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Butler groups and Shelah's Singular Compactness

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Abstract. A torsion-free group is a B_2-group if and only if it has an axiom-3 family $\mathfrak{C}$ of decent subgroups such that each member of $\mathfrak{C}$ has such a family, too. Such a family is called $SL_{\aleph_0}$-family. Further, a version of Shelah's Singular Compactness having a rather simple proof is presented. As a consequence, a short proof of a result [R1] stating that a torsion-free group B in a prebalanced and TEP exact sequence $0 \to K \to C \to B \to 0$ is a B_2-group provided K and C are so.

Keywords: B_1-group, B_2-group, prebalanced subgroup, torsion extension property, decent subgroup, axiom-3 family
Classification: 20K20

All groups in this paper are additively written abelian. If x is an element of a torsion-free group G then $\mathbf{t}_G(x) = \mathbf{t}(x)$ will denote the type of x in G. By a *smooth* (*ascending*) *union of a group* G we mean a collection of pure subgroups G_α indexed by an initial segment of ordinals with the property that $G_\beta \leq G_\alpha$ when $\beta < \alpha$ and $G_\alpha = \cup_{\beta<\alpha} G_\beta$ whenever α is a limit ordinal. For unexplained terminology and notations see [F1].

An exact sequence $E : 0 \to H \to G \xrightarrow{\beta} K \to 0$ with K torsion-free is *balanced* if the induced map $\beta_* :$ Hom$(J, G) \to$ Hom(J, K) is surjective for each rank one torsion-free group J. Equivalently, E is balanced if all rank one (completely decomposable) torsion-free groups are projective with respect to E. A torsion-free group B is said to be a B_1-*group* (*Butler group*) if Bext$(B, T) = 0$ for all torsion groups T, where Bext is the subfunctor of Ext consisting of all balanced-exact extensions. A subgroup H of a torsion-free group G is said to be *prebalanced* if, for each $g \in G \setminus H$, there are elements $h_0, \dots, h_n \in H$ and a non-zero integer m such that $\mathbf{t}(g + H) = \cup_{i=0}^{n} \mathbf{t}(mg + h_i)$.

Another relevant concept in the study of infinite rank Butler groups is the *torsion extension property* (TEP). A pure subgroup H of a torsion-free group G is said to have TEP in G, or briefly, H is TEP(-subgroup) in G, if every homomorphism $H \to T$ with T torsion extends to a homomorphism $G \to T$.

Let G be a torsion-free group and H a pure subgroup of corank 1 in G. The types $\mathbf{t}(J)$ of those pure rank 1 subgroups J of G which are not contained in H generate a lattice ideal $\mathcal{P}_{G|H}$ in the lattice $\mathcal{T}$ of all types. We say that H is

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preseparative (*$\aleph_0$-prebalanced* in the terminology of [BF]) in G, if the ideal $\mathcal{P}_{G|H}$ is countably generated. Equivalently, H is preseparative in G, if for each $g \in G \setminus H$ there is a countable subset $\{h_0, h_1, \dots\} \subseteq H$ such that for each $h \in H$ there are $m, n < \omega$, $m \neq 0$, with $t(g+h) \leq t(mg+h_0) \cup t(mg+h_1) \cup \dots \cup t(mg+h_n)$. In this case we shall also say that *$\{h_0, h_1, \dots\}$ is a preseparative set for g over H*, or *H-preseparative set for g*. If the corank of H in G is > 1, then H is defined to be *preseparative in G* if the ideal $\mathcal{P}_{K|H}$ is countably generated for every pure subgroup K of G that contains H as a corank 1 subgroup.

Let H be a pure subgroup of a torsion-free group G. We say that a smooth ascending union $G = \bigcup_{\alpha<\mu} H_\alpha$ is a *B-filtration from $H = H_0$ to G* if $H_{\alpha+1} = H_\alpha + B_\alpha$ for each $\alpha + 1 < \mu$, where B_α is a Butler group of finite rank, i.e. a pure subgroup of a completely decomposable group, or, equivalently [B], a torsion-free homomorphic image of a completely decomposable group of finite rank. If $H_0 = 0$, then we speak simply about the *B-filtration of G*. A torsion-free group G is called a *B_2-group* if it has a B-filtration.

Recall, that an *axiom-3 family* of a torsion-free group G over its subgroup H is a collection $\mathfrak{C}$ of pure subgroups of G containing H such that $H, G \in \mathfrak{C}$ and (i) $\sum_{\alpha\in I} H_\alpha \in \mathfrak{C}$ whenever H_α belongs to $\mathfrak{C}$ for each $\alpha \in I$; (ii) for every $U \in \mathfrak{C}$ and every countable subset X of G there is $V \in \mathfrak{C}$ such that $U \cup X \subseteq V$ and $|V/U| \leq \aleph_0$. Moreover, (see [DHR]) such a collection $\mathfrak{C}$ is said to be a $G(\lambda)$-family if, instead of (i), $\mathfrak{C}$ is closed under arbitrary smooth ascending unions of members of $\mathfrak{C}$ and in (ii) the countability is replaced by the infinite cardinal λ.

In this note we start with some known results on decent subgroups to obtain a slight generalization of a characterization of B_2-groups by showing that each such a group has an axiom-3 family of decent subgroups "hereditarily", as observed by [R1]. Several results on infinite rank Butler groups are based on Shelah's Singular Compactness. Our second purpose is to present a rather simple version of this principle having a short proof. This result is then applied to obtain a new proof of a result of Rangaswamy [R1; Theorem 3] stating that for a prebalanced and TEP B_2-subgroup K of a B_2-group C the factor-group C/K is a B_2-group, again. This theorem seems to be the most important result using the Shelah's Singular Compactness in the sense that all other results requiring this principle can be derived easily from this one.

Note [AH], that a pure subgroup H of a torsion-free group G is said to be *decent*, if for each finite rank pure subgroup $L/H \leq G/H$ there is a finite rank Butler group B such that $L = H + B$.

1. Lemma. *Let $K \leq H$ be pure subgroups of a torsion-free group G.*

(i) *If K is decent in G, then K is decent in H;*

(ii) *if H is decent in G, then H/K is decent in G/K;*

(iii) *if K is prebalanced and H/K is decent in G/K then H is decent in G.*

Proof: (i) Let $L/K \leq H/K$ be a finite rank pure subgroup. There is a finite rank Butler group $B \leq G$ with $L = K + B$. Now $L = K + (B \cap H)$ and $B \cap H$ is finite rank Butler as a pure subgroup of B.

(ii) Let $L/H \leq G/H$ be a finite rank pure subgroup. There is a finite rank Butler group $B \leq G$ with $L = H + B$. Hence $L/K = H/K + (B+K)/K$, where the last subgroup is finite rank Butler, being isomorphic to $B/B \cap K$.

(iii) If L/H is a finite rank pure subgroup of G/H then $L/K = H/K + \tilde{B}/K$ with $\tilde{B}/K$ finite rank Butler. The prebalancedness of K yields $\tilde{B} = K + B$, hence $L = H + B$, where B is a finite rank Butler group. □

2. Lemma. *If H is a decent subgroup of at most countable corank in a torsion-free g roup G, then there is a B-filtration from H to G and H is TEP in G.*

Proof: Expressing $G/H = \bigcup_{n<\omega} L_n/H$ as an ascending union of finite rank pure subgroups with $L_0 = H$, the decency yields $L_n = H + B_n$ for a finite rank Butler group B_n. Consequently, $L_{n+1} = L_n + B_{n+1}$ for each $n < \omega$ and $G = \bigcup_{n<\omega} L_n$ is the desired B-filtration.

Every pure subgroup of a finite rank Butler group is TEP by [B1; Theorem 4]. So, if $L = H + B$ where B is finite rank Butler and if $\varphi : H \to T, T$ torsion, is any homomorphism, then the restriction of φ to $H \cap B$ extends to $\varrho : B \to T$ and $\psi : L \to T$ given by $\psi(h+b) = \varphi(h) + \varrho(b)$ is an extension of φ and the assertion follows by the induction. □

Let $G = \bigcup_{\alpha<\mu} H_\alpha$ be a B-filtration of a torsion-free group G, $H_{\alpha+1} = H_\alpha + B_\alpha$ with B_α finite rank Butler for each $\alpha + 1 < \mu$. Recall, that a subset $S \subseteq \mu$ is said to be *closed* provided $H_\beta \cap B_\beta \leq \langle B_\gamma \mid \gamma \in S, \gamma < \beta \rangle$ for each $\beta \in S$. Moreover, for any subset $S \subseteq \mu$ we put $G(S) = \sum_{\gamma \in S} B_\gamma$. Finally, for each $0 \neq g \in G$ we define $\nu(g) = \nu$ if $g \in H_{\nu+1} \setminus H_\nu$.

3. Lemma. *If $\bar{S} \subseteq \mu$ is a closed subset, then every element $\lambda \in \bar{S}$ lies in a finite closed subset contained in $\bar{S}$.*

Proof: Proving indirectly, let us assume that $\lambda \in \bar{S}$ is the smallest ordinal which is not in a finite closed subset of $\bar{S}$. The intersection $H_\lambda \cap B_\lambda$ is of finite rank and we can select any its maximal linearly independent subset $x_1, \dots, x_l \in B_\lambda$. Obviously, $\nu(x_i) = \lambda_i < \lambda$ and we claim $\lambda_i \in \bar{S}$. If not, then we can write $x_i = y + z$, where $y \in \langle B_\varrho \mid \varrho \in \bar{S}, \varrho < \lambda_i \rangle$ and $z \in \langle B_\varrho \mid \varrho \in \bar{S}, \varrho > \lambda_i \rangle$. Hence $z = z_1 + \dots + z_k$, $z_i \in B_{\varrho_i}$, $\varrho_1 < \dots < \varrho_k$, $\varrho_i \in \bar{S}$ and ϱ_k can be chosen as small as possible. Now $z_k = x_i - y - z_1 - \dots - z_{k-1} \in B_{\varrho_k} \cap H_{\varrho_k} \leq \langle B_\varrho \mid \varrho \in \bar{S}, \varrho < \varrho_k \rangle$ gives $z = 0$ and so $x_i = y \in \langle B_\varrho \mid \varrho \in \bar{S}, \varrho < \lambda_i \rangle$, which yields a contradiction $\nu(x_i) < \lambda_i$. The choice of λ gives the existence of a finite closed subset S_i of $\bar{S}$ with $x_i \in G(S_i)$ for each $i = 1, \dots, l$. The set $S = \bigcup_{i=1}^{l} S_i$ is closed and so is $S \cup \{\lambda\}$ owing to the fact that $G(S)$ is G-pure and contains a maximal linearly independent subset of $H_\lambda \cap B_\lambda$. □

4. Lemma. *If $S \subseteq \mu$ is closed, then $G(S)$ is a decent subgroup of G.*

Proof: The purity of $G(S)$ follows by the standard argument (see e.g. [AH], [DHR]). If $L/G(S) \leq G/G(S)$ is finite rank pure subgroup, then Lemma 3 yields the existence of a finite closed subset $T \subseteq \mu$ containing a set of representatives of a maximal linearly independent subset of $L/G(S)$. It is a routine to check that $L \subseteq G(S \cup T)$ which yields $L = G(S) + (L \cap G(T))$, the last intersection being finite rank Butler as a pure subgroup of $G(T)$. $\square$

The following families of subgroups have been introduced in [B4].

5. Definition. Let λ be an infinite cardinal and H be a subgroup of a torsion-free group G. A collection $\mathfrak{C} = \mathfrak{C}_\lambda(H, G)$ of pure subgroups of G containing H is said to be an *SL_λ-family of H in G* if $H, G \in \mathfrak{C}$ and (i) $\sum_{\alpha \in I} H_\alpha \in \mathfrak{C}$ whenever H_α belongs to $\mathfrak{C}$ for each $\alpha \in I$; (ii) if $V \subseteq \tilde{V}$ are elements of $\mathfrak{C}$ and $X \subseteq \tilde{V}$ is any subset with $|X| \leq \lambda$ then there is $U \in \mathfrak{C}$ such that $V \cup X \subseteq U \leq \tilde{V}$ and $|U/V| \leq \lambda$.

If, instead of (i), a weaker condition stating that $\mathfrak{C}$ is closed under arbitrary smooth ascending unions is satisfied, then we say that $\mathfrak{C}$ is a *WL_λ-family of H in G*. In both cases we shall also speak about families of G over H. Especially, for $H = 0$ we shall speak simply about families of G.

Furthermore, we say that a smooth ascending union $G = \bigcup_{\alpha<\mu} H_\alpha$ of G-pure subgroups is a *λ-chain from H_0 to G* if $|H_{\alpha+1}/H_\alpha| \leq \lambda$ for each $\alpha + 1 < \mu$. If all H_α's are G-preseparative, then we speak about a *λ-preseparative chain from H_0 to G*. Especially, for $H_0 = 0$, we shall speak simply about a *λ-chain* or a *λ-preseparative chain of G*.

6. Theorem. *The following conditions are equivalent for a torsion-free group G:*

(i) *G is a B_2-group;*
(ii) *G has an $SL_{\aleph_0}$-family of decent subgroups;*
(iii) *G has a $WL_{\aleph_0}$-family of decent subgroups;*
(iv) *G has an axiom-3 family of decent subgroups;*
(v) *G has a $G(\aleph_0)$-family of decent subgroups;*
(vi) *G has an $\aleph_0$-chain of decent subgroups.*

Moreover, every member of any of these families is a B_2-group and is TEP in G.

Proof: If $G = \bigcup_{\alpha<\mu} H_\alpha$ is a B-filtration, then it follows from Lemmas 3 and 4 that the set $\mathfrak{C} = \{G(S) \mid S \subseteq \mu, S \text{ closed}\}$ is an $SL_{\aleph_0}$-family of decent subgroups. Moreover, if $G = \bigcup_{\alpha<\mu} H_\alpha$ is an $\aleph_0$-chain of decent subgroups of G, then Lemma 2 yields the existence of a B-filtration of G and the rest is obvious. For the additional assertions apply Lemma 2 and [B4; Theorem 4.3]. $\square$

7. Lemma. *Let a torsion-free group G be a smooth ascending union $G = \bigcup_{\alpha<\tau} G_\alpha$ of its subgroups such that G_α is preseparative in $G_{\alpha+1}$ for each $\alpha < \tau$. Then G_α is preseparative in G for each $\alpha < \tau$.*

Proof: The purity of G_α in G is obvious. Proving indirectly, let us assume that $\alpha < \tau$ is the first ordinal such that G_α is not G-preseparative and let $g \in G \setminus G_\alpha$ be an element without a G_α-preseparative set with $\nu(g) = \beta$ as small as possible. If $\{g_n \mid n < \omega\}$ is a G_β-preseparative set for g and $\{g_{nk} \mid k < \omega\}$ is a G_α-preseparative set for g_n, $n < \omega$, then it is a routine to show (cf. [B4]) that

$\{-kg_{ij} \mid k,i,j < \omega, k \neq 0\}$ is a G_α-preseparative set for g — a contradiction finishing the proof. $\square$

8. Lemma. *A pure subgroup H of a B_2-group G is a B_2-group if and only if there is a preseparative chain from H to G.*

PROOF: If there is a preseparative chain from H to G, then there is an $SL_{\aleph_0}$-family of preseparative subgroups of G over H by [B4; Theorem 4.2] and H is a B_2-group by [B4; Corollary 3.12].

To prove the converse, we borrow an idea from [F2]. Let $0 \to K \to H \oplus C \to G \to 0$ be any relative balanced-projective resolution. Since G is a B_2-group, K has a preseparative chain in $H \oplus C$ and it is consequently a B_2-group by [B4; Corollary 3.12] and so it has an $SL_{\aleph_0}$-family $\mathfrak{C}$ of decent subgroups by Theorem 6. Using the usual back-and-forth argument we can construct the filtrations $K = \bigcup_{\alpha<\mu} K_\alpha$, $C = \bigcup_{\alpha<\mu} C_\alpha$, $G = \bigcup_{\alpha<\mu} G_\alpha$ such that $G_0 = H, G_\alpha$ are pure in G, C_α are summands of C, K_α belong to $\mathfrak{C}$, the factor-groups $C_{\alpha+1}/C_\alpha$ are countable and the sequences $0 \to K_\alpha \to H \oplus C_\alpha \to G_\alpha \to 0$ are prebalanced-exact. We can write $C_{\alpha+1} = C_\alpha \oplus C'$ with C' completely decomposable countable and we have the following commutative diagram

$$\begin{tikzcd}
 & K_\alpha \arrow[r, equal] \arrow[d] & K_\alpha \arrow[d] & & \\
0 \arrow[r] & K_{\alpha+1} \arrow[r] \arrow[d] & H \oplus C_\alpha \oplus C' \arrow[r] \arrow[d] & G_{\alpha+1} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L \arrow[r] & G_\alpha \oplus C' \arrow[r, "\pi"] & G_{\alpha+1} \arrow[r] & 0
\end{tikzcd}$$

with natural maps, where $L = \mathrm{Ker}\, \pi$, the first column is exact by the 3×3-lemma and consequently L is a B_2-group by Lemmas 2 and 1. If U/G_α is a rank one pure subgroup of $G_{\alpha+1}/G_\alpha$ then we have the exact sequence $0 \to L \to G_\alpha \oplus \tilde{C} \to U \to 0$, where $\tilde{C}$ is a B_2-group as a pure subgroup of countable completely decomposable group C'. So G_α is preseparative in $G_{\alpha+1}$ by [BF; Proposition 3.3] and Lemma 7 finishes the proof. $\square$

<u>$(\kappa,\mathfrak{C})$-Shelah game.</u> Let κ be a regular uncountable cardinal, let G be a torsion-free group of cardinality $|G| > \kappa^+$ and let $\mathfrak{C}$ be a family of subgroups of G. We define the $(\kappa,\mathfrak{C})$-*Shelah game* on G in the following way: Player I picks subgroups G_{2i}, $i<\omega$, of cardinality κ and player II picks G_{2i+1} such that $G_i \subseteq G_{i+1}$ for all $i<\omega$. Player II wins if G_{2i+1} is a member of $\mathfrak{C}$ and it is TEP in G_{2i+3} for each $i<\omega$.

9. Definition. Let λ, κ be infinite cardinals and G be a torsion-free group of cardinality $|G| \geq \kappa$. A collection $\mathfrak{C}$ of pure subgroups of G is said to be a $G(\lambda,\kappa)$-

family, if $0 \in \mathfrak{C}$ and

(i) if $H \in \mathfrak{C}$ and $X \subseteq G$ is any subset with $|X| \leq \lambda$, then $H \cup X$ is contained in a member K of $\mathfrak{C}$ with $|K/H| \leq \lambda$;
(ii) $\mathfrak{C}$ is closed under smooth ascending unions $\bigcup_{\alpha<\mu} H_\alpha$ with $|\mu| \leq \kappa$.

10. Lemma. *Let κ be a regular uncountable cardinal and G a torsion-free group of cardinality $|G| > \kappa^+$. If G has a $G(\kappa, \kappa^+)$-family $\mathfrak{C}$ of B_1-subgroups, then player II has a winning strategy in the (κ, C)-Shelah game.*

Proof: In view of Lemma 1.2 in [H], $(\kappa, \mathfrak{C})$-Shelah game is determined and so we are going to show that player I has no winning strategy. By way of contradiction let us assume that I has a winning strategy s and he has picked G_0. Take H_0 to be any member of $\mathfrak{C}$ containing G_0 and assume that H_β, $\beta < \alpha$, have been already defined for some $0 < \alpha < \kappa^+$. For α limit we simply set $H_\alpha = \bigcup_{\beta<\alpha} H_\beta$, while for $\alpha = \beta + 1$ we select H_α to be any member of $\mathfrak{C}$ containing H_β and all $s(H_{\alpha_0}, \dots, H_{\alpha_n})$, $\alpha_0 < \dots < \alpha_n < \alpha$, $n < \omega$. The union $H = \bigcup_{\alpha<\kappa^+} H_\alpha$ belongs to $\mathfrak{C}$ by the hypothesis and [B3; Lemma 4] yields the existence of a cub U in κ^+ such that H_α is TEP in H for each $\alpha \in U$.

Now when player I has chosen G_{2i} in the $(\kappa, \mathfrak{C})$-Shelah game, then player II picks G_{2i+1} to be H_α, where α is the least non-limit element of U containing G_{2i}. $\square$

Looking at the proof of Theorem 6 we see that to a given B-filtration of a B_2-group G it is associated an $SL_{\aleph_0}$-family $\mathcal{F}(G)$ of decent, TEP and B_2-subgroups of G in the natural way, given by the closed subsets of the corresponding ordinal number. It is natural to speak about an *$SL_{\aleph_0}$-family of decent subgroups corresponding to a given B-filtration* of G. Obviously, it follows from Lemma 3 that if $G = \bigcup_{\alpha<\mu} H_\alpha$ is a B-filtration of G and $G = \bigcup_{\alpha<\lambda} K_\alpha$ is any smooth ascending union consisting of members of the given B-filtration of G, then $\mathcal{F}(K_\beta) \subseteq \mathcal{F}(K_\alpha)$ whenever $\beta \leq \alpha$ and $\bigcup_{\beta<\alpha} \mathcal{F}(K_\beta) \subseteq \mathcal{F}(K_\alpha)$, α limit. Moreover, if $H \leq K$ are members of $\mathcal{F}(G)$, then using Lemma 2 we can easily prove the existence of a B-filtration from H to K.

11. Theorem. *Let G be a torsion-free group of singular cardinality κ. If, for some cardinal $\lambda < \kappa$, G has a $G(\lambda)$-family $\mathfrak{C}$ of B_1-subgroups such that each member of $\mathfrak{C}$ of cardinality $< \kappa$ is a B_2-group and there is a B-filtration from H to K whenever $H \leq K$ are members of $\mathfrak{C}$ of cardinalities $< \kappa$ and H is TEP in K, then G is a B_2-group.*

Proof: There is a smooth ascending union $\kappa = \cup_{\alpha<\mu} \kappa_\alpha$ with $\kappa_0 > \mu = \operatorname{cof} \kappa$, $\kappa_0 > \lambda$ and κ_α regular whenever α is non-limit. Further, let $G = \cup_{\alpha<\mu} G_\alpha$ be a smooth union with $|G_\alpha| = \kappa_\alpha$.

Set $G^0_\alpha = G_\alpha$ for each $\alpha < \mu$ and assume that G^n_α has been already defined for some $n < \omega$ and all $\alpha < \mu$. For α limit or 0 set $H^n_\alpha = G^n_\alpha$ and for α successor take H^n_α according to the $(\kappa_\alpha, \mathfrak{C})$-Shelah game $G^0_\alpha, H^0_\alpha, G^1_\alpha, H^1_\alpha, \dots$, the hypotheses of Lemma 10 being obviously satisfied. For each $\alpha < \mu$ let $\{h^j_\alpha \mid j < \kappa_\alpha\}$ be any list of the elements of H_α^n. Moreover, H_α^n has an $SL_{\aleph_0}$-family $\mathcal{F}(H_\alpha^n)$ of decent and TEP subgroups corresponding to a given B-filtration of H_α^n. The routine set-theoretical arguments lead to the conclusion that we can select G_α^{n+1} in such a way that it has cardinality κ_α, contains $H_\alpha^n \cup \{h_\gamma^j \mid \gamma < \mu, j < \kappa_\alpha\}$ and $G_\alpha^{n+1} \cap H_{\alpha+1}^n \in \mathcal{F}(H_{\alpha+1}^n)$.

Now for each α non-limit H_α^n is TEP in H_α^{n+1} by Lemma 10, hence the B-filtration of H_α^n extends to that of H_α^{n+1} and consequently $\mathcal{F}(H_\alpha^n) \subseteq \mathcal{F}(H_\alpha^{n+1}) \subseteq \mathcal{F}(H_\alpha)$, where $H_\alpha = \bigcup_{n<\omega} H_\alpha^n$. Moreover, for $\alpha < \mu$ arbitrary we have $H_\alpha = H_\alpha \cap H_{\alpha+1} = \bigcup_{n<\omega}(H_\alpha^n \cap H_{\alpha+1}^n) \leq \bigcup_{n<\omega}(G_\alpha^{n+1} \cap H_{\alpha+1}^n) \leq \bigcup_{n<\omega}(H_\alpha^{n+1} \cap H_{\alpha+1}^{n+1}) = H_\alpha$, hence $H_\alpha \in \bigcup_{n<\omega} \mathcal{F}(H_{\alpha+1}^n) \subseteq \mathcal{F}(H_{\alpha+1})$ and H_α is TEP in $H_{\alpha+1}$ by Theorem 6. By hypothesis, there is a B-filtration from H_α to $H_{\alpha+1}$ and consequently it remains to show that the union $G = \bigcup_{\alpha<\mu} H_\alpha$ is smooth.

Let $\alpha < \mu$ be a limit ordinal and let $h \in H_\alpha$ be arbitrary. Then $h \in H_\alpha^n$ for some $n < \omega$ and consequently $h = h_\alpha^j$ for some $j < \kappa_\alpha$. Thus $j < \kappa_\beta$ for some $\beta < \alpha$, the chain $\{\kappa_\alpha \mid \alpha < \mu\}$ being assumed smooth. This yields $h \in G_\beta^{n+1} \leq H_\beta$ and the proof is complete. □

12. Theorem. *If $E : 0 \to K \to C \xrightarrow{\pi} B \to 0$ is a prebalanced and TEP exact sequence where K and C are B_2-groups, then B is a B_2-group, too.*

PROOF: With respect to Theorem 6 let $\mathfrak{C}_K$ and $\mathfrak{C}_C$ be $WL_{\aleph_0}$-families of decent and TEP B_2-subgroups of K and C, respectively. By the usual back-and-forth argument we can construct a $WL_{\aleph_0}$-family $\mathfrak{C} = \{H \in \mathfrak{C}_C \mid H \cap K \in \mathfrak{C}_K, \pi(H) \text{ pure in } B\}$. Clearly, $H \cap K$ is TEP in C, hence in H and consequently $\pi(H) \cong H/H \cap K$ is a B_1-group. So, B has $WL_{\aleph_0}$-family $\pi(\mathfrak{C})$ of B_1-subgroups.

Assume now, that $|B| = \kappa$ is the smallest cardinality for which B is not a B_2-group. If $H \in \mathfrak{C}$ is such that $|\pi(H)| < \kappa$, then $0 \to H \cap K \to K \to \pi(H) \to 0$ is TEP and prebalanced-exact and consequently $\pi(H)$ is a B_2-group by the choice of κ. Let $U \leq V$ be elements of $\pi(\mathfrak{C})$ such that $|V| < \kappa$ and $F : 0 \to U \to V \to V/U \to 0$ is TEP. Then U, V are B_2-groups, hence F is preseparative by [R2; Theorem 2] and consequently it is prebalanced by [B2; Lemma 3.5] in view of the existence of $\aleph_0$-chain of B_2-subgroups from U to V. The choice of κ yields that V/U is a B_2-group and consequently there is a B-filtration from U to V. Hence κ is regular uncountable cardinal owing to [BS] and Theorem 11. From $\pi(\mathfrak{C})$ we can construct a κ-filtration $B = \bigcup_{\alpha<\kappa} B_\alpha$ consisting of B_2-groups by the choice of κ. However, with respect to [B3; Lemma 4] the B_α's can be assumed TEP in B and we are through. □

One part of the following result has been proved in [DHR; Proposition 3.9], while the second one was proved in [R1; Theorem 8] under (CH).

13. Theorem. *If $E : 0 \to K \to C \to G \to 0$ is a prebalanced exact sequence, where G is a B_2-group, then K is a B_2-group if and only if C is.*

PROOF: There is a preseparative chain from K to C, G being a B_2-group. If C

is a B_2-group, then K is so by Lemma 8. Conversely, if K is a B_2-group, then the B-filtration of K extends to that of C by Lemma 1. □

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