field on $T \times X$. The set D is always considered with the trace σ-field $\mathcal{D} = \{D \cap A \mid A \in \mathcal{T} \otimes \mathcal{B}(X)\}$.

Let $\mathcal{R}$ be a family of sets. By $S(\mathcal{R})$ we denote the family of all sets obtained from $\mathcal{R}$ by the Souslin operation. If $S(\mathcal{R}) = \mathcal{R}$, we say $\mathcal{R}$ is a Souslin family. If the σ-field $\mathcal{T}$ is complete with respect to a σ-finite measure, then $\mathcal{T}$ is a Souslin family. We refer to Wagner [14] and Levin [10, Theorem D.7] for other sufficient conditions for $S(\mathcal{T}) = \mathcal{T}$.

We shall use the following projection theorem.

Theorem 1.1 ([4, Theorem 1.3], [10, Theorem D.3])**.** *Suppose $\mathcal{T}$ is a Souslin family and X is a Souslin space. Then $\operatorname{proj}_T A \in \mathcal{T}$ for each $A \in S(\mathcal{T} \otimes \mathcal{B}(X))$.*

Let $\psi : T \to \mathcal{P}(Y)$, where Y is a topological space and $\mathcal{P}(Y)$ is the family of all subsets of Y. The set-valued map ψ is measurable if

$$\psi^{-1}(V) = \{t \in T \mid \psi(t) \cap V \neq \emptyset\} \in \mathcal{T}$$

for each open $V \subset Y$ (note that Himmelberg [5] calls such a mapping weakly measurable).

By D_t we denote t-section of D, i.e. $D_t = \{x \in X \mid (t, x) \in D\}$, $t \in T$. The set D may be treated as a graph of the multifunction $t \to D_t$. We say that D has a Castaing representation if there exists a countable family U of measurable functions $u : T \to X$ such that for each $t \in T$, $u(t) \in D_t$ and the set $\{u(t) \mid u \in U\}$ is dense in D_t.

The set D has a Castaing representation provided one of the following conditions is satisfied:

(i) $D = T \times X$ and X is separable.

(ii) There is a countable subset $E \subset X$ such that $E \cap D_t$ is dense in D_t for $t \in T$, and $D^x = \{t \in T \mid (t,x) \in D\}$ belongs to $\mathcal{T}$ for $x \in E$.

(iii) X is a Souslin space, $\mathcal{T}$ is a Souslin family and $D \in S(\mathcal{T} \otimes \mathcal{B}(X))$ (see e.g. [10, Theorem D.4]).

(iv) X is separable and metrizable, D_t are complete, and the multifunction $t \to D_t$ is measurable (see [5, Theorem 5.6]).

Throughout this paper we deal with extended real-valued functions $f : D \to \mathbb{R} \cup \{-\infty\}$. By a set-valued map associated to f we mean $\phi : T \to \mathcal{P}(X \times \mathbb{R})$ defined by

$$\phi(t) = \{(x,r) \in X \times \mathbb{R} \mid (t,x) \in D \ \text{ and } \ f(t,x) \geq r\}.$$

Note that $\phi(t)$ is the subgraph of $f(t,\cdot)$. We say that such a function f is a Carathéodory integrand if it is finite, measurable (with respect to $\mathcal{D}$), and for each $t \in T$, $f(t,\cdot)$ is continuous on D_t. It is well known that if X has a countable base and $f : T \times X \to \mathbb{R}$ is measurable in t and continuous in x, then f is product measurable (i.e. f is a Carathéodory integrand).

We shall study inclusions between the following classes of functions:

$F_1(D) = \{f : D \to \mathbb{R} \cup \{-\infty\} \mid f$ is measurable and for each $t \in T$, $f(t,\cdot)$ is upper semicontinuous on $D_t\}$,

$F_2(D) = \{f : D \to \mathbb{R} \cup \{-\infty\} \mid f$ is the limit of a decreasing sequence of Carathéodory integrands$\}$,

$F_3(D) = \{f : D \to \mathbb{R} \cup \{-\infty\} \mid$ the set-valued map associated to f is measurable and for each $t \in T$, $f(t,\cdot)$ is upper semicontinuous on $D_t\}$.

Elements of $F_3(D)$ are called normal integrands (cf. Rockafellar [12]; note that in [7] we use a different terminology).

The study of these functional classes is motivated by their applications in optimization and mathematical economy. In particular, they appear when we deal with the following problem: Let f be a real-valued function on D. We ask under which assumptions the function

$$v(t) = \sup\{f(t,x) \mid x \in D_t\}, \quad t \in T, \tag{1.1}$$

is measurable. Suppose for each $t \in T$ this supremum is attained. Does there exist measurable $u : T \to X$ such that $u(t) \in D_t$ and $v(t) = f(t,u(t))$, $t \in T$? Such a function u is called an optimal measurable selection. The following theorem holds:

Theorem 1.2 ([13], [3]). *Suppose X is separable and metrizable. If the multifunction $t \to D_t$, $t \in T$, is measurable and compact-valued, and $f \in F_2(D)$, then there exists an optimal measurable selection.*

In general, the assumption $f \in F_2(D)$ cannot be replaced by the weaker condition $f \in F_1(D)$ (cf. [3]).

2. Main result

We start with two auxiliary lemmata. Remind that we have assumed $\operatorname{proj}_T D = T$.

Lemma 1. *Suppose D has a Castaing representation. If $A \subset D$ is such that $A \in \mathcal{D}$ and A_t is open in D_t for each $t \in T$, then $\operatorname{proj}_T A \in \mathcal{T}$.*

PROOF: Let U be a Castaing representation of D. Since A_t are open in D_t,

$$\operatorname{proj}_T A = \{t \in T \mid u(t) \in A_t \text{ for some } u \in U\} = \bigcup_{u \in U} \{t \in T \mid (t, u(t)) \in A\}.$$

The observation that the function from T to D given by $t \to (t, u(t))$ is measurable, completes the proof.

□

The next lemma is a slight generalization of a result from [8, Lemma], but for the sake of completeness we give its proof.

Lemma 2. *Let $f : D \to \mathbb{R} \cup \{-\infty\}$, and ϕ be the set-valued map associated to f. Then:*

(i) *If ϕ is measurable then the function v defined by* (1.1) *is measurable.*
(ii) *If f is a Carathéodory integrand and D has a Castaing representation, then f is a normal integrand.*
(iii) *If X is separable and metric, ϕ is measurable and $g : X \to \mathbb{R}$ is uniformly continuous, then the set-valued map ψ associated to h, $h(t,x) = f(t,x) - g(x)$, $(t,x) \in D$, is also measurable.*

PROOF: Observe that for any $V \subset X$, $a, b \in \mathbb{R}$, $a < b$, we have

$$\phi^{-1}(V \times (a,b)) = \phi^{-1}(V \times (a,\infty)) = \operatorname{proj}_T(f^{-1}((a,\infty)) \cap (T \times V)). \tag{2.1}$$

Now the assertion (i) follows from the equalities

$$v^{-1}((a,\infty)) = \{t \in T \mid f(t,x) > a \text{ for some } x \in D_t\} = \\ = \operatorname{proj}_T f^{-1}((a,\infty)) = \phi^{-1}(X \times (a,\infty)).$$

If $f(t,\cdot)$ is continuous, then the t-section of $f^{-1}((a,\infty) \cap (T \times V))$ is open in D_t for each open $V \subset X$. The application of Lemma 1 together with the equality (2.1) prove the assertion (ii).

In order to prove (iii), take for each $n \in \mathbb{N}$ a number $\delta_n > 0$ such that $|g(x) - g(y)| < \frac{1}{n}$ provided $d(x,y) < \delta_n$, where d is a metric on X. Let $E \subset X$ be countable and dense. It is not difficult to check that for open $V \subset X$ and $a \in \mathbb{R}$ we have

$$\{(t,x) \in D \cap (T \times V) \mid h(t,x) > a\} = \\ = \bigcup_{n \in \mathbb{N}} \bigcup_{e \in V \cap E} \Big\{(t,x) \in D \cap (T \times B(e,\delta_n)) \mid f(t,x) > g(e) + a + \frac{1}{n}\Big\},$$

where $B(e, \delta_n)$ is the open ball with center e and radius δ_n. This equality together with (2.1) imply the measurability of ψ, which completes the proof. □

The following theorem summarizes our knowledge of relations between classes $F_i(D)$, $i = 1, 2, 3$. Some of these inclusions were already known. We refer to Remark 2 for the comparison of our theorem with previous results.

Theorem 2.1. *Let X be separable and metrizable, and $D \subset T \times X$ such that $\operatorname{proj}_T D = T$. Then:*

(i) $F_3(D) \subset F_2(D) \subset F_1(D)$.

(ii) *If $\mathcal{T}$ is a Souslin family, X a Souslin space and $D \in S(\mathcal{T} \otimes \mathcal{B}(X))$, then $F_1(D) = F_2(D) = F_3(D)$.*

(iii) *If T and X are Polish spaces, $\mathcal{T} = \mathcal{B}(T)$ and $D \in S(\mathcal{T} \otimes \mathcal{B}(X))$, then $F_1(D) = F_2(D)$.*

(iv) *If X is σ-compact, and D has a Castaing representation and closed t-sections D_t, $t \in T$, then $F_2(D) = F_3(D)$.*

Proof: (i) The inclusion $F_2(D) \subset F_1(D)$ is obvious, thus we prove $F_3(D) \subset F_2(D)$. Let h be an increasing homeomorphism of $\mathbb{R} \cup \{-\infty\}$ and $[-1, 1)$. It is immediate that if $f \in F_3(D)$ then $h \circ f \in F_3(D)$. Similarly, if $g : D \to \mathbb{R}$ is a Carathéodory integrand such that $|g(t, x)| < 1$, $(t, x) \in D$, then $h^{-1} \circ g$ is a Carathéodory integrand too. Hence, it suffices to prove that any $f \in F_3(D)$ which satisfies $-1 \leq f(t, x) < 1$ is the limit of a decreasing sequence of Carathéodory integrands with values in the interval $(-1, 1)$.

We adopt the classical proof of the theorem of Baire on the approximation of a semicontinuous function by a monotone sequence of continuous ones (see e.g. [1, p. 390]). Let the functions $f_n : T \times X \to [-1, 1)$ and $g_n : T \times X \to (-1, 1)$ be defined by the formulae

$$f_n(t, x) = \sup\{f(t, y) - nd(x, y) \,|\, y \in D_t\},$$
$$g_n(t, x) = \max\Big\{f_n(t, x),\ -1 + \frac{1}{n}\Big\}, \quad n \in \mathbb{N},$$

where d is a metric compatible with the topology of X. By Lemma 2, the functions f_n are measurable in t. Consequently, g_n are also measurable in t. From the proof of the theorem of Baire we know that $g_n(t, \cdot)$ are continuous, and the sequence $g_n \,|\, D$ is convergent to f. Being measurable in t and continuous in x the functions g_n are product measurable. Hence, $g_n \,|\, D$ are also measurable. It means that $f \in F_2(D)$.

(ii) It suffices to prove that $F_1(D) \subset F_3(D)$. Note that under our assumptions $\mathcal{D} \subset S(\mathcal{T} \otimes \mathcal{B}(X))$. If $f \in F_1(D)$ then $f^{-1}((a, \infty)) \in \mathcal{D}$ for each $a \in \mathbb{R}$. Now (2.1) together with Theorem 1.1 imply the measurability of the set-valued map ϕ associated to f (cf. [10, Theorem D.6]).

(iii) This is a consequence of Theorem 3.1 from [7].

(iv) We prove the inclusion $F_2(D) \subset F_3(D)$. Any $f \in F_2(D)$ is the limit of a decreasing sequence $\{f_n \mid n \in \mathbb{N}\}$ of Carathéodory integrands. Denote by ϕ and ϕ_n, respectively, the set-valued maps associated to f and f_n. It is not difficult to check that

$$\phi(t) = \bigcap \{\phi_n(t) \mid n \in \mathbb{N}\}.$$

By Lemma 2 (ii), each ϕ_n is measurable (and closed-valued). Since X is σ-compact, it implies the measurability of ϕ ([5, Corollary 4.2]). It means that f is a normal integrand, which completes the proof. □

Remarks. 1. Theorem 2.1 is a generalization of the main result from [8], where we studied the case $D = T \times X$.

2. We shall discuss some previous results, but note that the definition of the normal integrand varies from paper to paper. An analogous result to (ii) for $D = T \times X$ was already given by Berliocchi and Lasry ([2, Theorem 2 and Theorem 2′]). In Theorem 2 they studied the case when T is a locally compact Polish space endowed with a Radon measure, and the corresponding properties of $f(t,\cdot)$ are required for almost all $t \in T$. Theorem 2′ for an abstract measure space was given without proof. Rockafellar ([12, Theorem 2A]) proved that $F_1(T \times \mathbb{R}^n) = F_3(T \times \mathbb{R}^n)$, under assumption that the σ-field $\mathcal{T}$ is complete. The equality $F_1(T \times X) = F_2(T \times X)$ was given by Pappas ([11, Corollary 1]) for the case, when $\mathcal{T}$ is complete and X is a locally compact Polish space. Levin ([9, Theorem 7]) gave the equality $F_2(T \times X) = F_3(T \times X)$ for compact X, but without proof. Related result to (ii) for $D = T \times X$ was obtained by Zygmunt ([15, Theorem 3.4]).

3. If there is a function $f : D \to \mathbb{R} \cup \{-\infty\}$ such that its associated set-valued map ϕ is measurable, then D is the graph of a measurable multifunction. In the proof of this fact we may assume that $-1 \le f(t,x) < 1$ for $(t,x) \in D$ (cf. the proof of (i)). Then for any open $V \subset X$,

$$\{t \in T \mid D_t \cap V \neq \emptyset\} = \{t \in T \mid (t,x) \in D \text{ for some } x \in V\} = \phi^{-1}(V \times \mathbb{R}) \in \mathcal{T}.$$

Hence, $t \to D_t$, $t \in T$, is a measurable multifunction.

3. Examples

In this section we give two examples which show that in general, the classes $F_i(D)$, $i = 1, 2, 3$, do not coincide.

Example 1. Recently the first author ([6]) gave an example of a non-Borel function $g : T \to [0,1]$ with the graph W being a G_δ-set in $T \times [0,1]$, where T is a coanalytic subset of the plane. It is based on the Sierpiński example from 1931. Let X be the interval $[0,1]$, $\mathcal{T} = \mathcal{B}(T)$ and $D = T \times X$. We show that $F_1(D) \neq F_2(D)$.

Let f be the characteristic function of the set W. It is obvious that $f \in F_1(D)$. We claim that f does not belong to $F_2(D)$. If not, there is a decreasing sequence of Carathéodory functions f_n, which converges to f. Replacing f_n by $\min\{f_n, 1\}$, we may assume that $0 \le f_n(t,x) \le 1$, $(t,x) \in D$, and $f_n(t,x) = 1$ for $(t,x) \in W$. Denote

$$\begin{aligned} A_n &= f_n^{-1}\Big(\Big(\frac{1}{2}, 1\Big]\Big), \quad B_n = f_n^{-1}\Big(\Big[\frac{1}{2}, 1\Big]\Big), \\ \overline{A}_n &= \{(t,x) \in T \times X \mid x \in \mathrm{cl}\,(A_n)_t\}. \end{aligned}$$

We have

$$W \subset A_n \subset \overline{A}_n \subset B_n, \quad n \in \mathbb{N}. \tag{3.1}$$

It is easy to see that

$$W = \bigcap\{B_n \mid n \in \mathbb{N}\}. \tag{3.2}$$

Since vertical sections of A_n are open in $[0,1]$, the set-valued map $t \to (A_n)_t$ is measurable. Indeed, for each open $V \subset X$,

$$\big\{t \in T \mid (A_n)_t \cap V \ne \emptyset\big\} = \mathrm{proj}_T(A_n \cap T \times V) \in \mathcal{T},$$

because of Lemma 1. Consequently, $\overline{A}_n$ is a graph of a measurable multifunction too. It follows from (3.1) and (3.2) that

$$W = \bigcap\big\{\overline{A}_n \mid n \in \mathbb{N}\big\}.$$

The intersection of countably many measurable multifunctions with compact values is a measurable set-valued map ([5, Theorem 4.1]). Hence W is a graph of a Borel function, which is a contradiction.

This example gives a negative answer to the question from [7]. Recently Burgess and Maitra [3] constructed a function $f \in F_1(T \times X)$, where X is a compact metric space, for which there is no optimal measurable selection. It follows from Theorem 1.2 that such a function does not belong to $F_2(T \times X)$.

Example 2. Let X be the set of irrationals, T the interval $[0,1]$, $\mathcal{T} = \mathcal{B}(T)$ and $D = T \times X$. Let $A \subset T \times X$ be closed and such that $\mathrm{proj}_T A$ is not Borel. Finally, let f be the characteristic function of A. It is immediate that $f \in F_1(D)$, and the function v corresponding to f by (1.1) is the characteristic function of $\mathrm{proj}_T A$. It follows from Lemma 2 (i) that $f \notin F_3(D)$. Thus $F_1(D) = F_2(D) \ne F_3(D)$.

Note that in Example 1 we have $F_1(D) \ne F_2(D) = F_3(D)$. Therefore it is interesting to construct a set D such that $F_1(D) \ne F_2(D) \ne F_3(D)$. It can be done by combining Examples 1 and 2; we omit the details.

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