Constructions of smooth and analytic cocycles over irrational circle rotations

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Abstract. We define a class of step cocycles (which are coboundaries) for irrational rotations of the unit circle and give conditions for their approximation by smooth and real analytic coboundaries. The transfer functions of the approximating (smooth and real analytic) coboundaries are close (in the supremum norm) to the transfer functions of the original ones. This result makes it possible to construct smooth and real analytic cocycles which are ergodic, ergodic and squashable (see [Aaronson, Lemańczyk, Volný]), of type III_0, or which are coboundaries with nonintegrable transfer functions. The cocycles are constructed as sums of coboundaries.

Keywords: smooth cocycle, real analytic cocycle, transfer function, type III_0, ergodic and squashable, distributions of a cocycle

Classification: 28D05, 11K50, 60F05

1. Introduction

Let us represent the unit circle $\mathbb{T}$ as the interval $[0,1)$ and its irrational rotation by the transformation $T : x \mapsto x + \alpha \bmod 1$ (where $\alpha \in (0,1)$ is an irrational number). By λ we denote the Lebesgue (probability) measure on $[0,1)$. For any measurable function f, the transformation $T_f : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$ defined by

$$T_f(x,y) = (Tx, y + f(x))$$

preserves the product Lebesgue measure on $\mathbb{T} \times \mathbb{R}$. Let us denote $S_n(f) = \sum_{i=0}^{n-1} f \circ T^i$, $n = 0,1,\dots$. The mapping $\mathbb{Z} \times \mathbb{T} \to \mathbb{T}$ defined by $(i,x) \mapsto S_i(f)(x)$ for $i \geq 0$ and $(i,x) \mapsto -S_{-i}(f)(x)$ for $i < 0$, is called a cocycle. Here (as usually) we call cocycle the function f. If f can be represented by $f = g - g \circ T$ where g is measurable, we say that f is a coboundary and g its transfer function.

A real number a is called essential value of the cocycle f iff for every $\epsilon > 0$ and every set $B \subset [0,1)$ of positive measure there exists a positive integer n such that

$$\lambda(B \cap T^{-n}B \cap \{S_n(f) \in \mathcal{U}_\epsilon(a)\}) > 0$$

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where $\mathcal{U}_\epsilon(a)$ denotes the ϵ-neighbourhood of a. The cocycle f is ergodic iff all real numbers are its essential values and it is of type III_0 iff ∞ and 0 are the only essential values (see [Schmidt]).

We shall present a method of construction of smooth or real analytic cocycles F with a special limit behaviour of $S_n(F)$.
First, in the next Section, we define a special class of coboundaries f which are step cocycles (see (5)). The main results (Theorems 1 and 2) show that the step cocycles defined by (5) can be well approximated by coboundaries F which are $\mathcal{C}^\infty$ functions or zero mean real trigonometric polynomials, so that the transfer functions g, G, of f, F, are also close to each other. Therefore, the limit properties of $S_n(F)$ are similar to those of $S_n(f)$.
Then, in the last Section, we apply the approximation results in constructing smooth and real analytic cocycles F

- for which the distributions of $S_n(F)$ converge along subsequences to all probability laws,
- which are of type III_0,
- which are coboundaries with nonintegrable transfer functions,
- in [Aaronson, Lemańczyk, Volný], ergodic and squashable smooth and real analytic cocycles F are found.

Theorem 1 is partially included in the proofs in the paper [Liardet, Volný], where the result on the convergence of $S_n(F)$ was proved for $F \in \mathcal{C}^p$, $1 \le p \le \infty$ (here we show the simplification of the proof enabled by the use of Theorem 1 and we extend the result to real analytic functions). The construction of smooth and real analytic coboundaries with nonintegrable transfer functions generalizes results from [Baggett, Merrill]. The construction of the ergodic and squashable cocycles in [Aaronson, Lemańczyk, Volný] implicitly uses Theorem 1 (Theorem 2 enables us to extend the results to real analytic cocycles).

Recall that the irrational number α can be represented by the continued fraction expansion $\alpha = [0; a_1, a_2, \dots]$ where the positive integers a_n are called partial quotients. The convergents p_n/q_n defined by the recurrent formulas

$$\begin{aligned} p_0 &= 0, \quad p_1 = 1, \quad p_n = a_n p_{n-1} + p_{n-2}, \\ q_0 &= 1, \quad q_1 = a_1, \quad q_n = a_n q_{n-1} + q_{n-2} \end{aligned} \tag{1}$$

give an approximation of α.

In the constructions we shall largely use the

<u>Rokhlin towers</u>.

By $\{x\}$ we denote the fractional part of x, i.e. $\{x\} = x - [x]$. For $x \in [0,1)$, $\|x\|$ denotes $\min\{x, 1-x\}$.
From the continued fraction expansion we get two Rokhlin towers: If n is odd, we have

$[\{j\alpha\}, \{(q_{n-1}+j)\alpha\})$, $j = 0, \dots, q_n - 1$ and

$[\{q_n\alpha\}, 1), [\{(j+q_n)\alpha\}, \{j\alpha\}),\ j = 1, \dots, q_{n-1}-1,$
for n even we have
$[\{q_{n-1}\alpha\}, 1), [\{(j+q_{n-1})\alpha\}, \{j\alpha\}),\ j = 1, \dots, q_n - 1,$
$[\{j\alpha\}, \{(q_n+j)\alpha\}),\ j = 0, \dots, q_{n-1}-1.$
In the next we shall for simplicity suppose that n is odd (the cases with n even are similar).
For $0 \le x < 1 - \|q_{n-1}\alpha\|$ we thus have $T^{q_{n-1}}x = x + \|q_{n-1}\alpha\|$. Let us denote

$$I_0 = [0, \|q_{n-1}\alpha\|),\ \ I_i = T^i I_0,\ \ i = 1, \dots, q_n - 1,$$

(2)
$$J_u = [\{u\alpha\}, \{u\alpha\} + a_n\|q_{n-1}\alpha\|) = \bigcup_{i=0}^{a_n-1} I_{u+iq_{n-1}}, \quad u = 0, \dots, q_{n-1}-1.$$

Notice that

(2′)
$$\frac{1}{2q_n} \le \|q_{n-1}\alpha\| \le \frac{1}{q_n},$$
$$\frac{1}{2q_{n-1}} \le a_n\|q_{n-1}\alpha\| \le \frac{1}{q_{n-1}}.$$

We shall consider the rotations α with unbounded partial quotients, i.e. with

$$\limsup_{n\to\infty} q_{n+1}/q_n = \infty.$$

For such rotations there exist $\mathcal{C}^1$ cocycles which are nontrivial, i.e. are not coboundaries (we assume zero mean). This does not hold in the case of bounded partial quotients; for such rotations there exist nontrivial absolutely continuous cocycles (cf. [Liardet, Volný]) but none which would also be ergodic and squashable, or of type III_0, has been found.
The condition

(3)
$$\limsup_{n\to\infty} q_{n+1}/q_n^p = \infty$$

is necessary and sufficient for the existence of nontrivial cocycles in the space $\mathcal{C}^p$; if it holds, the set of ergodic (hence nontrivial) cocycles is in $\mathcal{C}^p$ dense and G_δ (cf. e.g. [Baggett, Merrill], [Liardet, Volný]).
The validity of (3) for all positive integers p is necessary and sufficient for the existence of a nontrivial (and ergodic as well) $\mathcal{C}^\infty$ cocycle ([Liardet, Volný]).
The last assumption on the rotation we shall use is

(4)
$$\limsup_{n\to\infty} \log q_{n+1}/q_n = \infty.$$

The positivity of the lim sup is necessary and sufficient for the existence of real analytic nontrivial cocycles ([Herman]); if they exist, an ergodic one exists, too. If $\limsup_{n\to\infty} \log q_n/q_{n-1} < \infty$, we do not know whether a real analytic cocycle F can also be squashable, whether the distributions of $S_n(F)$ can converge along subsequences to all probability laws, whether F can be of type III_0.

2. Approximations of step cocycles

All integrable cocycles will be assumed to have a zero mean. Let $\bar{f}$ be a step function on $[0,1)$, i.e.

$$\bar{f} = \sum_{j=1}^{m} b_j \chi(B_j)$$

where $\{B_1, \ldots, B_m\}$ is a partition of $[0,1)$, $B_i = [x_i, x_{i+1})$, $0 = x_1 < x_2 < \cdots < x_{m+1} = 1$, let $\int_0^1 \bar{f}(x)\,dx = 0$. Define

$$ {}^n f(x) = \sum_{i=0}^{q_{n-1}-1} \chi_{J_i} \bar{f}\Big(\frac{T^{-i}x}{a_n \|q_{n-1}\alpha\|}\Big). \tag{5}$$

Suppose that for a positive integer n, all the numbers x_i, $0 \le i \le m+1$, can be expressed as fractions with denominators a_n, i.e. $a_n x_i$ are integers. Then ${}^n f$ is constant on the sets I_i, $0 \le i \le q_n - 1$, and zero out of their union. Since $\int_0^1 \bar{f}\,d\lambda = 0$, we thus have $\sum_{i=0}^{q_{n-1}-1} {}^n f(T^i x) = 0$ for every $x \in I_0$. Therefore, ${}^n f$ is a coboundary with a transfer function ${}^n g$ where

$$\begin{aligned} {}^n g(T^i x) &= -\sum_{j=0}^{i-1} {}^n f(T^j x) \quad \text{for} \quad x \in I_0,\ 1 \le i \le q_n - 1, \\ {}^n g(x) &= 0 \qquad\qquad\qquad\quad \text{for} \quad x \in I_0 \ \text{and for}\ x \notin \bigcup_{i=0}^{q_n-1} I_i. \end{aligned} \tag{6}$$

Moreover, we can derive

$$\frac{1}{2} b a_n q_{n-1} \max |{}^n f| \le \max_k |S_k({}^n f)(x)| \le 2 q_n \max |{}^n f|$$

where b is the length of the interval B_j on which $|\bar{f}|$ attains its maximal value.

Let ${}^n f$ be the cocycle defined by (5), ${}^n g$ its transfer function defined by (6).

Theorem 1. *For every $\epsilon, \delta > 0$ there exist positive integers k_0, $K > 0$, such that if $a_n > k_0 q_{n-1}^{p-1}$, then there exists a cocycle $F \in \mathcal{C}^p$,*

(i) *$F = {}^n f$ on a set of measure at least $1-\epsilon$, and $|F| \le (1+\epsilon)\max|\bar{f}|$,*
(ii) *$\|F\|_{\mathcal{C}^p} < K q_{n-1}^p$, and*
(iii) *for transfer functions ${}^n g$, G, of the cocycles ${}^n f$, F, we have $|{}^n g - G| < \delta \sup |{}^n g| \le \delta q_n$.*

Theorem 2. *For every $\epsilon, \delta > 0$ there exists a positive integer k_0 such that if $\log q_n/q_{n-1} > k_0$, then there exists a real trigonometric polynomial*

$$F(x) = \sum_{\ell=-s}^{s} c_\ell e^{2\pi i \ell q_{n-1} x},$$

- (i′) *$|F - {}^n f| < \epsilon$ on a set of measure greater or equal $1-\epsilon$, and $|F| \le (1+\epsilon)\max|\bar{f}|$,*
- (ii′) *for $M = \sum_{\ell=-s}^{s} |c_\ell|$ we have $M e^{s q_{n-1}}/q_n < \delta$,*
- (iii) *for the transfer functions ${}^n g$, G, of ${}^n f$, F, we have $\sup|{}^n g - G| < \delta q_n$.*

Notice that for any coboundary F with a transfer function G, if there does not exist $d > 0$ with $G > d$ or $G < -d$, then

$$\begin{aligned} &\sup_m \sup |S_m(F)| \le 2 \sup |G|, \\ &\sup |G| \le \sup_m \sup |S_m(F)|. \end{aligned} \tag{7}$$

The first inequality immediately follows from $F = G - G \circ T$, the second from $\sum_{i=0}^{n-1} F \circ T^i = G - G \circ T^n$ and ergodicity of T.

Using (iii) and this observation we can approximate the partial sums $S_n(F)$ by $S_n(f)$.

Proof of Theorem 1:

I. First we shall prove the theorem for $p = 1$.

Let h be a nonnegative $\mathcal{C}^\infty$ function on $\mathbb{R}$, $h \le 1$, $h(0) = h'(0) = 0 = h(1) = h'(1)$, $h(x) = 0$ for $x \in \mathbb{R} \setminus (0,1)$, $\int_0^1 h(x)\,dx = 1/c$.
H is a periodic function on $\mathbb{R}$ with period 2 defined by

$$H(t) = \begin{cases} \int_0^t h(x)\,dx & \text{on} \quad [0,1] \\ \int_0^1 h(x)\,dx - \int_1^t h(x-1)\,dx & \text{on} \quad [1,2]. \end{cases}$$

There exists a positive integer ν such that

$$d = \int_0^1 (\frac{1}{c} - H(x))\,dx \Big/ \int_0^{2\nu} H(x)\,dx = \left(\frac{1}{c} - \int_0^1 H(x)\,dx\right) \Big/ \int_0^{2\nu} H(x)\,dx < \epsilon.$$

For k_0 we choose an integer bigger than $(4\nu+2)m/\epsilon$. We shall suppose that n is fixed and $a_n > k_0$. Without loss of generality we can (and shall) assume that $\epsilon/((4\nu+2)m)$ is a positive fraction of the type k/a_n. The functions ${}^n f$, ${}^n g$, will be denoted by f, g.

We define

$$\begin{aligned}
\tilde f(x) &= cb_i\frac{(4\nu+2)m}{\epsilon}h(\frac{(4\nu+2)m}{\epsilon}(x-x_i)) \\
&\quad \text{on the intervals} \quad [x_i, x_i+\frac{\epsilon}{(4\nu+2)m}),\ \ i=1,\dots,m, \\
\tilde f(x) &= (-1)^{j+1}cdb_i\frac{(4\nu+2)m}{\epsilon}h(\frac{(4\nu+2)m}{\epsilon}(x-x_i-\frac{j\epsilon}{(4\nu+2)m})) \\
&\quad \text{on the intervals} \quad [x_i+\frac{j\epsilon}{(4\nu+2)m}, x_i+\frac{(j+1)\epsilon}{(4\nu+2)m}), \\
&\quad i=1,\dots,m,\ \ j=1,\dots,2\nu, \\
\tilde f(x) &= (-1)^{j}cdb_i\frac{(4\nu+2)m}{\epsilon}h(\frac{(4\nu+2)m}{\epsilon}(x-x_{i+1}+\frac{(2\nu+1-j)\epsilon}{m(4\nu+2)})) \\
&\quad \text{on the intervals} \quad [x_{i+1}-\frac{(2\nu-j+1)\epsilon}{(4\nu+2)m}, x_{i+1}-\frac{(2\nu-j)\epsilon}{(4\nu+2)m}), \\
&\quad i=2,\dots,m+1,\ \ j=0,\dots,2\nu-1, \\
\tilde f(x) &= \ -cb_i\frac{(4\nu+2)m}{\epsilon}h(\frac{(4\nu+2)m}{\epsilon}(x-x_{i+1}+\frac{\epsilon}{(4\nu+2)m})) \\
&\quad \text{on the intervals} \quad [x_{i+1}-\frac{\epsilon}{(4\nu+2)m}, x_{i+1}),\ \ i=2,\dots,m+1, \\
\tilde f(x) &= 0 \ \ \text{otherwise.}
\end{aligned}$$

For

$$\tilde F(x) = \int_0^x \tilde f(z)\,dz$$

and $i=1,\dots,m$ we have

$$\tilde F(x) = b_i \quad \text{on} \quad [x_i+\frac{\epsilon}{2m}, x_{i+1}-\frac{\epsilon}{2m}), \tag{8}$$

$$\int_{x_i}^{x_{i+1}} \tilde F(z)\,dz = \int_{x_i}^{x_{i+1}} \bar f(z)\,dz, \tag{9}$$

$$|\tilde F| \le (1+d)\max|\bar f| \le (1+\epsilon)\max|\bar f|. \tag{10}$$

Define

$$\tilde f^*(x) = \sum_{i=0}^{q_{n-1}-1} \frac{1}{a_n\|q_{n-1}\alpha\|}\chi_{J_i}\tilde f(\frac{T^{-i}x}{a_n\|q_{n-1}\alpha\|}),$$

$$F(x) = \int_0^x \tilde f^*(z)\,dz.$$

Let us define

$$B_{i,j} = T^j(a_n\|q_{n-1}\alpha\|B_i), \quad 1\le i\le m,\ \ 0\le j\le q_{n-1}-1.$$

Every $B_{i,j}$ equals a union $\bigcup_{\ell=r}^{s} I_{j+\ell q_{n-1}}$ where $0 \le r < s \le a_n$, i.e. $B_{i,j} = \bigcup_{\ell=0}^{s-r} T^{q_{n-1}\ell} I_{j+rq_{n-1}}$. For every $0 \le j \le q_{n-1}-1$, the sets $B_{i,j}$, $1 \le i \le m$, are disjoint subsets of J_j.
The functions f and F are supported by the sets $B_{i,j}$ and by (8),

$$\lambda(B_{i,j} \cap \{f \ne F\})/\lambda(B_{i,j}) \le \epsilon$$

for each $1 \le i \le m$, $0 \le j \le q_{n-1}-1$. By the definition, $\bigcup_{i=1}^{m} B_{i,j} = J_j$, $0 \le j \le q_{n-1}-1$, $\bigcup_{j=0}^{q_{n-1}-1} J_j$ is the support of both f and F. From this, the first part of (i) follows, and from (10) we get $|F| \le (1+\epsilon)\max|\bar{f}|$.

From the definition of $\tilde{f}$ we can see that
$|\tilde{f}^*| \le \max|\bar{f}|(c+cd)(4\nu+2)m/(\epsilon a_n\|q_{n-1}\alpha\|)$.
We have $a_n\|q_{n-1}\alpha\| \ge 1/(2q_{n-1})$, hence
$|\tilde{f}^*| \le \max|\bar{f}| \cdot 2(4\nu+2)(c+cd)mq_{n-1}/\epsilon$
which proves (ii) for $\mathcal{C}^1$ functions with $K = \max|\bar{f}|(c+cd)2(4\nu+2)m/\epsilon$.

The function f is a coboundary with a transfer function

$$
\begin{aligned}
g(T^i x) &= -\sum_{j=0}^{i-1} f(T^j x) \quad \text{for} \quad x \in I_0,\ 1 \le i \le q_n - 1,\\
g(x) &= 0 \qquad\qquad\qquad\quad \text{for} \quad x \in I_0 \ \text{ and for } \ x \notin \bigcup_{i=0}^{q_n-1} I_i
\end{aligned}
\tag{6}
$$

(recall that we denote ${}^n f$, ${}^n g$, by f, g); in the same way, with F at the place of f, we define a function

$$
\begin{aligned}
G(T^i x) &= -\sum_{j=0}^{i-1} F(T^j x) \quad \text{for} \quad x \in I_0,\ 1 \le i \le q_n - 1,\\
G(x) &= 0 \qquad\qquad\qquad\quad \text{for} \quad x \in I_0 \ \text{ and for } \ x \notin \bigcup_{i=0}^{q_n-1} I_i.
\end{aligned}
\tag{6'}
$$

For any $1 \le k \le m$ and $0 \le j \le q_{n-1}-1$, there exist $0 \le r \le s \le a_n$ such that $B_{k,j} = \bigcup_{i=r}^{s} I_{j+iq_{n-1}}$. The points $T^{j+\ell q_{n-1}}x$, $r \le \ell \le s$, $x \in I_0$, are the points from the orbit $(T^i x)_{i=0}^{q_n-1}$, which belong to the $B_{k,j}$.

For any $x \in I_0$ and $0 \le i \le q_n - 1$ we have

$$F(T^i x) - F(T^i 0) = \int_{T^i 0}^{T^i x} \tilde{f}^*(y)\, dy. \tag{11}$$

On the intervals

$$(x_i + \frac{\ell}{a_n}, x_i + \frac{\ell+1}{a_n}) \quad \text{and} \quad (x_{i+1} - \frac{\ell+1}{a_n}, x_{i+1} - \frac{\ell}{a_n}),$$
$$0 \le \ell/a_n \le \epsilon/(2m) - 1/a_n,$$

the values of $\tilde{f}$ differ just in the sign (i.e. for $t \in [0, \frac{1}{a_n})$, $\tilde{f}(x_i + \frac{\ell}{a_n} + t) = -\tilde{f}(x_{i+1} - \frac{\ell+1}{a_n} + t)$). For the function $\tilde{f}^*$, the intervals $(T^{j+iq_{n-1}}0, T^{j+iq_{n-1}}x)$, $r \le i \le s$, thus occur in pairs on which $\tilde{f}^*$ differ just in the sign (if it is nonzero). From this and from (11) it follows that

$$\sum_{i=r}^{s} \Big(F(T^{j+iq_{n-1}}0) - F(T^{j+iq_{n-1}}x) \Big) = 0 \quad \text{for every} \quad x \in I_0,$$

hence

$$\int_{B_{k,j}} F(y)\, dy = \lambda(I_0) \sum_{\ell=r}^{s} F(T^{j+\ell q_{n-1}}0)$$

The function f is constant on the sets $B_{k,j}$, therefore

$$\int_{B_{k,j}} f(y)\, dy = \lambda(I_0) \sum_{\ell=r}^{s} f(T^{j+\ell q_{n-1}}0).$$

By (9) we have

$$\int_{B_{k,j}} F(x)\, dx = \int_{B_{k,j}} f(x)\, dx$$

for every $1 \le k \le m$, $0 \le j \le q_{n-1} - 1$, hence for every $x \in I_0$,

$$\sum_{i=r}^{s} F(T^{j+iq_{n-1}}x) = \sum_{i=r}^{s} f(T^{j+iq_{n-1}}x).$$

From this and $S_{q_n}(f) = 0$ (recall $f = {}^n f$) it follows that $S_{q_n}(F)(x) = 0$ for every $x \in I_0$, hence F is a coboundary with transfer function G (defined by $(6')$). By the definition of F we have $|F| \le (1+d)\max|f|$ and by (i), $\lambda(F \ne f) \le \epsilon$, hence by (6), $(6')$ we get $|G - g| \le \epsilon(1+d)\max|f| \cdot q_n$. From this, (iii) follows.

II. Next we shall prove the theorem for a general p.

Let us define a class of $\mathcal{C}^\infty$ functions

$\mathcal{H} = \{h \in \mathcal{C}^\infty$ on $\mathbb{R}$, $h \le 1$, $h(0) = h'(0) = 0 = h(1) = h'(1)$, $h(x) = 0$ for $x \in \mathbb{R} \setminus (0,1)$, $\int_0^1 h(x)\, dx > 0\}$.

$\mathcal{P}$ is a set of positive integers p such that

(12) For all $h \in \mathcal{H}$ there exist numbers $0 < \eta_1 < \eta_2$, $\eta_2 < t_1 < t_2 \cdots < t_k < 1 - \eta_2$, $t_{i+1} - t_i \ge \eta_2$ for $i = 1, \dots, k-1$, $\exists k_0 \in \mathbb{N}, K > 0$, $\forall a_n > k_0 q_{n-1}^{p-1}$ $\exists\, \gamma_1, \dots, \gamma_k \in \mathbb{R}$, $0 < \eta_1 < \eta < \eta_2$, $a_n \eta \in \mathbb{N}$, such that for the function $\tilde{f}$ on $[0,1]$ defined by

$$\tilde{f}(t) = \sum_{i=1}^{k} \gamma_i h((t - t_i)/\eta)$$

the p-th indefinite integral F of

$$\tilde{f}^*(t) = \left(\frac{1}{a_n \|q_{n-1}\alpha\|}\right)^p \sum_{i=0}^{q_{n-1}-1} \chi_{J_i}(t) \tilde{f}\left(\frac{T^{-i}t}{a_n \|q_{n-1}\alpha\|}\right) \tag{13}$$

satisfies (i)-(iii):

(i) $F = {}^n f$ on a set of measure at least $1 - \epsilon$ and $|F| \le (1+\epsilon) \max |\bar{f}|$,

(ii) $\|F\|_{\mathcal{C}^p} < K q_{n-1}^p$,

(iii) for transfer functions ${}^n g$, G, of the cocycles ${}^n f$, F, we have $|{}^n g - G| < \delta \sup |{}^n g| \le \delta q_n$.

In the first part of the proof we showed that $1 \in \mathcal{P}$. By induction we shall prove that all positive integers belong to $\mathcal{P}$.

Suppose that $p \ge 1$, $p \in \mathcal{P}$. Let $h \in \mathcal{H}$,

$$h_1(x) = \begin{cases} h(2x) & \text{for} \quad x \in [0, 1/2) \\ -h(2x-1) & \text{for} \quad x \in [1/2, 1) \\ 0 & \text{for} \quad x \in \mathbb{R} \setminus (0,1), \end{cases}$$

$$H(t) = \int_0^t h_1(x)\, dx, \quad t \in \mathbb{R}.$$

Then $H \in \mathcal{H}$. By the assumptions there exist numbers $\eta > 0$, $\eta < t_1 < \cdots < t_k < 1 - \eta$, and $\gamma_1, \dots, \gamma_k \in \mathbb{R}$, such that for

$$\tilde{f}(t) = \sum_{i=1}^{k} \gamma_i H((t - t_i)/\eta)$$

and $\tilde{f}^*$ derived from $\tilde{f}$ by (13), the p-th indefinite integral of $\tilde{f}^*$ satisfies (i)-(iii) with a constant $K = K(H)$ in (ii).
We have

$$\tilde{f}^{*\prime} = \frac{1}{a_n\|q_{n-1}\alpha\|}\tilde{f}^{\prime *},$$

hence the $(p+1)$-st indefinite integral of $\tilde{f}^{\prime *}/(a_n\|q_{n-1}\alpha\|)$ satisfies (i)-(iii); the constant K in (ii) is less or equal to $2K(H)$ (cf. $(2')$).

It remains to show that $\tilde{f}'$ can be expressed like in (12). Suppose that $a_n\eta/2 \in \mathbb{N}$ (here we have to increase k_0) and define $\eta' = \eta/2$. We define

$$t'_i = t_{(i+1)/2} \quad \text{for } i \text{ odd}, \quad t'_i = t_{i/2} + \frac{\eta}{2} \quad \text{for } i \text{ even},$$
$$\gamma'_i = \eta\gamma_{(i+1)/2} \quad \text{for } i \text{ odd}, \quad \eta\gamma'_i = t_{i/2} + \frac{\eta}{2} \quad \text{for } i \text{ even},$$

$1 \le i \le 2k$.
Then

$$\begin{aligned}
\tilde{f}'(t) &= \sum_{i=1}^{k} \gamma_i H'((t-t_i)/\eta) \\
&= \sum_{i=1}^{k} \Big(\frac{1}{\eta} h(2(t-t_i)/\eta)\chi_{\{0\le (t-t_i)/\eta \le 1/2\}} - \\
&\qquad - \frac{1}{\eta} h(2(t-(t_i+\eta/2))/\eta)\chi_{\{1/2\le (t-t_i)/\eta\le 1\}}\Big) \\
&= \sum_{i=1}^{2k} \gamma'_i h((t-t'_i)/\eta').
\end{aligned}$$

Therefore, $p+1 \in \mathcal{P}$. This finishes the proof. □

Proof of Theorem 2: All functions used here are to be understood as 1-periodic functions on $\mathbb{R}$. Let

$$P(x) = \sum_{\ell=-s}^{s} c_\ell e^{2\pi i \ell x},$$

where $c_0 = 0$, $c_{-\ell} = \bar{c}_\ell$, i.e. P is a real, zero mean trigonometric polynomial. P is chosen so that

$$\lambda(|P - \bar{f}| \ge \epsilon) < \epsilon/2, \quad |P| \le (1+\epsilon)\max|\bar{f}|.$$

There exists k_0 such that if $\log q_n/q_{n-1} > k_0$, then (ii$'$) $Me^{sq_{n-1}}/q_n < \delta$, where $M = \sum_{\ell=-s}^{s}|c_\ell|$, and also

$$\text{if } \log q_n/q_{n-1} > k_0, \text{ then } 4sq_{n-1}M < \epsilon q_n \text{ and } 2q_{n-1}\sup|\bar{f}| < \epsilon q_n.$$

The function F is defined by

$$F(x) = P(xq_{n-1}) = \sum_{\ell=-s}^{s} c_\ell e^{2\pi i \ell q_{n-1} x}.$$

For $\tilde{f}(x) = \bar{f}(xq_{n-1})$ we thus have

$$\lambda(|F - \tilde{f}| \geq \epsilon) < \epsilon/2, \quad |F| \leq (1+\epsilon) \max |\bar{f}|.$$

Each of the functions f, $\tilde{f}$, has at most $q_{n-1}(m+1)$ points of discontinuity. By [Khinchine],

$$|\alpha - \frac{p_n}{q_n}| < \frac{1}{q_n q_{n+1}},$$

hence the points $\{i\alpha\}$, $0 \leq i \leq q_n - 1$ differ from ip_n/q_n by less than $1/q_n$ ($\{x\}$ denotes the fractional part of x, i.e. $x - [x]$). The points of discontinuity of f, $\tilde{f}$, are thus distant by less than $1/q_n$, hence

$$\lambda(|f - \tilde{f}| \geq \epsilon) < q_{n-1} \frac{2(m+1)}{q_n};$$

for n sufficiently big we have

$$\lambda(|F - f| \geq \epsilon) < \epsilon,$$

i.e. (i$'$) holds.

It is well known that for trigonometric polynomials the equation $F = G - G \circ T$ has a solution

$$G(x) = \sum_{\ell=-s}^{s} \frac{c_\ell}{1 - e^{2\pi i \ell \alpha}} e^{2\pi i \ell x},$$

hence F is a coboundary.
We are going to prove (iii), i.e. to give an upper estimate for $|G - g|$.
Recall that

$$\begin{aligned} &\sup_m \sup |S_m(F)| \leq 2 \sup |G|, \\ &\sup |G| \leq \sup_m \sup |S_m(F)|. \end{aligned} \tag{7}$$

We shall prove that

$$\forall\, m\ \exists\, 0 \leq m' \leq q_n - 1 \quad |S_m(F) - S_{m'}(F)| \leq 4 s q_{n-1} M, \tag{14}$$

$$\forall\, u\ \exists\, 0 \leq u' \leq q_n - 1 \quad |g - g \circ T^{u-u'}| \leq 2 q_{n-1} \sup |f| = 2 q_{n-1} \sup |\bar{f}|. \tag{15}$$

Then we can estimate $|G - g|$ using suprema of the sums $S_k(F)$, $S_k(f)$, $0 \leq k \leq q_n - 1$.

Let us prove (14):
From the equalities $q_m = a_m q_{m-1} + q_{m-2}$ it follows that for every integer $u \geq q_n$ there exist denominators $q_{n+r-1} \leq u < q_{n+r}$ and $0 \leq u' \leq q_n - 1$ and nonnegative

integers $0 \le b_1 \le a_{n+1}$, $0 \le b_2 \le a_{n+2}, \ldots, 0 \le b_r \le a_{n+r}$ where a_i are partial quotients of α and $1 \le r$, such that

$$u = u' + b_1 q_n + \cdots + b_r q_{n+r-1}$$

(this is a special case of the Ostrowski's formula, cf. also [Herman, p. 101]). Because $|1 - e^{2\pi i q_m \alpha}| \le 1/q_{m+1}$ (see [Kuipers, Niederreiter, pp. 122–123]), for any positive integer $m \ge n$, $-sq_{n-1} \le j \le sq_{n-1}$, and $0 \le v \le a_{m+1}$, we have

$$|1 - e^{2\pi i j v q_m \alpha}| \le \frac{sq_{n-1} a_{m+1}}{q_{m+1}},$$

hence

$$|S_{v \cdot q_m + \ell}(e^{2\pi i j x}) - S_\ell(e^{2\pi i j x})| = |\frac{1 - e^{2\pi i j v q_m \alpha}}{1 - e^{2\pi i j \alpha}}| \le \frac{1}{|1 - e^{2\pi i j \alpha}|} \frac{s q_{n-1} a_{m+1}}{q_{m+1}}.$$

From this and from $q_{m+1} \ge q_m a_{m+1}$ (cf. (1)) we get

$$|S_{v \cdot q_m + \ell}(e^{2\pi i j x}) - S_\ell(e^{2\pi i j x})| \le \frac{1}{|1 - e^{2\pi i j \alpha}|} \frac{s q_{n-1}}{q_m},$$

hence

$$|S_u(e^{2\pi i j x}) - S_{u'}(e^{2\pi i j x})| \le \frac{s q_{n-1}}{|1 - e^{2\pi i j \alpha}|} (\frac{1}{q_n} + \frac{1}{q_{n+1}} + \ldots).$$

For a positive constant $L = 1/q_n + 1/q_{n+1} + \ldots$ we thus have $|S_u(F) - S_{u'}(F)| \le sq_{n-1} M q_n (1/q_n + 1/q_{n+1} + \ldots) \le L s q_{n-1} M$. From (1) it follows that $L \le 4$. This finishes the proof of (14).

Proof of (15):

Recall that f is suppposed to be constant on the sets I_i, $0 \le i \le q_n - 1$ and

$$(6) \qquad \begin{aligned} g(T^v x) &= -\sum_{i=0}^{v-1} f(T^i x) && \text{for } x \in I_0, \quad v = 0, \ldots, q_n - 1 \\ g(x) &= 0 && \text{for } x \in [0,1) \setminus \bigcup_{i=0}^{q_n - 1} I_i. \end{aligned}$$

We have $|S_u(f) - S_{u'}(f)| = |g \circ T^u - g \circ T^{u'}| = |g - g \circ T^{u-u'}|$.

Following [Stewart]

$$\|(b_1 q_n + \cdots + b_r q_{n+r-1})\alpha\| \le \|q_{n-1}\alpha\|.$$

Therefore, $x \mapsto T^{u-u'}x = x + \beta \bmod 1$ where $\beta \le \|q_{n_k - 1}\alpha\| \le 1/q_{n_k}$.

For $x \in I_0$ we have $\sum_{i=0}^{q_n - 1} f(T^i x) = 0$, I_i and $I_{i+q_{n-1}}$ are adjacent, $0 \le i \le$

$q_n - q_{n-1} - 1$, and the sets from the smaller Rokhlin tower are adjacent to I_i with $i \le q_{n-1} - 1$ or $i \ge q_n - q_{n-1} - 1$.
Therefore, $|S_u(f) - S_{u'}(f)| \le 2q_{n-1} \sup |f|$, which proves (15).

Because $|F - f| \le \epsilon$,

$$\max_{0 \le k \le q_n - 1} |S_k(F - f)(x)| \le q_n \epsilon.$$

From this, (14), and (15), it follows that

$$\max_{0 \le k} |S_k(F - f)(x)| \le q_n \epsilon + 4sq_{n-1}M + 2q_{n-1} \sup |\bar{f}|.$$

For $k_0 \le \log q_n/q_{n-1}$ we have $4sq_{n-1}M < \epsilon q_n$ and $2q_{n-1} \sup |\bar{f}| < \epsilon q_n$, hence $\max_{0 \le k} |S_k(F - f)(x)| \le 3q_n \epsilon$, which proves (iii) and thus finishes the proof of the theorem. □

3. Applications

The set of applications of Theorems 1 and 2 which we exhibit here is rather aimed to show the possibilities than to exhaust them.

The coboundary ${}^n f$ defined by (5) has a specific feature: If $k < jq_{n-1}$, ${}^n f \circ T^k$ is the function ${}^n f$ shifted by less than j/q_n (cf. (2′)), hence for k/q_n small, $S_k({}^n f)$ is close to $k^n f$.

Theorem 3. *Let the irrational number α satisfy one of the assumptions:*

(3) $\limsup_{n\to\infty} q_{n+1}/q_n^p = \infty$ *where p is a positive integer,*
(3′) $\limsup_{n\to\infty} q_{n+1}/q_n^p = \infty$ *for all positive integers p,*
(4) $\limsup_{n\to\infty} \log q_{n+1}/q_n^p = \infty$.

Then there exists a cocycle F such that the distributions of the partial sums $S_n(F)$ weakly converge along subsequences (n_k) to all probability laws. If (3′) holds, F can be found in $\mathcal{C}^p$, and if (4) holds, F can be found real analytic.

Proof: The set of all probability measures on $\mathbb{R}$ with finite supports is a dense set in the space of all probability measures on $\mathbb{R}$ equipped with the topology of weak convergence. The space is separable, hence there exists a sequence $\bar{f}_k$, $k = 1, 2, \ldots$, of step functions on $[0, 1)$ the distributions $\lambda \circ \bar{f}_k^{-1}$ of which form a dense set. For a sequence $n_k \to \infty$ of positive integers, let the functions ${}^{n_k} f_k$ be defined by (5). We shall simplify the notation and denote them by f_k. $J_0, \ldots, J_{q_{n-1}-1}$ are the sets on $\mathbb{T}$ defined by (2). Define λ_{J_j} by $\lambda_{J_j}(E) = \lambda(J_j \cap E)/\lambda(J_j)$. Then the distributions $\lambda_{J_j} \circ f_k^{-1}$ are the same as the distributions $\lambda \circ \bar{f}_k^{-1}$. For a_{n_k} big, the smaller Rokhlin tower (see the Introduction) has small measure. For h_k converging in the measure to h, the distributions $\lambda \circ h_k^{-1}$ weakly converge to $\lambda \circ h^{-1}$ (cf. e.g. [Billingsley]). For $a_{n_k} \to \infty$ and $\epsilon_k \to 0$, for $\lambda(h_{n_k} = {}^{n_k} f_k) > 1 - \epsilon_k$, the sequence $\lambda \circ h_{n_k}^{-1}$, $k = 1, 2, \ldots$, is thus dense in the space of all probability measures.

Let $1 \le p < \infty$. We shall recursively costruct sequences of numbers $n_k \to \infty$, $\epsilon_k \to 0$, and coboundaries F_k, such that (16.1)-(16.4) hold.
Theorem 1 guarantees the existence of γ_k such that if $q_n/q_{n-1}^{p-1} > \gamma_k$, then there exists a number $K_k > 0$ and a coboundary F_k for which

(i) $F_k = {}^n f$ on a set of measure at least $1 - \epsilon_k$,
(ii) $\|F_k\|_{\mathcal{C}^p} < K_k q_{n-1}^p$, and
(iii) for transfer functions ${}^n g_k$, G_k, of ${}^n f_k$, F_k, and $\delta_k = \epsilon_k^2$, we have $|{}^n g_k - G_k| < \delta_k \sup |{}^n g_k| \le \delta_k q_n$.

We choose $\gamma_k > K_k/\epsilon_k^2$.
Let us put $n_1 = 1 = \epsilon_1$ and $F_1 = 0$. Suppose that $k \ge 2$ and $\epsilon_1, \dots, \epsilon_{k-1}$, $n_1, \dots, n_{k-1}$, and coboundaries $F_1, \dots, F_{k-1}$ were defined. $G_1, \dots, G_{k-1}$ are their transfer functions. We define ϵ_k, n_k, so that

$$\epsilon_k \le \epsilon_{k-1}, \quad \epsilon_k \le 1/2^{k+1}, \quad \epsilon_k \epsilon_{k-1} a_{n_{k-1}} q_{n_{k-1}-1} < 1/2^k \tag{16.1}$$

$$\epsilon_k a_{n_k} \text{ is an integer.} \tag{16.2}$$

$\sum_{j=1}^{k-1} F_j$ is a coboundary with a transfer function $\sum_{j=1}^{k-1} G_j$, $T^{\epsilon_k a_{n_k} q_{n_k-1}}$ represents a rotation by less than ϵ_k, hence we can choose ϵ_k sufficiently small so that

$$\lambda(|S_{\epsilon_k a_{n_k} q_{n_k-1}}(\sum_{j=1}^{k-1} \frac{F_j}{\epsilon_j a_{n_j} q_{n_j-1}})| \ge 1/2^k) < 1/2^k. \tag{16.3}$$

We choose n_k such that

$$n_k > n_{k-1}, \quad q_{n_k}/q_{n_k-1}^p > \gamma_k. \tag{16.4}$$

From this and $q_{n_k} \le 2a_{n_k} q_{n_k-1}$ (see (1)) it follows

$$\|\frac{F_k}{\epsilon_k a_{n_k} q_{n_k-1}}\|_{\mathcal{C}^p} \le \frac{K_k q_{n_k-1}^p}{\epsilon_k a_{n_k} q_{n_k-1}} \le \frac{2K_k q_{n_k-1}^p}{\epsilon_k q_{n_k}} \le \frac{2K_k}{\epsilon_k \gamma_k} < 2\epsilon_k,$$

hence the sum $F = \sum_{k=1}^\infty F_k/(\epsilon_k a_{n_k} q_{n_k-1})$ converges in $\mathcal{C}^p$. We also have $|F_k/(\epsilon_k a_{n_k} q_{n_k-1})| \le 2\epsilon_k$. From this and from (16.1) we get

$$|S_{\epsilon_k a_{n_k} q_{n_k-1}}(\sum_{j=k+1}^{\infty} \frac{F_j}{\epsilon_j a_{n_j} q_{n_j-1}})| \le \sum_{j=k+1}^{\infty} \frac{1}{2^j} \le \frac{1}{2^k}.$$

From this inequality and from (16.3) it follows that

$$\lambda(|S_{\epsilon_k a_{n_k} q_{n_k-1}}(F - \frac{F_k}{\epsilon_k a_{n_k} q_{n_k-1}})| \ge \frac{1}{2^{k-1}}) < \frac{1}{2^k}. \tag{17}$$

From the definition of f_k (see (5)) and (2), (2′) we get

$$\lambda(S_{\epsilon_k a_{n_k} q_{n_k-1}}(f_k) = \epsilon_k a_{n_k} q_{n_k-1} f_k) \geq 1 - \epsilon_k. \tag{18}$$

By Theorem 1 (iii) and (7),

$$\sup_m \sup \frac{1}{\epsilon_k a_{n_k} q_{n_k-1}} |S_m(f_k - F_k)| \leq \frac{\delta_k q_{n_k}}{\epsilon_k a_{n_k} q_{n_k-1}} \leq 2\epsilon_k \to 0, \tag{19}$$

hence the distributions of $S_{\epsilon_k a_{n_k} q_{n_k-1}}(F_k/(\epsilon_k a_{n_k} q_{n_k-1}))$ form a dense set in the space of all probability measures.
Therefore (cf. (17)), the distributions of $S_{\epsilon_k a_{n_k} q_{n_k-1}}(F)$ form a dense set in the space of all probability measures as well. This finishes the proof for $1 \leq p < \infty$.

If $\limsup_{n\to\infty} q_{n+1}/q_n^p = \infty$ for all positive integers p, we can construct the functions F_k so that as before, the distributions of $S_{\epsilon_k a_{n_k} q_{n_k-1}}(F)$ weakly converge to all probability measures and for each positive integer p we have $\sum_{k=1}^{\infty} \|F_k\|_{\mathcal{C}^p} < \infty$, hence $F \in \mathcal{C}^\infty$.

If the assumption (4) holds, Theorem 2 enables us to construct similarly as before a sum F of trigonometric polynomials F_k such that the distributions of the partial sums $S_n(F)$ weakly converge along subsequences to all probability laws:
For $n_k \to \infty$ and $\epsilon_k \to 0$ we approximate the functions ${}^{n_k}f_k$ by real trigonometric polynomials $F_k(x) = \sum_{\ell=-s_k}^{s_k} c_\ell e^{2\pi i \ell x}$ and define
$F(x) = \sum_{k=1}^{\infty} F_k(x)/(\epsilon_k a_{n_k} q_{n_k-1})$.

By Theorem 2 (ii′) for $M_k = \sum_{\ell=-s_k}^{s_k} |c_\ell|$, $M_k e^{s_k q_{n_k-1}}/q_{n_k} < \epsilon_k$, we can choose ϵ_k and n_k so that $F(x) = \sum_{k=1}^{\infty} F_k(x) = \sum_{\ell=-\infty}^{\infty} c_\ell e^{2\pi i \ell x}$ where $\sum_{\ell=-\infty}^{\infty} |c_\ell| e^{|\ell|} < \infty$, hence F is real analytic.
Similarly as before, choosing recursively the numbers ϵ_k we can guarantee

$$|S_{\epsilon_k a_{n_k} q_{n_k-1}}(\sum_{j=k+1}^{\infty} \frac{F_j}{\epsilon_j a_{n_j} q_{n_j-1}})| \leq \frac{1}{2^k}.$$

Any trigonometric polynomial (of zero mean) is a coboundary, hence we can similarly as before guarantee (16.3). Therefore, (17) holds. We have (18) and by Theorem 2 (iii) we get (19). The distributions of $S_{\epsilon_k a_{n_k} q_{n_k-1}}(F_k/(\epsilon_k a_{n_k} q_{n_k-1}))$ thus form a dense set in the space of all probability measures, hence the same holds for F. □

In the case (3), F can be found absolutely continuous (but not Lipschitz) or Lipschitz (but not $\mathcal{C}^1$). An easy way of doing so is to use in the proof of Theorem 1 step or integrable functions for h (instead of a $\mathcal{C}^\infty$ function). As it was said in the introduction, for absolutely continuous, Lipschitz, and $\mathcal{C}^p$, $1 \leq p \leq \infty$, cocycles the result was proved in [Liardet, Volný]. The construction is based on the same ideas there. It can be, moreover, easily proved that the cocycles whose existence is given by Theorem 3, form in the spaces $\mathcal{C}^p$, $1 \le p \le \infty$, as well as in the spaces of absolutely continuous and of Lipschitz functions, a dense G_δ set. (The proof is based on the fact that in those spaces, the coboundaries are dense; the same idea was used for L^p spaces in [Volný].) As shown in [Liardet, Volný], Theorem 3 has immediate implication to the rate of convergence in the ergodic theorem. It is shown there, moreover, that the bounds given by Theorem 3 are the best possible. This gives an impression that in cases where Theorem 3 cannot be applied (e.g. for rotations with bounded partial quotients), the result does not hold. The same might be the case of the next theorem as well.

Theorem 4. *Let the assumption* (3) *or* (3′) *or* (4) *of Theorem* 3 *hold. Then there exists a cocycle F of type III_0; if* (3′) *holds, F can be found in $\mathcal{C}^\infty$, and if* (4) *holds, F can be found real analytic. In the case* (3), *F can be found absolutely continuous* (*but not Lipschitz*) *or Lipschitz* (*but not $\mathcal{C}^1$*).

Theorem 5. *Let the assumption* (3) *or* (3′) *or* (4) *of Theorem* 3 *hold. Then there exists a coboundary F with non-integrable transfer function; if* (3′) *holds, F can be found in $\mathcal{C}^\infty$, and if* (4) *holds, F can be found real analytic. In the case* (3), *F can be found absolutely continuous* (*but not Lipschitz*) *or Lipschitz* (*but not $\mathcal{C}^1$*).

Remark. In [Liardet, Volný, Theorem 5, (13)], it is shown that if for a positive integer p we have $\limsup_{n\to\infty} q_{n+1}/q_n^p < \infty$, then for $F \in \mathcal{C}^p$ with zero mean the integrals $\int_0^1 (S_n(F))^2\, d\lambda$ are uniformely bounded, hence (see [Parry, Tuncel]) F is a coboundary with a square integrable transfer function. Hence, for $1 \le p \le \infty$, in $\mathcal{C}^p$ there exists a nontrivial cocycle if and only if there exists a coboundary with a nonintegrable transfer function.

Proof of Theorem 4: We shall construct the cocycles as sums of coboundaries $F = \sum_{k=1}^\infty F_k$.

Let us suppose that (3) holds, $1 \le p < \infty$.
Let k be a positive integer. Take n such that $q_n > q_{n-1}^p$ and define $m_n = [a_n/2^k]$ where $[x]$ denotes the integer part of x. Define

$$\bar{f}_{k,n} = \begin{cases} 1 & \text{on} \quad [0, m_n/a_n) \\ -1 & \text{on} \quad [[a_n/2]/a_n, [a_n/2]/a_n + m_n/a_n) \\ 0 & \text{otherwise.} \end{cases}$$

Then we can define $f_{k,n}$ in the same way as in (5); $f_{k,n}$ is then constant on the sets I_i, $0 \le i \le q_n - 1$, from the Rokhlin tower, and zero out of their union. Moreover, $\sum_{i=0}^{q_n-1} f_{k,n}(x) = 0$ for all $x \in I_0$, hence

$$f_{k,n} = g_{k,n} - g_{k,n} \circ T$$

where

$$(6') \qquad \begin{aligned} g_{k,n}(T^v x) &= -\sum_{i=0}^{v-1} f_{k,n}(T^i x) && \text{for } x \in I_0, \quad v = 0, \dots, q_{n_k} - 1 \\ g_{k,n}(x) &= 0 && \text{for } x \in [0,1) \setminus \bigcup_{i=0}^{q_{n_k}-1} I_i. \end{aligned}$$

Let $\epsilon_k = 1/12^k$. Similarly as in the proof of Theorem 1 we can define

$B_{1,j} = T^j \bigcup_{\ell=0}^{m_n-1} I_{q_{n-1}\ell}$,

$B_{2,j} = T^j \bigcup_{\ell=m_n}^{[a_n/2]-1} I_{q_{n-1}\ell}$,

$B_{3,j} = T^j \bigcup_{\ell=[a_n/2]}^{[a_n/2]=m_n-1} I_{q_{n-1}\ell}$,

$B_{4,j} = T^j \bigcup_{\ell=[a_n/2]+m_n}^{a_n-1} I_{q_{n-1}\ell}$,

$j = 0, \dots q_{n-1} - 1$.

For a_n sufficiently big we thus have

$$(20) \qquad \lambda(g_{k,n} = m_n q_{n-1}) > \frac{1}{2} - \epsilon_k, \qquad \lambda(g_{k,n} = 0) > \frac{1}{2} - \epsilon_k.$$

By Theorem 1 there exists a number K_k such that for a_n/q_{n-1}^p sufficiently big we can find $F_{k,n} \in \mathcal{C}_0^p$, $F_{k,n} = G_{k,n} - G_{k,n} \circ T$,

(i) $\lambda(F_{k,n} = f_{k,n}) > 1 - \epsilon_k$,
(ii) $\|F_{k,n}\|_{\mathcal{C}^p} < K_k q_{n-1}^p$,
(iii) $|G_{k,n} - g_{k,n}| < \epsilon_k q_n$.

Let $n_0 = 1$ and for $k = 1, 2, \dots$ let n_k be the smallest n for which $12^k K_k < q_n/q_{n-1}^p$, and (20) holds true. Define

$$d_k = 6^k/q_{n_k}, \;\; F_k = d_k F_{k,n_k}, \;\; G_k = d_k G_{k,n_k}, \;\; f_k = d_k f_{k,n_k}, \;\; g_k = d_k g_{k,n_k}.$$

We then have

$$\|F_k\|_{\mathcal{C}^p} \le K_k q_{n_k-1}^p \frac{6^k}{q_{n_k}} \le \frac{1}{2^k}$$

and for the transfer function $g_k = d_k g_{k,n_k}$ of f_k we have

$$|G_k - g_k| < d_k \epsilon_k q_{n_k} = \frac{1}{2^k}.$$

The sum $F = \sum_{k=1}^{\infty} F_k$ converges in $\mathcal{C}^p$. In the case (3′) we can have $\|F_k\|_{\mathcal{C}^k} \leq 1/2^k$, hence $F \in \mathcal{C}^{\infty}$.

Recall (see [Schmidt]) that a real number a is an essential value of the cocycle F iff for every $\epsilon > 0$ and every set $B \subset [0,1)$ of positive measure there exists a positive integer n such that

$$\lambda(B \cap T^{-n}B \cap \{S_n(F) \in \mathcal{U}_{\epsilon}(a)\}) > 0$$

(where $\mathcal{U}_{\epsilon}(a)$ denotes the ϵ-neighbourhood of a).
Let us consider the set

$$B = \{x : \forall k\, (g_k(x) = m_{n_k} q_{n_k - 1} \text{ or } g_k(x) = 0)\}.$$

From (20) it follows that $\lambda(B) > 0$. From the definition of g_k it follows that on B,

$$g_k \in (-\frac{1}{2^{k-1}}, \frac{1}{2^{k-1}}) \cup (3^k - \frac{1}{2^{k-1}}, 3^k + \frac{1}{2^{k-1}});$$

from this and Theorem 1 (iii) we get

$$G_k \in (-\frac{1}{2^{k-2}}, \frac{1}{2^{k-2}}) \cup (3^k - \frac{1}{2^{k-2}}, 3^k + \frac{1}{2^{k-2}}).$$

It is well known (see [Schmidt]) that two cohomologous cocycles (i.e. differing in a coboundary) have the same sets of essential values.
Since each of the cocycles F_k is a coboundary, we can consider the essential values of $F_{(k)} = \sum_{j=k}^{\infty} F_j$ instead of F. We have $S_n(F_j) = G_j - G_j \circ T^n$, hence on $B \cap T^{-n}B$ the sum $|S_n(F_{(k)})|$ is either smaller than $1/2^{k-3}$ or bigger than $3^k - 1/2^{k-2}$, hence F cannot have any finite essential value other than 0. On the other hand, along a rigid sequence $(r_k = [a_{n_k}/2] q_{n_k - 1})$ F is attaining values bigger than $3^k/2$ on the sets $I_{j + i q_{n_k - 1}}$, $0 \leq j \leq q_{n_k - 1} - 1$, $0 \leq i \leq [a_{n_k}/2]$. By the EVC criterium from [Aaronson, Lemańczyk, Volný], ∞ is thus an essential value of F. Therefore, F is of the type III_0.

Let us suppose that (3′) holds. We can do the same construction. The functions F_k are from $\mathcal{C}^{\infty}$ and we can guarantee that for every $1 \leq p < \infty$, $\sum_{k=1}^{\infty} \|F_k\|_{\mathcal{C}^p} < \infty$, hence $F \in \mathcal{C}^{\infty}$.

Let the assumption (4) be fulfilled. We then define $f_{k,n}$ and $g_{k,n}$ in the same way as before. Theorem 2 guarantees the existence of $n = n_k$ and a real trigonometric polynomial

$$F_{k,n}(x) = \sum_{\ell = -s_k}^{s_k} c_{\ell} e^{2\pi i \ell q_{n-1} x}$$

with $c_{-\ell} = \bar{c}_{\ell}$ such that for $\epsilon_k = 1/12^k$, (ii′) for $M_k = \sum_{\ell=-s_k}^{s_k} |c_\ell|$ we have $M_k e^{s_k q_{n-1}}/q_n < \epsilon_k$, and

(iii) for the transfer functions $g_{k,n}$ (of $f_{k,n}$), and $G_{k,n}$ (of $F_{k,n}$), we have $\sup |g_{k,n} - G_{k,n}| < \epsilon_k q_n$.

We define $d_k = 6^k/q_{n_k}$ and put $F = F_{k,n_k}$, $f_k = f_{k,n_k}$ as before, hence $G_k = d_k G_{k,n_k}$ is a transfer function of F_k and $g_k = d_k g_{k,n_k}$ is a transfer function of f_k. Similarly as before we have

$$|G_k - g_k| < d_k \epsilon_k q_{n_k} = \frac{1}{2^k}.$$

where $\lambda(g_k = m_{n_k} q_{n_k-1}) > 1/2 - \epsilon_k$, $\lambda(g_k = 0) > 1/2 - \epsilon_k$, $m_{n_k} q_{n_k-1} > [a_{n_k}/2^k] q_{n_k-1} > q_{n_k}/2^k - 2q_{n_k-1}$. Using the same arguments as before, we can see that F is of type III_0. The function $F = \sum_{k=1}^{\infty} F_k$ is a trigonometric polynomial $F(x) = \sum_{\ell=-\infty}^{\infty} c_\ell e^{2\pi i \ell x}$ with $c_0 = 0$, $c_{-\ell} = \bar{c}_\ell$, $\sum_{\ell=0}^{\infty} |c_\ell| e^\ell < \infty$, hence F is a real analytic function. □

Proof of Theorem 5: Similarly as in the proof of Theorem 4 we define

$$\bar{f}_{k,n} = \begin{cases} 1 & \text{on} \quad [0, m_n/a_n) \\ -1 & \text{on} \quad [(1+k)m_n/a_n, (1+k)m_n/a_n + m_n/a_n) \\ 0 & \text{otherwise,} \end{cases}$$

$f_{k,n}$ is defined in the same way as in the previous case. The transfer function $g_{k,n}$ of $f_{k,n}$ has a support of measure less or equal than $(k+2)/2^k$ (cf. (4′)). We define the functions F_k, G_k in the same way as before. We have

$$\lambda(|G_k| > 1/2^k) \leq \frac{k+2}{2^k}, \ \ \lambda(G_k \geq 3^k/2) \geq \frac{k}{2^k}.$$

Hence, by the Borel-Cantelli Lemma, the sum $G = \sum_{k=1}^{\infty} G_k$ converges almost surely; $F = G - G \circ T$ (where $F = \sum_{k=1}^{\infty} F_k$). From $G_k \geq 0$, $\mathrm{E}\, G_k \geq (3/2)^k/2$ it follows that G is not integrable. □

Remarks.

The type III_0 cocycles are constructed also in the papers [Lemańczyk] and [Hamachi] in this volume.

The constructions from [Kwiatkowski, Lemańczyk, Rudolph] and [Kwiatkowski, Lemańczyk, Rudolph, II] allow to construct $\mathcal{C}^\infty$ and analytic cocycles cohomologous to step cocycles. This way, analytic cocycles which are ergodic or of type III_0 can be constructed (see [Lemańczyk]). Analytic cocycles which are ergodic and squashable can be constructed that way, too. The set of rotations for which the results hold is in all cases smaller than when using Theorem 2 ([Lemańczyk II]).

As M. Lemańczyk informed me, the existence of analytic coboundaries with nonintegrable transfer functions is also proved in [Katok].

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