Comment.Math.Univ.Carolin. 36,4 (1995)727–735

Analytic nonregular cocycles over irrational rotations

M. Lemańczyk*

Abstract. Analytic cocycles of type III_0 over an irrational rotation are constructed and an example of that type is given, where all corresponding special flows are weakly mixing.

Keywords: cocycle, special flow, weak mixing

Classification: 28D05

Introduction

Assume that $T : (X, \mathcal{B}, \mu) \longrightarrow (X, \mathcal{B}, \mu)$ is an ergodic automorphism of a standard Borel space. Each measurable function $f : X \longrightarrow \mathbf{R}$ is called a *cocycle*. In fact, the cocycle corresponding to f and a $\mathbf{Z}$-action of T is defined as $f^{(n)}(x) = \sum_0^{n-1} f(T^k x)$ if $n \geq 0$ and $f^{(n)}(x) = -\sum_n^{-1} f(T^k x)$ if $n < 0$. Let $\overline{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$ be the one-point Alexandroff compactification of $\mathbf{R}$. Then $r \in \overline{\mathbf{R}}$ is said to be an *extended essential value* of f (see [9]) if for each open neighbourhood $U(r)$ of r (in $\overline{\mathbf{R}}$) and an arbitrary set C of positive measure, there exists an integer n such that

$$\mu(C \cap T^{-n}C \cap \{x \in X : \ f^{(n)}(x) \in U(r)\}) > 0.$$

The set of extended essential values will be denoted by $\overline{E}(f)$. The set $E(f) = \overline{E}(f) \cap \mathbf{R}$ is called the set of *essential values* of f and it is a closed subgroup of $\mathbf{R}$. The skew product

$$T_f : (X \times \mathbf{R}, \tilde{\mathcal{B}}, \tilde{\mu}) \longrightarrow (X \times \mathbf{R}, \tilde{\mathcal{B}}, \tilde{\mu}), \quad T_f(x, r) = (Tx, f(x) + r)$$

is said to be a *cylinder flow*. Here by $\tilde{\mu}$ we denoted the product measure of μ and infinite Lebesgue measure λ on the line. A cylinder flow is ergodic iff $E(f) = \mathbf{R}$ ([9]); in this case the cocycle f will be called *ergodic*. A necessary condition for an integrable f to be ergodic is $\int_X f \, d\mu = 0$. One case, where T_f is far from being ergodic is the case of f *coboundary*, i.e. f equal to $g - gT$ for a certain measurable $g : X \longrightarrow \mathbf{R}$. One has f is a coboundary iff $\overline{E}(f) = \{0\}$. It may happen that $E(f)$ is $\{0\}$ but f is not a coboundary. According to [9], such cocycles are said to be *of type* III_0. A cocycle f is said to be *regular* if the quotient cocycle $f^* : X \longrightarrow \mathbf{R}/E(f)$ is a coboundary. It is not hard to see that

*Research partially supported by KBN grant 2 P301 031 07 (1994).

nonregular cocycles are exactly those of type III_0. It is also clear that a cocycle f is a coboundary if and only if $\infty \notin \overline{E}(f)$ (that is there exists a set B of positive measure and a compact set K in $\mathbf{R}$ such that $f^{(n)}(x) \in K$ whenever $x,\ T^n x \in B$).

In this note, we introduce a subclass of type III_0 cocycles namely those with unbounded gaps. This gives rise to a new condition for a real cocycle to be nonregular. Using it and the idea of almost analytic constructions from [7] we conclude with constructions of analytic cocycles of type III_0.

As an application, we give an answer to A. Katok's question. Given an ergodic automorphism T and a real zero mean cocycle f which is assumed to be in L^1, A. Katok in [4, Section 12.4] considers the special flows over T built under the function $f_{k,l} = f + 2\pi\alpha k + 2\pi l$ for any integers k and l (here we assume that $e^{2\pi i\alpha}$ is an eigenvalue of T and that $f_{k,l}$ is positive). If the cylinder flow T_f is ergodic then all these flows are weakly mixing. According to [2] an *ergodic* real cocycle is weakly mixing (meaning no L^∞-eigenfunctions for T_f) iff the only measurable solution $\xi : \mathbf{T} \longrightarrow \mathbf{T}$ to the equation

$$e^{2\pi i r f(x)} = c\frac{\xi(Tx)}{\xi(x)} \tag{1}$$

(where $r \in \mathbf{R}$, $|c| = 1$) exists for $r = 0$ (and then c must be an eigenvalue of T). Examples of ergodic weak mixing cocycles are contained in [2] (such are, for example, all ergodic squashable cocycles). A. Katok asks whether it is possible to have a nonergodic cocycle f such that (1) has no nontrivial solution. If $E(f) = r\mathbf{Z}$ with $r \neq 0$ then f is necessarily regular and it follows from [9] that f is cohomologous to a cocycle taking values in $r\mathbf{Z}$, whence we have a solution to (1) for $1/r$ and $c = 1$. Combining our methods of constructing nonregular cocycles and [7] we obtain however that there exist type III_0 analytic cocycles over certain irrational rotations for which there is no nontrivial solution of (1) and in particular, an answer to the Katok's question is obtained (see also [1], [3], from which one can deduce a similar answer but only for the case $c = 1$).

A complete discussion of the existence of smooth type III_0 cocycles over an irrational rotation is given in [10].

1. Type III_0 cocycles. A general condition

Let $\tau : (Y, \mathcal{C}, \nu) \longrightarrow (Y, \mathcal{C}, \nu)$ be an ergodic automorphism and $f : Y \longrightarrow \mathbf{R}$ a cocycle. Then f is called *a cocycle with unbounded gaps* if there exists a sequence of open intervals P_n such that $|P_n| \longrightarrow \infty$ and

$$\{f^{(k)}(y) :\ y \in Y, k \in \mathbf{Z}\} \cap P_n = \emptyset$$

for all $n \geq 1$.

Let $T : (X, \mathcal{B}, \mu) \longrightarrow (X, \mathcal{B}, \mu)$ be an ergodic automorphism. Denote by

$$\mathcal{R}(T) = \{(x, T^k x) :\ x \in X, k \in \mathbf{Z}\}$$

the *relation* generated by T. An *orbit cocycle* is any measurable map $\tilde{\varphi}:\mathcal{R}(T)\longrightarrow\mathbf{R}$ satisfying

$$\tilde{\varphi}(x,y)+\tilde{\varphi}(y,z)-\tilde{\varphi}(x,z)=0$$

for all $(x,y),(y,z)\in\mathcal{R}(T)$ ($\tilde{\varphi}(x,T^nx)=\varphi^{(n)}(x)$, for a measurable $\varphi:X\longrightarrow\mathbf{R}$ is an example of an orbit cocycle). If $B\in\mathcal{B}$ then put

$$\mathcal{R}_B(T)=\mathcal{R}(T)\cap B\times B.$$

The corresponding *restricted orbit cocycle* $\tilde{\varphi}_B$ is defined as

$$\tilde{\varphi}_B=\tilde{\varphi}|_{\mathcal{R}_B(T)}.$$

Lemma 1. *Let $\varphi:X\longrightarrow\mathbf{R}$ be a cocycle. If there exists $B\in\mathcal{B}$ such that the restricted orbit cocycle $\tilde{\varphi}_B$ is a cocycle with unbounded gaps then φ is of type III_0 provided that it is not a coboundary.*

PROOF: Suppose that $r\in\mathbf{R}\setminus\{0\}$ is an essential value of φ. Choose n so that there exists an integer l satisfying $lr\in P_n$. Now, $lr\in E(\varphi)$ so given $\varepsilon>0$ there exists N such that

$$\mu(B\cap T^{-N}B\cap[\varphi^{(N)}\in B(lr,\varepsilon)])>0$$

which leads to an easy contradiction with the fact that $\tilde{\varphi}_B$ has unbounded gaps. □

Proposition 1. *Suppose that $T:(X,\mathcal{B},\mu)\longrightarrow(X,\mathcal{B},\mu)$ is ergodic and that $\varphi:X\longrightarrow\mathbf{R}$ is a cocycle whose certain restriction has unbounded gaps. Then*

$$\sup\left(\{\mu(B):B\in\mathcal{B},\ \tilde{\varphi}_B\ \ \textit{has unbounded gaps}\}\right)=1.$$

PROOF: Fix $\varepsilon>0$ and let $C\in\mathcal{B}$, $\mu(C)>0$ be such that $\tilde{\varphi}_C$ has unbounded gaps with a sequence (P_n), $P_n=(a_n,b_n)$ of the corresponding intervals. Since T is ergodic, we can find $K\geq 1$ so that if we put $B_1=\bigcup_{i=0}^{K-1}T^iC$ then $\mu(B_1)>1-\varepsilon/2$. Consider $\varphi^{(s)}$, $s=-K,\dots,0,\dots,K$ as $2K+1$ measurable functions on X. Then we can find a constant $W>0$ and a set $Y\subset X$ such that $\mu(Y)>1-\varepsilon/2$ and

$$|\varphi^{(s)}(y)|\leq W\quad\text{for all}\quad y\in Y\quad\text{and}\quad s=-K,\dots,K. \tag{2}$$

Finally put $B=B_1\cap Y$. It remains to show that $\tilde{\varphi}_B$ has unbounded gaps. Suppose that $x,T^Nx\in B$. If $|N|\leq K$ then $|\varphi^{(N)}(x)|$ is simply bounded by W since $x\in Y$. We can hence suppose that $|N|>K$. We have $x\in T^iC$, $T^Nx\in T^jC$, where $0\leq i,j\leq K-1$. Since $|N|>K$, the signs of N and $N+i-j$ are the same and

$$\varphi^{(N)}(x)=\varphi^{(N+i-j)}(T^{-i}x)+\varphi^{(-i)}(x)-\varphi^{(-j)}(T^Nx),$$

where $z=T^{-i}x\in C$ and $T^{N+i-j}(z)=T^{N-j}x\in C$. Since at the same time $x,T^Nx\in Y$, it follows from (2) that $\tilde{\varphi}_B$ has unbounded gaps with a corresponding sequence $(Q_n)_{n\geq n_0}$, where $Q_n=(a_n+2W,b_n-2W)$ for n large enough. □

Remark 1. In the next section, we will construct some type III_0 cocycles, where the supremum in Proposition 1 is achieved (i.e. φ itself is a cocycle with unbounded gaps). In general however the supremum is not achieved (we will construct type III_0 analytic, hence continuous cocycles over irrational rotations and such cocycles cannot have unbounded gaps).

Notice that if a restriction $\tilde{\varphi}_B$ of a cocycle φ has unbounded gaps then for each cohomologous cocycle $\psi = \varphi + f - fT$ and $\varepsilon > 0$ we can find a subset $B_\varepsilon \subset B$ with $\mu(B_\varepsilon) > (1-\varepsilon)\mu(B)$ and such that $\tilde{\psi}_{B_\varepsilon}$ has unbounded gaps.

Remark 2. For $\varphi \in L^1(X,\mu)$ recurrent (i.e. φ of zero mean), T. Hamachi has found examples of type III_0 cocycles whose no restriction has unbounded gaps.

Notice moreover that since the notion of a cocycle is in fact a notion depending only on orbits of T, by a standard argument involving Dye theorem, we obtain that each ergodic automorphism admits a recurrent cocycle with unbounded gaps once there exists an ergodic automorphism with such a property. In the next section we slightly strengthen this observation.

2. Abstract constructions of nonregular cocycles

First construction. We will now present a detailed construction of a cocycle with unbounded gaps over T admitting a special sequence of Rokhlin towers. This can be directly applied to any irrational rotation by α, where α has unbounded partial quotients (see Appendix in [6]). An advantage of this kind of constructions is that if α is sufficiently fast approximated by rationals then cocycles similar to those presented below are cohomologous to smooth ones ([6], [7]).

Step 1. Given $a_1 \in \mathbf{R}^+$ and $n_1 \geq 2$, $n_1 \in \mathbf{N}$, denote

$$E_1 = \{0, \pm a_1, \ldots, \pm n_1 a_1\}.$$

Let $b_1 > 0$ be a number which is a multiple of any element of E_1.

Step 2. Given $a_2 \in \mathbf{R}^+$ and $n_2 \geq 2$, $n_2 \in \mathbf{N}$ satisfying certain additional conditions, denote

$$E_2 = \{e_1 \pm j a_2 :\ j = 0, \ldots, n_2,\ e_1 \in E_1\}.$$

We require that for each $e_1 \in E_1$ and $j = 1, \ldots, n_2$,

$$|e_1 \pm j a_2| > b_1.$$

Finally, fix a positive number b_2 which is a multiple of all elements of E_2.

Step $k+1$. Given $a_{k+1} \in \mathbf{R}^+$ and $n_{k+1} \geq 2$, $n_{k+1} \in \mathbf{N}$ satisfying certain additional conditions, denote

$$E_{k+1} = \{e_k \pm j a_{k+1} :\ j = 0, \ldots, n_{k+1},\ e_k \in E_k\}.$$

We require that for each $e_k \in E_k$ and $j = 1, \dots, n_{k+1}$,

$$|e_k \pm j a_{k+1}| > b_k. \tag{3}$$

Finally, fix a positive number b_{k+1} which is a multiple of all elements of E_{k+1}.

By the construction, we obtain that

$$E_1 \subset E_2 \subset \ldots \subset E_k \subset \ldots, \quad k \geq 1.$$

Moreover, in view of (3), for each $k \geq 2$, $a, b \in E_k$

$$|a - b| \geq \frac{1}{2} b_{k-1} \tag{4}$$

if $a \neq b$ and either $a \notin E_{k-1}$ or $b \notin E_{k-1}$.

Assume that $T : (X, \mathcal{B}, \mu) \longrightarrow (X, \mathcal{B}, \mu)$ is an ergodic automorphism of a standard probability Borel space. We assume that T admits a special sequence of Rokhlin towers

$$\mathcal{R}_k = \{I_k, TI_k, \dots, T^{q_k - 1} I_k\} \quad (k \geq 1),$$

where $\mu(\bigcup_{i=0}^{q_k-1} T^i I_k) \geq 1 - \varepsilon_k$ with $\varepsilon_k \longrightarrow 0$; moreover

$$I_k = J_0^{(k)} \cup J_1^{(k)} \cup \ldots \cup J_{n_k+1}^{(k)} \quad \text{(a disjoint union)},$$

where $T^{q_k} J_i^{(k)} = J_{i+1}^{(k)}$ for $i = 0, 1, \dots, n_k$. Furthermore, we assume that

$$I_{k+1} \subset J_0^{(k)}, \quad k \geq 1.$$

Definition of a real cocycle. For each $k \geq 1$ let $\varphi_k : X \longrightarrow \mathbf{R}$ be defined by the following formula

$$\varphi_k(x) = \begin{cases} 0 & x \in J_0^{(k)} \\ a_k & x \in J_1^{(k)} \cup \ldots \cup J_{n_k}^{(k)} \\ -n_k a_k & x \in J_{n_k+1}^{(k)} \\ 0 & \text{otherwise.} \end{cases}$$

Finally, put

$$\varphi(x) = \sum_{k \geq 1} \varphi_k(x), \quad x \in X.$$

Notice that φ_k's have disjoint supports, so φ is a well defined real cocycle. Denote

$$B_k = \bigcup_{i=1}^{q_k - 1} T^i (I_k \setminus (J_{n_k}^{(k)} \cup J_{n_k+1}^{(k)})).$$

Clearly, $\mu(B_k) \geq 1 - (\varepsilon_k + \frac{2}{n_k} + \frac{1}{q_k})$.

Proposition 2. *Assume that $n_k \longrightarrow \infty$. If (q_k) is a rigidity time for T (i.e. if $fT^{q_k} - f \longrightarrow 0$ in measure for each measurable $f : X \longrightarrow \mathbf{R}$) such that for each $k \geq 1$*

$$\sum_{i\geq 1} 2\frac{q_k}{q_{k+i}} < \frac{1}{2}$$

then φ is not a coboundary.

Proof: We have that $\varphi_k^{(q_k)} = a_k$ on B_k, while for all $i = 1, \dots, k-1$ and $x \in X$ we have $\sum_{i=1}^{k-1} \varphi_i^{(q_k)}(x) \in E_{k-1}$. Hence

$$|(\sum_{i=1}^{k} \varphi_i)^{(q_k)}(x)| \geq a_k - \sup E_{k-1} > b_{k-1} > 1 \quad \text{on} \quad B_k.$$

For each $i \geq 1$ the cocycle $\varphi_{k+i}^{(q_k)} = 0$ on a set of measure at least $1 - 2\frac{q_k}{q_k+i}$. Hence, because of our standing assumption $(\sum_{i\geq 1} \varphi_{k+i})^{(q_k)} = 0$ on a set of measure at least $1/2$. Since the measure of B_k tends to 1, $\varphi^{(q_k)}$ does not go to zero in measure, so φ cannot be a coboundary. □

Proposition 3. *Under the assumptions of Proposition 2, φ is of type III_0. In fact φ itself is a cocycle with unbounded gaps.*

Proof: A simple use of Borel-Cantelli lemma shows that given N for a.e. $x \in X$ there exists $k = k(x)$ such that $\varphi_{k+i}^{(N)}(x) = 0$ for all $i \geq 1$ (i.e. with probability 1, the trajectory $x, \dots, T^{N-1}x$ does not cross I_{k+i}). We have then that $\varphi^{(N)}(x) = \sum_{i=1}^{k} \varphi_i^{(N)}(x)$, whence $\varphi^{(N)}(x) \in E_k$. We have shown that $\varphi^{(N)}$ takes values only in $\bigcup_{k\geq 1} E_k$. It follows now from (4) that φ has unbounded gaps. □

Second construction. We assume now that $T : (X, \mathcal{B}, \mu) \longrightarrow (X, \mathcal{B}, \mu)$ is an ergodic automorphism. Let $a_1 \in \mathbf{R}^+$ and put $F_1 = \{-a_1, 0, a_1\}$. Suppose that sets $F_i = \{-a_i, 0, a_i\}$ with $a_i \in \mathbf{R}^+$, $i = 1, \dots, n$ are already defined. Choose $a_{n+1} \in \mathbf{R}^+$ so that

$$\inf_{e_n \in E_n} |\pm a_{n+1} - e_n| \geq n + 1, \tag{5}$$

where $E_n = F_1 + \ldots + F_n$. Let (h_n) be an increasing sequence of natural numbers such that

$$h_n(\frac{1}{h_{n+1}} + \frac{1}{h_{n+2}} + \ldots) \longrightarrow 0. \tag{6}$$

Given n, find a Rokhlin tower $\mathcal{R}_n$ of height h_n, i.e.

$$\mathcal{R}_n = (I_n, TI_n, \dots, T^{h_n-1}I_n)$$

with $\mu(\bigcup_{i=0}^{h_n-1} T^i I_n) > 1 - 1/2^n$. Finally define (a coboundary)

$$\varphi_n(x) = \begin{cases} a_n & \text{if } x \in T^{[h_n/2]} I_n \\ -a_n & \text{if } x \in T^{h_n-1} I_n \\ 0 & \text{otherwise.} \end{cases}$$

It follows from (6) that $\sum_{n\geq 1} \mu(\operatorname{supp} \varphi_n) < +\infty$ hence by Borel-Cantelli lemma the cocycle $\varphi(x) = \sum_{n\geq 1} \varphi_n(x)$ is well-defined and moreover for a.e. $x \in X$ and all $N \in \mathbf{Z}$, $\varphi^{(N)}(x) \in \bigcup_{k\geq 1} E_k$. Therefore, in view of (5), φ has unbounded gaps. Notice however that if we represent

$$\varphi^{([h_n/2])}(x) = \psi_1^{([h_n/2])}(x) + \psi_2^{([h_n/2])}(x),$$

where

$$\psi_1(x) = \sum_{k=1}^{n} \varphi_k(x) \quad \text{and} \quad \psi_2(x) = \sum_{k\geq n+1} \varphi_k(x)$$

then $\psi_1^{([h_n/2])}(x) \geq a_n - \sup E_{n-1} \geq n$ for x from a set of measure at least $1/3$ while

$$\operatorname{supp}(\psi_2^{([h_n/2])}) \subset \bigcup_{s=-[h_n/2]}^{[h_n/2]} T^s(\operatorname{supp} \varphi_{n+1} \cup \operatorname{supp} \varphi_{n+2} \cup \ldots).$$

Hence, $\mu(\operatorname{supp} \psi_2^{([h_n/2])}) \leq 2h_n(\frac{1}{h_{n+1}} + \frac{1}{h_{n+2}} + \ldots)$ and in view of (6) we conclude that $\varphi^{([h_n/2])}$ is bigger that n on a set of measure at least $1/4$ and therefore φ cannot be a coboundary.

Notice that if in addition

$$\sum_{n\geq 1} \frac{a_n}{h_n} < +\infty \tag{7}$$

then the φ which we construct is integrable. Therefore

Proposition 4. *For each ergodic automorphism T there exists a recurrent cocycle $\varphi \in L^1(X,\mu)$ which is of type III_0 and of unbounded gaps.*

□

Remark 3. We can easily strengthen the above result to each $L^p(X,\mu)$, $p < +\infty$, while for $p = +\infty$ it is no longer true (obviously, φ with zero mean bounded as a function cannot have unbounded gaps as a cocycle unless it is a coboundary). However, using well-known results concerning cohomology of L^1-cocycles with bounded ones (e.g. [4], see also [5]), we obtain that each ergodic T admits a recurrent cocycle bounded as a function and whose certain restriction has unbounded gaps.

3. Nonregular analytic cocycles over irrational rotations

In Construction 1 of the previous section there is a lot of freedom. Our idea was to construct φ as a series of coboundaries $\sum_{k\geq 1}\varphi_k$, in such a way that $\varphi^{(N)}$ takes values in $\bigcup_{k\geq 1} E_k$ and E_k is a set of the form $F_1+\ldots+F_k$, where F_s is a (finite) set of the values assumed by $\varphi_s^{(N)}$, $N\in\mathbf{Z}$. In such a construction if we know that the smallest nonzero value in F_k is much bigger than the sup E_{k-1} we are sure that φ is a cocycle with unbounded gaps. Therefore, if in Construction 1 we play with values

$$a_0^{(k)},\ldots,a_{n_k+1}^{(k)},$$

where $a_i^{(k)}$ is the constant value of φ_k on $J_i^{(k)}$, $i=0,\ldots,n_k+1$ with

$$\sum_{i=0}^{n_k+1} a_i^{(k)}=0,\quad a_0^{(k)}=0$$

then we will obtain a cocycle with unbounded gaps whenever the smallest nonzero value from $F_k:=\{\sum_{i=s}^{j}a_i^{(k)}\ :\ s,j\geq 0\}$ (if $j<s$ this sum is understood as the sum from s to n_k+1 and then from 0 to j) will be sufficiently big with respect to any number from E_{k-1}. Obviously many of the $a_s^{(k)}$'s can be equal to zero. It is now clear that a construction of nonregular cocycles can be carried out using the idea of an a.a.c.c.p. from [7] (with (q_k) a subsequence of denominators of an irrational number α). As a corollary, we obtain that

Corollary 1. *If α can be approximated sufficiently fast by rationals* (*so that irrational rotation by α admits an a.a.c.c.p. construction*) *then for the rotation by α there exists an analytic type III_0 cocycle.* □

In [7], there is a construction of an analytic real cocycle f such that the corresponding special flows are weakly mixing. In fact, it is clear from the proof of Proposition 3 of that paper that f itself is weakly mixing (*i.e.* (1) is satisfied). It is also easy to see that there is no special restriction on the growth of the parameters in the corresponding a.a.c.c.p.. This gives rise to the proof of the following proposition (which, in particular is an answer to the Katok's question).

Proposition 5. *There is an irrational rotation T and an analytic type III_0 cocycle f such that the only measurable solution $\xi:\mathbf{T}\longrightarrow\mathbf{T}$ to the equation*

$$e^{2\pi i r f(x)}=c\frac{\xi(Tx)}{\xi(x)}$$

exists only for $r=0$. □

Acknowledgments. The author thanks the referee and S. Sinelshchikov for remarks improving the paper.

References

[1] Aaronson J., Hamachi T., Schmidt K., *Associated actions and uniqueness of cocycles*, to appear in Proc. of Okayama Conference, 1992.

[2] Aaronson J., Lemańczyk M., Volný D., *A salad of cocycles*, preprint.

[3] Golodets V.Ya., Sinel'shchikov S.D., *Classification and structure of cocycles of amenable ergodic equivalence relations*, preprint.

[4] Katok A.B., *Constructions in Ergodic Theory*, unpublished lecture notes.

[5] Kočergin A.W., *On the homology of functions over dynamical systems*, Dokl. AN SSSR **281** (1976).

[6] Kwiatkowski J., Lemańczyk M., Rudolph D., *On the weak isomorphism of measure-preserving diffeomorphisms*, Isr. J. Math. **80** (1992), 33–64.

[7] Kwiatkowski J., Lemańczyk M., Rudolph D., *A class of cocycles having an analytic coboundary modification*, Isr. J. Math. **87** (1994), 337–360.

[8] Moore C.C., Schmidt K., *Coboundaries and homomorphisms for non-singular actions and a problem of H. Helson*, Proc. London Math. Soc. (3) **40** (1980), 443–475.

[9] Schmidt K., *Cocycles of Ergodic Transformation Groups*, Lect. Notes in Math. Vol. 1, Mac Millan Co. of India, 1977.

[10] Volný D., *Constructions of smooth and analytic cocycles over irrational circle rotations*, Comment. Math. Univ. Carolinae **36.4** (1995), 745–764.

Department of Mathematics and Computer Science, Nicholas Copernicus University, ul. Chopina 12/18, 87-100 Toruń, Poland

E-mail: mlem@mat.uni.torun.pl