Characterizing realcompact spaces as limits of approximate polyhedral systems

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Abstract. Realcompact spaces can be characterized as limits of approximate inverse systems of Polish polyhedra.

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1. Introduction

In a series of recent papers various classes of spaces were characterized as spaces which admit resolutions (commutative or approximate), consisting of special classes of polyhedra. By a polyhedron we mean the carrier of a simplicial complex, endowed with the CW-topology. For pseudocompact spaces this was achieved in [6], for Lindelöf and strongly paracompact spaces in [1], for n-dimensional spaces in [8] and [18] and for finitistic spaces in [12]. In this paper we obtain such characterizations for the class of realcompact spaces. In the literature one can already find two characterizations of realcompact spaces in terms of inverse systems. The first one characterizes realcompact spaces as limits of inverse systems of Lindelöf spaces ([15, Theorem 23]). The second one characterizes realcompact spaces as limits of inverse systems of Polish spaces ([3, Proposition 3.2.17]). Recall that a realcompact space can be defined as a space homeomorphic to a closed subspace of a product of copies of the real line $\mathbb{R}$. Since $\mathbb{R}$ is topologically complete and topological completeness is preserved under direct products and closed subsets, every realcompact space is topologically complete. A separable space is called Polish if it is completely metrizable. A regular space is called Lindelöf if its open coverings have countable refinements. Note that Polish spaces are Lindelöf ([2, Theorem 3.8.1]) and Lindelöf spaces are realcompact ([2, Theorem 3.11.12]). In both cases the inverse systems used to expand the space did not consist of polyhedra.

Our aim is to expand realcompact spaces into polyhedral inverse systems. However, these systems will be approximate inverse systems. The notions of an approximate inverse system and its limit were recently introduced by S. Mardešić, L. Rubin and T. Watanabe ([8], [11]). We state their basic definitions.

Definition 1.1. An approximate (inverse) system is a collection $\mathcal{X} = (X_a, \mathcal{U}_a, p_{aa'}, A)$ consisting of:

- a preordered set $A = (A, <)$ which is directed and unbounded;
- for each $a \in A$, a (topological) space X_a and a normal covering (mesh) $\mathcal{U}_a$ of X_a;
- for each comparable pair $a < a'$ in A, a (continuous) mapping $p_{aa'} : X_{a'} \to X_a$, $p_{aa} = 1_{X_a}$ is the identity mapping on X_a.

These data must also satisfy the following three conditions:

(A1) $(p_{aa'} p_{a'a''}, p_{aa''}) < \mathcal{U}_a$, whenever $a < a' < a''$;

(A2) $(\forall\, a \in A)\ (\forall \mathcal{U} \in Cov(X_a))\ (\exists\, a' > a)\ (\forall\, a_2 > a_1 > a')$ $(p_{aa_1} p_{a_1a_2}, p_{aa_2}) < \mathcal{U}$;

(A3) $(\forall\, a \in A)\ (\forall \mathcal{U} \in Cov(X_a))\ (\exists\, a' > a)\ (\forall\, a'' > a')\ \mathcal{U}_{a''} < p^{-1}_{aa''}\,\mathcal{U}$.

Here, for any two mappings $f, g : X \to Y$ and any covering $\mathcal{V}$ of Y, $(f, g) < \mathcal{V}$ means that, for every $x \in X$, there exists a $V \in \mathcal{V}$ such that $f(x), g(x) \in V$. For coverings $\mathcal{U}$, $\mathcal{U}'$ of X, $\mathcal{U}' < \mathcal{U}$ means that $\mathcal{U}'$ refines $\mathcal{U}$.

A normal (also called numerable) covering of a space X is any open covering of X which admits a subordinate partition of unity. The set of all normal coverings of X is denoted by $Cov(X)$.

Definition 1.2. An approximate map q from a space Y into an approximate system $\mathcal{X}$, $q : Y \to \mathcal{X}$, is any collection $q = \{q_a \,|\, a \in A\} = (q_a)$ of mappings $q_a : Y \to X_a$ (called projections) such that:

(AS) For every $a \in A$ and every $\mathcal{U} \in Cov(X_a)$ there exists an $a' > a$ such that $(q_a, p_{aa''} q_{a''}) < \mathcal{U}$, whenever $a'' > a'$.

Definition 1.3. An approximate map $p = (p_a) : X \to \mathcal{X}$ is called a limit of $\mathcal{X}$ provided it has the following universal property:

(UL) For any approximate map $q : Y \to \mathcal{X}$ there exists a unique mapping $q : Y \to X$ satisfying $p_a g = q_a$, for every $a \in A$.

Since a limit space X is determined up to a unique homeomorphism, we often speak of the limit X of $\mathcal{X}$ and we write $X = \lim \mathcal{X}$.

Let POL denote the collection of all polyhedra.

Definition 1.4. An approximate resolution of a space X is an approximate map $p : X \to \mathcal{X} = (X_a, \mathcal{U}_a, p_{aa'}, A)$ satisfying the following two conditions:

(R1) $(\forall\, P \in \mathrm{POL})\ (\forall\, \mathcal{V} \in Cov(P))\ (\forall\, f : X \to P)\ (\exists\, a \in A)\ (\forall\, a' > a)$ $(\exists\, g : X_{a'} \to P)\ (g p_{a'}, f) < \mathcal{V}$;

(R2) $(\forall\, P \in \mathrm{POL})\ (\forall\, \mathcal{V} \in Cov(P))\ (\exists\, \mathcal{V}' \in Cov(P))\ (\forall\, a \in A)$ $(\forall\, g, g' : X_a \to P)\ (g p_a, g' p_a) < \mathcal{V}' \Rightarrow (\exists\, a' > a)\ (\forall\, a'' > a')$ $(g p_{aa''}, g' p_{aa''}) < \mathcal{V}$.

An approximate resolution of a space X can be characterized by conditions of a different kind. Instead of (R1) and (R2), which are often difficult to verify, more convenient are the following two equivalent conditions ([11, Theorem 2.8]).

(B1)* $(\forall \mathcal{U} \in Cov(X))\ (\exists\, a \in A)\ (\exists\, \mathcal{V} \in Cov(X_a))\ p_a^{-1}\mathcal{V} < \mathcal{U}$.
(B2)* $(\forall\, a \in A)\ (\forall \mathcal{U} \in Cov(X_a))\ (\exists\, a' > a)\ p_{aa'}(X_{a'}) \subseteq st\,(p_a(X),\mathcal{U})$.

Now we can state our main results.

Theorem 1.5. *For a paracompact space X the following statements are equivalent.*

(i) *X is Lindelöf.*
(ii) *X admits an approximate resolution $p : X \to \mathcal{X} = (X_a, \mathcal{U}_a, p_{aa'}, A)$, where all X_a are Polish polyhedra, all bonding maps $p_{aa'}$ are PL-mappings and all projections p_a are surjections.*
(iii) *X admits an approximate resolution $p : X \to \mathcal{X} = (X_a, \mathcal{U}_a, p_{aa'}, A)$, where all X_a are Lindelöf polyhedra.*

Theorem 1.6. *For a topological space X the following statements are equivalent.*

(i) *X is realcompact.*
(ii) *X is the limit of an approximate inverse system $\mathcal{X} = (X_a, \mathcal{U}_a, p_{aa'}, A)$, where all X_a are Polish polyhedra and all bonding maps $p_{aa'}$ are PL.*
(iii) *X is the limit of an approximate inverse system $\mathcal{X} = (X_a, \mathcal{U}_a, p_{aa'}, A)$, where all X_a are realcompact polyhedra.*

Since each paracompact space X is topologically complete, every polyhedral approximate resolution $p : X \to \mathcal{X} = (X_a, \mathcal{U}_a, p_{aa'}, A)$ of X is a limit of $\mathcal{X}$ ([11, Theorem 3.1]). Therefore an approximate resolution in Theorem 1.5 is also a limit.

2. Polish, Lindelöf and realcompact polyhedra

For a complex $\mathbb{K}$, let $\mathbb{K}^0$ denote the set of all vertices of $\mathbb{K}$.

Proposition 2.1. *Let $P = |\mathbb{K}|$ be a polyhedron. Then the following statements are equivalent.*

(i) *$\mathbb{K}^0$ is countable.*
(ii) *P is separable.*
(iii) *P is Lindelöf.*

Proof: (i) $\Rightarrow$ (ii). If $\mathbb{K}^0$ is countable, then $|\mathbb{K}|$ is a countable union of simplices. Since each simplex is a separable space we obtain (ii).

(ii) $\Rightarrow$ (iii) is obvious, since each separable paracompact space is Lindelöf ([9, Corollary 3, Appendix 1] and [2, Corollary 5.1.26]).

(iii) $\Rightarrow$ (i). Suppose the contrary, i.e. that $P = |\mathbb{K}|$ is Lindelöf, but $\mathbb{K}^0$ is uncountable. Then the open covering $\mathcal{S} = \{st\,(v,\mathbb{K}) : v \in \mathbb{K}^0\}$ of P, formed by all stars of the vertices of $\mathbb{K}$, has a countable refinement $\mathcal{U}$. This implies the existence of a member $U \in \mathcal{U}$, which contains an uncountable subset of $\mathbb{K}^0$. Since $\mathcal{U}$ refines $\mathcal{S}$ and each member of $\mathcal{S}$ contains only one vertex of $\mathbb{K}$, we obtain a contradiction. $\square$

Proposition 2.2. *A polyhedron $P = |\mathbb{K}|$ is Polish if and only if $\mathbb{K}$ is countable and locally finite.*

Proof: Let $P = |\mathbb{K}|$ be Polish. Proposition 2.1 implies that $\mathbb{K}$ is a countable set. On the other hand, metrizability of P implies that $\mathbb{K}$ is locally finite ([9, Example 1, Appendix 1]).

Suppose now that $\mathbb{K}$ is countable and locally finite. Then $P = |\mathbb{K}|$ is separable and metrizable. The standard metric on $|\mathbb{K}|$ is defined by using the barycentric coordinates ([9, Appendix 1, § 1.3]). Since $\mathbb{K}$ does not contain any "infinite simplex", this metric is complete ([5, Chapter 3, Lemma 11.5]), which shows that P is Polish. $\square$

Note that each Polish polyhedron is itself a Polish ANR, i.e. a completely metrizable separable ANR. Therefore the family pPOL of all Polish polyhedra is contained in the family pANR of all Polish ANR's.

Remark 2.3. Let $P = |\mathbb{K}|$ be a separable polyhedron and let $\mathcal{U} \in Cov(P)$ be an open covering of P. Let $\mathbb{L}$ be a subdivision of $\mathbb{K}$ such that the closed stars $\overline{st\,(v, \mathbb{L})}$ of the vertices v of $\mathbb{L}$ refine $\mathcal{U}$, i.e. $\overline{\mathcal{S}} = \{\overline{st\,(v, \mathbb{L})} : v \in \mathbb{L}^0\} < \mathcal{U}$ ([9, Theorem 4, Appendix 1]). Since $\mathbb{L}$ is a subdivision of $\mathbb{K}$ and card $\mathbb{K}^0 \leq \omega$, it follows that card $\mathbb{L} \leq \omega$ and therefore $|\mathbb{L}|_m$ (the carrier of $\mathbb{L}$ with the standard metric topology) is a separable ANR ([5, Chapter 3, Theorem 11.3 and Lemma 11.4]). Let $i : P \to |\mathbb{L}|_m = Q$ be the identity mapping. It is well known that i has a homotopy inverse $j : Q \to P$ such that $(ji, id_P) < \overline{\mathcal{S}} < \mathcal{U}$ ([9, Theorem 10, Appendix 1]). This shows that the family sPOL of all separable polyhedra is approximately dominated by the family sANR of all separable ANR's ([11, p. 599]). On the other hand, since each open covering of a Lindelöf space has a countable star-finite refinement ([14, Chapter 5, 4B]), it can be proved that the family sANR is approximately dominated by the family pPOL. Therefore the families sPOL, sANR, pPOL and pANR are approximately equivalent.

Proposition 2.4. *A polyhedron $P = |\mathbb{K}|$ is realcompact if and only if $\tau =$ card $(\mathbb{K}^0)$ is a non-measurable cardinal number.*

Proof: Let $P = |\mathbb{K}|$ be realcompact. Since $\mathbb{K}^0$ is a closed subspace of P, it follows that $\mathbb{K}^0$ is also realcompact ([2, Theorem 3.11.4]). Therefore, $\mathbb{K}^0$ is discrete realcompact space. However, a discrete space is realcompact if and only if its cardinal is non-measurable ([4, Theorem 12.2]).

Conversely, let τ be non-measurable. A paracompact space X is realcompact if and only if each closed discrete subspace of X is realcompact ([13]). Since each polyhedron is paracompact, it is sufficient to prove that each closed discrete subspace of P is realcompact. First note that card $\mathbb{K} = \tau$. Let $D \subseteq P$ be any closed discrete subspace of P. For each simplex σ of $\mathbb{K}$, $D \cap \sigma$ is a closed discrete subspace of σ and card $(D \cap \sigma)$ is finite. Therefore, card $(D \cap \text{Int } \sigma)$ is also finite. Note that D is the disjoint union $D = \bigcup_{\sigma \in \mathbb{K}} (D \cap \text{Int } \sigma)$. Therefore, card D is a non-measurable sum of finite cardinals, which is again a non-measurable cardinal ([4, Theorem 12.5]). Since card D is non-measurable, it follows that D is realcompact. □

Remark 2.5. Since ω is a non-measurable cardinal ([4, Chapter 12]), one can view Proposition 2.4 as a generalization of Proposition 2.1.

3. Approximate resolutions mod sPOL

In this section we introduce the notion of an approximate resolution modulo a subclass $\mathcal{P}$ of POL, by restriction conditions (R1) and (R2) by replacing the condition $P \in$ POL by $P \in \mathcal{P}$. We obtain results analogous to those of [11, § 2 and § 3] for $\mathcal{P} =$ POL.

Definition 3.1. Let $\mathcal{P} \subseteq$ POL be a subclass of POL. An approximate resolution mod $\mathcal{P}$ of a space X is an approximate map $p : X \to \mathcal{X} = (X_a, \mathcal{U}_a, p_{aa'}, A)$ satisfying the following two conditions:

(R1) $(\forall P \in \mathcal{P})$ $(\forall \mathcal{V} \in Cov(P))$ $(\forall f : X \to P)$ $(\exists a \in A)$ $(\forall a' > a)$
$(\exists g : X_{a'} \to P)$ $(gp_{a'}, f) < \mathcal{V}$;

(R2) $(\forall P \in \mathcal{P})$ $(\forall \mathcal{V} \in Cov(P))$ $(\exists \mathcal{V}' \in Cov(P))$ $(\forall a \in A)$
$(\forall g, g' : X_a \to P)$ $(gp_a, g'p_a) < \mathcal{V}' \Rightarrow (\exists a' > a)$ $(\forall a'' > a')$
$(gp_{aa''}, g'p_{aa''}) < \mathcal{V}$.

If $\mathcal{C}$ is a class of spaces and all X_a, $a \in A$, belong to $\mathcal{C}$, we speak of an approximate $\mathcal{C}$-resolution mod $\mathcal{P}$.

If $\mathcal{P} =$ POL, Definition 3.1 coincides with the definition of an approximate resolution in [11]. An approximate resolution is an approximate resolution mod $\mathcal{P}$, for all classes $\mathcal{P} \subseteq$ POL, and therefore we can consider approximate resolutions mod $\mathcal{P}$ as a weakening of the notion of approximate resolution. Let $s\,Cov(X) \subseteq Cov(X)$ denote the family of all normal coverings of a space X, which have a *countable* normal refinement. If $\mathcal{U} \in s\,Cov(X)$, then there exists a refinement $\mathcal{V} \in s\,Cov(X)$ of $\mathcal{U}$ such that $|\mathbb{N}(\mathcal{V})| \in s$POL, where $\mathbb{N}(\mathcal{V})$ denotes the nerve of $\mathcal{V}$ and $|\mathbb{N}(\mathcal{V})|$ denotes its geometric realization.

Approximate resolutions mod sPOL can be also characterized by conditions of a different nature.

(sB1)* $(\forall \mathcal{U} \in s\,Cov(X))$ $(\exists a \in A)$ $(\exists \mathcal{V} \in Cov(X_a))$ $p_a^{-1}\mathcal{V} < \mathcal{U}$.

(B2)* $(\forall a \in A)$ $(\forall \mathcal{U} \in Cov(X_a))$ $(\exists a' > a)$ $p_{aa'}(X_{a'}) \subseteq st\,(p_a(X), \mathcal{U})$.

Theorem 3.2. *An approximate map $p : X \to \mathcal{X} = (X_a, \mathcal{U}_a, p_{aa'}, A)$ is an approximate resolution* mod sPOL *of the space X if and only if it satisfies* (sB1)* *and* (B2)*.

Proof: ((R1) for sPOL) $\Rightarrow$ (sB1)* and ((sB1)* and (B2)*) $\Rightarrow$ ((R1) for sPOL) are proved in the similar way as Lemma 2.11 and Lemma 2.13 in [11]. Since (B2)* $\Leftrightarrow$ ((R2) for POL) we need only to establish ((R2) for sPOL) $\Rightarrow$ (B2)*. From the proof of Lemma 2.12 in [11], it follows that ((R2) for $\{I = [0,1]\}$) $\Rightarrow$ (B2)*. Since $I \in s$POL, it follows that also ((R2) for sPOL) $\Rightarrow$ (B2)*. Note also that ((R2) for sPOL) $\Leftrightarrow$ ((R2) for POL). □

Theorem 3.3. *Let a Tychonoff space X be complete with respect to the uniformity generated by all countable normal coverings of X. If $p : X \to \mathcal{X} = (X_a, \mathcal{U}_a, p_{aa'}, A)$ is an approximate resolution* mod sPOL *such that all spaces X_a are Tychonoff spaces, then p is a limit of $\mathcal{X}$.*

Proof: The proof is a modification of the proof of Theorem 3.1 in [11], which uses Theorem 3.2. □

Corollary 3.4. *If $p : X \to \mathcal{X} = (X_a, \mathcal{U}_a, p_{aa'}, A)$ is an approximate resolution* mod sPOL *such that all spaces X_a are Tychonoff spaces and X is a realcompact space, then p is a limit of $\mathcal{X}$.*

Proof: A Tychonoff space is realcompact if and only if it is complete with respect to the uniformity generated by all countable normal coverings ([16, Theorem 1]). Now, the assertion follows from Theorem 3.3. □

As we just saw, realcompact spaces and approximate resolutions mod sPOL are analogues of topologically complete spaces and approximate resolutions.

4. Approximate semi-projection mod $\mathcal{P}$

Recently, T. Watanabe introduced the notion of an approximate semi-projection of a topological space ([18]). Here, we introduce the notion of an approximate semi-projection modulo a subclass $\mathcal{P}$ of POL.

Definition 4.1. Let X be a topological space, $\mathcal{P}$ a subclass of POL and let $\mathcal{F}$ be a class of mappings $f : X \to P$, where $P \in \mathcal{P}$. We call $\mathcal{F}$ an approximate semi-projection mod $\mathcal{P}$ of X provided for any mapping $f' : X \to P'$, $P' \in \mathcal{P}$, and any open covering $\mathcal{U} \in Cov(P')$, there exist a member $f : X \to P$ of $\mathcal{F}$ and a mapping $p : P \to P'$ such that $(pf, f') < \mathcal{U}$.

An approximate semi-projection $\mathcal{F}$ mod POL of X is just an approximate semi-projection in the sense of Watanabe.

Definition 4.2. Let $\mathcal{P}$ be a subclass of POL. We say that $\mathcal{P}$ is closed if it is approximately equivalent to a class $\mathcal{C}$ of topological spaces, which is closed with respect to finite products, i.e. $C_1, \dots, C_n \in \mathcal{C}$ implies $C_1 \times \cdots \times C_n \in \mathcal{C}$.

POL is approximately equivalent to ANR, so POL is closed. sPOL is also closed, since it is approximately equivalent to sANR.

Proposition 4.3. *Let $\mathcal{F}$ be an approximate semi-projection* mod $\mathcal{P}$ *of X and let $\mathcal{P}$ be closed. Then for any $n \in \mathbb{N}$, any mappings $f_i : X \to P_i$, $P_i \in \mathcal{P}$, $i = 1, \dots, n$, and any open coverings $\mathcal{U}_i \in Cov(P_i)$, $i = 1, \dots, n$, there exists a member $f : X \to P$ of $\mathcal{F}$ and mappings $p_i : P \to P_i$, $i = 1, \dots, n$, such that $(p_i f, f_i) < \mathcal{U}_i$ for every i. Furthermore, it is possible to achieve that p_i are PL-mappings.*

Proof: For every $i = 1, \dots, n$, choose an open covering $\mathcal{V}_i \in Cov(P_i)$ such that $st^2 \mathcal{V}_i < \mathcal{U}_i$. Here, for any open covering $\mathcal{V}$ of a topological space X, $st\,\mathcal{V}$

denotes the open covering of X consisting of open stars $st\,(V,\mathcal{V})$, $V \in \mathcal{V}$ and $st^2\mathcal{V} = st\,(st\,\mathcal{V})$. Let $\mathcal{P}$ be approximately equivalent to $\mathcal{C}$, which is closed with respect to finite products. Then, for every $i = 1,\dots,n$, there exist a $Q_i \in \mathcal{C}$ and mappings $\alpha_i : P_i \to Q_i$, $\beta_i : Q_i \to P_i$ such that $(\beta_i\alpha_i, id_{P_i}) < \mathcal{V}_i$. Put $Q = Q_1 \times \cdots \times Q_n \in \mathcal{C}$. Then there exist a $P' \in \mathcal{P}$ and mappings $\alpha : Q \to P'$ and $\beta : P' \to Q$ such that $(\beta\alpha, id_Q) < \mathcal{W}$, where $\mathcal{W}$ is an open covering of Q such that $st\,\mathcal{W}$ refines the covering $\beta_1^{-1}\mathcal{V}_1 \times \cdots \times \beta_n^{-1}\mathcal{V}_n \in Cov(Q)$. Let $g = \alpha(\alpha_1 f_1 \times \cdots \times \alpha_n f_n) : X \to P'$. Since $\mathcal{F}$ is an approximate semi-projection mod $\mathcal{P}$, there exist a member $f : X \to P$ of $\mathcal{F}$ and a mapping $p : P \to P'$, such that $(pf, g) < \beta^{-1}\mathcal{W}$. Put $p_i = \beta_i q_i \beta p : P \to P_i$, where $q_i : Q \to Q_i$ denotes the i-th projection.

Claim. $(p_i f, f_i) < \mathcal{U}_i$.

First note that $(\beta p f, \beta\alpha(\alpha_1 f_1 \times \cdots \times \alpha_n f_n)) < \mathcal{W}$. Since $(\beta\alpha, id_Q) < \mathcal{W}$, it follows that $(\beta p f, \alpha_1 f_1 \times \cdots \times \alpha_n f_n) < st\,\mathcal{W} < \beta_1^{-1}\mathcal{V}_1 \times \cdots \times \beta_n^{-1}\mathcal{V}_n$. Consequently, for every $i = 1,\dots,n$, $(q_i\beta p f, \alpha_i f_i) < \beta_i^{-1}\mathcal{V}_i$. Now, using the fact that $(\beta_i\alpha_i, id_{P_i}) < \mathcal{V}_i$, we obtain the desired conclusion $(\beta_i q_i \beta p f, f_i) < st\,\mathcal{V}_i < st^2\mathcal{V}_i < \mathcal{U}_i$.

If we want $p_i : P = |\mathbb{K}| \to P_i = |\mathbb{L}_i|$ to be PL, it is sufficient to replace the already constructed mapping p_i by a mapping p_i' obtained in the following way. Let $\mathbb{L}$ be a subdivision of $\mathbb{L}_i$ such that the closed stars $\overline{st\,(v,\mathbb{L})}$ of the vertices v of $\mathbb{L}$ refine $\mathcal{V}_i$. Let $\mathbb{K}'$ be a subdivision of $\mathbb{K}$ such that the closed stars $\overline{st\,(w,\mathbb{K}')}$ of the vertices w of $\mathbb{K}'$ refine $p_i^{-1}\mathcal{S}_i$, where $\mathcal{S}_i$ is the covering formed by the open stars of the vertices of $\mathbb{L}$. Then, there exists a simplicial approximation $\phi_i : \mathbb{K}' \to \mathbb{L}$ of p_i ([17, Chapter 3, Theorem 4.6]). Put $p_i' = |\phi_i| : |\mathbb{K}'| \to |\mathbb{L}|$. p_i' is PL and $(p_i', p_i) < \mathcal{V}_i$. Since $(p_i f, f_i) < st\,\mathcal{V}_i$, it follows that $(p_i' f, f_i) < st^2\mathcal{V}_i < \mathcal{U}_i$. p_i' is PL and satisfies all the required conditions. □

If a family $\mathcal{F} = \{p_a : X \to X_a \,|\, X_a \in \mathcal{P}, a \in A\}$ is an approximate resolution mod $\mathcal{P}$ of X then $\mathcal{F}$ is an approximate semi-projection mod $\mathcal{P}$ of X. Much more interesting is the converse question.

Proposition 4.4. *Let X be a topological space, $\mathcal{Q} \subseteq \mathcal{P} \subseteq$ POL and let $\mathcal{F} \subseteq \{f : X \to Q \,|\, Q \in \mathcal{Q}\}$ be an approximate semi-projection* mod $\mathcal{P}$ *of X. If $\mathcal{P}$ is closed and each $f \in \mathcal{F}$ is a surjection, then there exists an approximate $\mathcal{Q}$-resolution* mod $\mathcal{P}$ *$p : X \to \mathcal{X} = (X_a, \mathcal{U}_a, p_{aa'}, A)$ of the space X, such that each projection $p_a : X \to X_a$ belongs to $\mathcal{F}$ and all bonding maps $p_{aa'}$ are PL.*

Proof: Since $\mathcal{P}$ is closed, we can apply the inductive construction described in [7] using Proposition 4.3. This construction yields the desired approximate resolution mod $\mathcal{P}$. □

5. Approximate resolutions of Lindelöf spaces

Proposition 5.1. *Let X be a Lindelöf space and let $\mathcal{F}$ be the family of all surjections $f : X \to P$, where P is a Polish polyhedron. Then $\mathcal{F}$ is an approximate semi-projection of X.*

Proof: Let $P' = |\mathbb{K}| \in$ POL be a polyhedron, $f' : X \to P'$ be a mapping and let $\mathcal{U} \in Cov(P')$ be an open covering of P'. Choose a subdivision $\mathbb{L}$ of $\mathbb{K}$ such that the closed stars $\overline{st\,(v, \mathbb{L}}$ of the vertices v of $\mathbb{L}$ refine $\mathcal{U}$, i.e. $\overline{\mathcal{S}} = \{\overline{st\,(v, \mathbb{L})} : v \in \mathbb{L}^0\} < \mathcal{U}$. Since $(f')^{-1}\mathcal{S}$ is an open covering of X and X is a Lindelöf space, there exists a countable and star-finite open covering $\mathcal{V}$ of X such that $\mathcal{V} < (f')^{-1}\mathcal{S}$ ([14, Chapter 5, 4B]). Every Lindelöf space is paracompact ([2, Theorem 5.1.2]) , and therefore there exists a canonical mapping $\phi : X \to |\mathbb{N}(\mathcal{V})|$ of $\mathcal{V}$. In general, ϕ is not a surjection. However, there exists a subcomplex $\mathbb{N}$ of $\mathbb{N}(\mathcal{V})$ and an $\mathbb{N}(\mathcal{V})$-modification $f : X \to |\mathbb{N}|$ of ϕ ($\phi(x) \in \sigma \in \mathbb{N}(\mathcal{V})$ implies $f(x) \in \sigma$), which is irreducible and therefore surjective ([10, Corollary 1]). Clearly, $f : X \to |\mathbb{N}|$ belongs to $\mathcal{F}$.

We now define a mapping $p' : \mathbb{N}(\mathcal{V})^0 = \mathcal{V} \to \mathbb{K}^0$ by assigning to a vertex $V \in \mathbb{N}(\mathcal{V})^0$ we assign a vertex $v = p'(V) \in \mathbb{K}^0$ such that $V \subseteq (f')^{-1}(st\,(v, \mathbb{K}))$. It is readily seen that $p' : \mathbb{N}(\mathcal{V})^0 \to \mathbb{K}^0$ is a simplicial mapping and therefore induces a mapping $|p'| : |\mathbb{N}(\mathcal{V})| \to |\mathbb{K}|$. Put $p = |p'| \mid |\mathbb{N}| : |\mathbb{N}| \to |\mathbb{K}|$. It is easy to verify that $(f', pf) < \mathcal{U}$. □

Proposition 5.1 and Proposition 4.4 imply the following corollary.

Corollary 5.2. *Every Lindelöf space X admits an approximate resolution $p : X \to \mathcal{X} = (X_a, \mathcal{U}_a, p_{aa'}, A)$, where all X_a are Polish polyhedra and all bonding maps $p_{aa'}$ are PL-mappings. Moreover, every projection $p_a : X \to X_a$ is a surjection.*

Now, we can prove Theorem 1.5.

Corollary 5.2 proves (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i). Let $\mathcal{U}$ be any open covering of X and let $p : X \to \mathcal{X} = (X_a, \mathcal{U}_a, p_{aa'}, A)$ be an approximate resolution consisting of Lindelöf polyhedra. Since X is paracompact, $\mathcal{U}$ is normal. By (B1)* there exist an $a \in A$ and a countable open covering $\mathcal{V}$ of X_a such that $p_a^{-1}\mathcal{V} < \mathcal{U}$, which shows that X is Lindelöf. □

Remark 5.3. The equivalence (i) $\Rightarrow$ (iii) in Theorem 1.5 has been proved in a different way in [1, Theorem 3].

6. Approximate pPOL-resolution mod sPOL of a Tychonoff space

Definition 6.1. Let $X \subseteq Y$ be topological spaces. We say that X is C-embedded in Y, if for every continuous real valued function $f \in C(X, \mathbb{R})$ there exists a $\tilde{f} \in C(Y, \mathbb{R})$ such that $\tilde{f} \mid X = f$.

For a Tychonoff space X we introduce the following notation:

$A = C(X, \mathbb{R})$;

$2^A_\omega = \{B \subseteq A : \text{card } B \le \omega\} \subseteq 2^A$;

$\pi_B : \mathbb{R}^A \to \mathbb{R}^B$, $B \in 2^A_\omega$ and $\pi_{BC} : \mathbb{R}^C \to \mathbb{R}^B$, $B, C \in 2^A_\omega$, $B \subseteq C$, are the corresponding projections;
$p_B = \pi_B \,|\, X : X \to p_B(X) \subseteq \mathbb{R}^B$, $B \in 2^A_\omega$.

Clearly, $\mathbb{R}^A$ is the limit of the inverse system $\mathbb{S} = (\mathbb{R}^B, \pi_{BC}, 2^A_\omega)$. Since the Souslin number (also called cellularity) of $\mathbb{R}^A$ is $\leq \omega$ ([2, Corollary 2.3.18]), and $\pi_B : \mathbb{R}^A \to \mathbb{R}^B$ are open surjections, $\mathbb{S}$ has the following property: For every real function $f : \mathbb{R}^A \to \mathbb{R}$, there exist a $B \in 2^A_\omega$ and a mapping $g : \mathbb{R}^B \to \mathbb{R}$ such that $f = g\pi_B$. This is a consequence of the following fact ([3, Proposition 3.1.7]). If $(X_a, p_{aa'}, A)$ is an ω-complete inverse system of spaces, if the limit space X has the Souslin property, and all projections $p_a : X \to X_a$ are open surjections, then every mapping $f : X \to \mathbb{R}$ admits an $a \in A$ and a mapping $g : X_a \to \mathbb{R}$ such that $f = gp_a$.

Proposition 6.2. *Let X be a Tychonoff space, Y a separable metric space and let $f : X \to Y$ be a mapping. Then there exist a $B \in 2^A_\omega$ and a mapping $g : p_B(X) \to Y$ such that $f = gp_B$.*

PROOF: Since Y is a separable metric space, we may consider that Y is a subspace of $\mathbb{R}^\omega$. Therefore, we can represent $f : X \to Y \subseteq \mathbb{R}^\omega$ as a direct product $(\prod\limits_{n<\omega} f_n)$, where $f_n = \pi_n f : X \to \mathbb{R}$. X is C-embedded in $\mathbb{R}^A$ ([2, Theorem 2.3.20]) and therefore, for every $n < \omega$, there exists an extension $\tilde{f}_n : \mathbb{R}^A \to \mathbb{R}$ of f_n. Applying the above stated property to $\tilde{f}_n : \mathbb{R}^A \to \mathbb{R}$, we conclude that, for every $n < \omega$, there exist a $B_n \in 2^A_\omega$ and a mapping $g'_n : \mathbb{R}^{B_n} \to \mathbb{R}$ such that $\tilde{f}_n = g'_n \pi_{B_n}$.

Let $B = \bigcup\limits_{n<\omega} B_n \in 2^A_\omega$ and $g_n = g'_n \pi_{B_n B} : \mathbb{R}^B \to \mathbb{R}$. Note that $g_n \pi_B = g'_n \pi_{B_n B} \pi_B = g'_n \pi_{B_n} = \tilde{f}_n$, for every $n < \omega$. Let $g' = (\prod\limits_{n<\omega} g_n) : \mathbb{R}^B \to \mathbb{R}^\omega$ and put $g = g' \,|\, p_B(X) : p_B(X) \to \mathbb{R}^\omega$.

Claim 1. $f = gp_B$.

Let $x \in X$. Then

$$(gp_B)(x) = g'(\pi_B(x)) = (\prod_{n<\omega} g_n)(\pi_B(x)) =$$
$$= (g_n(\pi_B(x))) = (\tilde{f}_n(x)) = (f_n(x)) = f(x).$$

Claim 2. $g(p_B(X)) \subseteq Y$.

Let $z \in p_B(X)$. Since p_B is a surjection, there exists an $x \in X$ such that $p_B(x) = z$. Therefore, $g(z) = g(p_B(x)) = f(x) \in Y$. □

For every $B \in 2^A_\omega$, $p_B(X) \subseteq \mathbb{R}^B$ is a separable metric space, hence a Lindelöf space. By Theorem 1.5, there exists an approximate resolution $p = (p^B_a) : p_B(X) \to \mathcal{X}_B = (X^B_a, \mathcal{U}^B_a, p^B_{aa'}, A_B)$ of $p_B(X)$, consisting of Polish polyhedra and having surjective projections $p^B_a : p_B(X) \to X^B_a$. Put $\mathcal{F} = \{p^B_a p_B : X \to X^B_a \,|\, B \in 2^A_\omega, a \in A_B\}$.

Proposition 6.3. *The family* $\mathcal{F} = \{p_a^B p_B : X \to X_a^B \mid B \in 2_\omega^A, a \in A_B\}$ *is an approximate semi-projection* mod sPOL *of the Tychonoff space* X.

Proof: Let P be a separable polyhedron, let $f : X \to P$ be a mapping and let $\mathcal{U} \in Cov(P)$ be any open covering of P. Choose an open covering $\mathcal{V}$ of P such that $st\,\mathcal{V} < \mathcal{U}$. Since sPOL is approximately equivalent to sANR, there exist a $Q \in s$ANR and mappings $\alpha : P \to Q$, $\beta : Q \to P$ such that $(\beta\alpha, id_P) < \mathcal{V}$. By Proposition 6.2 there exist $B \in 2_\omega^A$ and a mapping $g : p_B(X) \to Q$ such that $\alpha f = g p_B$. Since $p = (p_a^B) : p_B(X) \to \mathcal{X}_B = (X_a^B, \mathcal{U}_a^B, p_{aa'}^B, A_B)$ is an approximate resolution of $p_B(X)$, there exist an $a \in A_B$ and a mapping $p : X_a^B \to P$ such that $(\beta g, pp_a^B) < \mathcal{V}$. Then $(\beta g p_B, pp_a^B p_B) < \mathcal{V}$, which implies $(\beta\alpha f, pp_a^B p_B) < \mathcal{V}$. Since $(\beta\alpha, id_P) < \mathcal{V}$, we obtain $(f, pp_a^B p_B) < st\,\mathcal{V} < \mathcal{U}$, which proves that $\mathcal{F}$ is an approximate semi-projection mod sPOL of X. $\square$

Proposition 6.4. *Let* X *be a Tychonoff space. Then* X *admits an approximate resolution* mod sPOL *consisting of Polish polyhedra and PL bonding maps.*

Proof: Put $\mathcal{F} = \{p_a^B p_B : X \to X_a^B \mid B \in 2_\omega^A, a \in A_B\}$. By Proposition 6.3 $\mathcal{F}$ is an approximate semi-projection of X mod sPOL of X. sPOL is closed, each $p_a^B p_B$ is a surjection and each X_a^B is a Polish polyhedron. Now, the assertion follows from Proposition 4.4. $\square$

Corollary 3.4 and Proposition 6.4 imply the next corollary.

Corollary 6.5. *Every realcompact space* X *is the limit of an approximate inverse system consisting of Polish polyhedra with PL bonding maps.*

Now, we can prove Theorem 1.6.

Corollary 6.5 proves (i) $\Rightarrow$ (ii). (ii) $\Rightarrow$ (iii) is obvious. Since realcompactness is preserved under products and closed subsets ([2, Theorem 3.11.4 and 3.11.5]), (iii) $\Rightarrow$ (i) is also obvious. $\square$

Remark 6.6. It is not known if there exist measurable cardinals. The assumption that every cardinal is non-measurable implies that every polyhedron is realcompact (Proposition 2.4) and therefore, since every topologically complete space is the limit of an approximate polyhedral system, by Theorem 1.6 (iii) that realcompact and topologically complete spaces coincide.

Note that in general it is not possible to obtain an approximate resolution of a realcompact space consisting of Polish or even separable polyhedra. Namely, in that case every normal covering of a realcompact space would have a countable refinement, which is obviously not true. E.g., a discrete space of cardinality 2^ω is realcompact, because 2^ω is non-measurable. However, it has an open (also normal) covering which does not admit a countable refinement.

Question. Is every realcompact space the limit of an usual (commutative) inverse system consisting of Polish polyhedra?

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