Remarks on some properties in the geometric theory of Banach spaces

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Abstract. The aim of this paper is to derive some relationships between the concepts of the property of strong (α') introduced recently by Hong-Kun Xu and the so-called characteristic of near convexity defined by Goebel and Sękowski. Particularly we provide very simple proof of a result obtained by Hong-Kun Xu.

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1. Introduction

Recently there has appeared a lot of papers discussing the geometry of Banach spaces from the viewpoint of compactness conditions (cf. [3], [12], [13] and references therein). Special attention has been paid to the concepts being generalizations of classical ones occurring in the geometry of Banach spaces such as convexity and smoothness, for instance [10], [11].

The aim of this paper is to study some relationships between the concept of the property strong (α') introduced by Hong-Kun Xu [9], the characteristic of near convexity defined by Goebel and Sękowski [8] and the modulus of near convexity which will be introduced below.

All these concepts are investigated in the geometric theory of Banach spaces involving compactness conditions.

In particular we are going to provide here very simple proof of a result obtained in [9]. Moreover, we indicate some incorrect statements from this paper and we raise a few open problems.

2. Notation, definitions and auxiliary facts

Let $(E, \|\cdot\|)$ be an infinite dimensional Banach space with the zero element θ. Denote by S_E the unit sphere in E and by B_E the closed unit ball in E.

Let E^* stand for the dual space of E. If X is a subset of E then the symbol $\operatorname{Conv} X$ denotes the closed convex hull of X. Moreover, the distance between a point x and the set X will be denoted $\operatorname{dist}(x, X)$.

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Now let us recall that if X is a nonempty bonded subset of the space E then the <u>*Kuratowski measure of noncompactness*</u> $\alpha(X)$ of X is defined in the following way

$$\alpha(X) = \inf \left\{ \varepsilon > 0 : \begin{array}{l} X \text{ can be covered by a finite family of subsets of } E \\ \text{having diameter smaller than } \varepsilon \end{array} \right\}.$$

For the properties of the function α and for other definitions of a measure of noncompactness we refer to [4], [6], for example.

In what follows we shall consider the function $\Delta_E : [0,2] \to [0,1]$ defined by the formula

$$\Delta_E(\varepsilon) = \inf \{1 - \operatorname{dist}(\theta, X) : X \subset B,\ X = \operatorname{Conv} X,\ \alpha(X) \geq \varepsilon\}.$$

This function was introduced first in [8] and is said the <u>*modulus of near convexity*</u> of the space E.

Observe that Δ_E is nondecreasing on the interval $[0,2]$ but up to now it is nothing known about its continuity on the interval $(0,2]$.

In connection with the function Δ_E we can define ([8]) the <u>*characteristic of near convexity*</u> of E as the number

$$\varepsilon_1(E) = \sup \{\varepsilon \geq 0 : \Delta_E(\varepsilon) = 0\}.$$

In the case $\varepsilon_1(E) = 0$ the space E is said to be <u>*nearly uniformly convex*</u> (cf. [8], [3]).

The characteristic of near convexity $\varepsilon_1(E)$ turns out to be very useful index in the geometry of Banach spaces [8]. For example, if $\varepsilon_1(E) < 1$ then E is reflexive and has normal structure.

3. Main results

In the sequel of the paper we will use another modulus of near convexity of the space E which is defined below.

First, for a given functional $f \in S_{E^*}$ and for a number $\varepsilon \in [0,1]$, denote by $F(f,\varepsilon)$ the slice

$$F(f,\varepsilon) = \{x \in B_E : f(x) \geq 1 - \varepsilon\}.$$

Next, let us define the function $\beta_E : [0,1] \to [0,2]$ in the following way

$$\beta_E(t) = \sup \{\alpha(X) : X \subset B_E,\ X = \operatorname{Conv} X,\ \operatorname{dist}(\theta, X) \geq 1 - t\}.$$

The function β_E turns out to be also a kind of modulus of near convexity.

Indeed, observe first that β_E is nondecreasing on the interval $[0,1]$ so there exists the limit

$$\beta_0(E) = \lim_{t \to 0} \beta_E(t).$$

Moreover, for further purposes we will use the following theorem.

Theorem 3.1. *For any $t \in [0,1]$ the following equality holds to be true*

$$\beta_E(t) = \sup\{\alpha(F(f,t)) : f \in S_{E^*}\}.$$

In order to prove the above theorem let us observe that the inequality

$$\sup\{\alpha(F(f,t)) : f \in S_{E^*}\} \leq \beta_E(t), \quad t \in [0,1]$$

is obvious. The converse inequality is a simple consequence of the separation theorem.

Now we can prove our main result.

Theorem 3.2. *The equality*

$$\beta_0(E) = \varepsilon_1(E)$$

is satisfied for every Banach space E.

Proof: Assume that $\varepsilon_1(E) > 0$. Take $\varepsilon > 0$ such that $\varepsilon < \varepsilon_1(E)$. Then $\Delta_E(\varepsilon) = 0$ which allows us to deduce that for any $\eta > 0$ we can find a subset X of B_E being closed, convex and such that $\alpha(X) \geq \varepsilon$ and $1 - \operatorname{dist}(\theta, X) \leq \eta$.

Hence

$$\operatorname{dist}(\theta, X) \geq 1 - \eta$$

and consequently

$$\sup\{\alpha(Y) : Y \subset B,\ Y = \operatorname{Conv} Y,\ \operatorname{dist}(\theta, Y) \geq 1 - \eta\} \geq \varepsilon.$$

This implies $\beta_E(\eta) \geq \varepsilon$ and yields the following inequality

$$\lim_{\eta \to 0} \beta_E(\eta) = \beta_0(E) \geq \varepsilon_1(E)$$

since ε was chosen arbitrarily to be smaller than $\varepsilon_1(E)$.

Now let us suppose that $\beta_0(E) > \varepsilon_1(E)$. Take δ such that $\varepsilon_1(E) < \delta < \beta_0(E)$. Then $\beta_E(t) > \delta$ for $t \in (0,1]$. Hence we infer that for every $t \in (0,1]$ there exists a subset X of B_E being closed, convex and such that $\operatorname{dist}(\theta, X) \geq 1 - t$ and $\alpha(X) \geq \delta$. This implies that $\Delta_E(\delta) \leq t$ for every $t \in (0,1]$. Consequently $\Delta_E(\delta) = 0$ which contradicts the inequality $\varepsilon_1(E) < \delta$. Thus we showed that $\varepsilon_1(E) = \beta_0(E)$ whenever $\varepsilon_1(E) > 0$.

Observe that the proof of the equality desired in the case $\varepsilon_1(E) = 0$ may be obtained in similar way as above.

This completes the proof. $\square$

Now let us provide a few remarks.

Remark 3.1. Notice that in the light of the above theorem a Banach space E is nearly uniformly convex if and only if $\beta_0(E) = 0$. This confirms the above assertion that the function β_E is a kind of the modulus of near convexity.

Remark 3.2. In the paper [2] it was obtained a similar result as that contained in Theorem 3.2 in the case when instead of the Kuratowski measure α we use the so-called Hausdorff measure of noncompactness χ.

In what follows let us recall the definition of the property strong (α') given in the paper [9].

Definition 3.1. A Banach space E is said to have the property strong (α') if there exists $\varepsilon \in (0,1)$ such that

$$\sup\{\alpha(F(f,\varepsilon)) : f \in S_{E^*}\} < 1.$$

Observe that in the light of Theorem 3.1 this property can be defined equivalently in the following way:

A Banach space E is said to have the property strong (α') whenever $\beta_0(E) < 1$.

On the other hand the above assertion implies that a space E has the property strong (α') if and only if $\varepsilon_1(E) < 1$. Indeed, it is a simple consequence of Theorem 3.2.

Let us mention that the above statement is one of the main results established in [9].

Thus, in virtue of the results from [8] mentioned above we have the following theorem.

Theorem 3.3. *If a Banach space E has the property strong (α') then it is reflexive and has normal structure.*

Finally let us mention that the condition $\varepsilon_1(E) < 1$ is equivalent to the condition $\Delta_E(1-) = \lim_{t\to 1-} \Delta_E(t) \neq 0$.

4. Final remarks

In this section we would like to pay attention to some unclear (and even incorrect) statements established in the paper [9]. Moreover, we shall raise some open problems in connection with those statements.

In the paper [7] the authors introduced the concept of the property strong (α'). We will not quote the definition of this concept given in [7] but we only recall its equivalent formulation (cf. also [7]).

Definition 4.1. A Banach space is said to have property strong (α') if and only if $\Delta_E(1) \neq 0$.

Thus, if a Banach space E has the property strong (α') then it has also the property (α'). It is not known if the reverse relation is true. In fact, this depends on the continuity of the function Δ_E at the point $t = 1$ (cf. [7]).

Now, let us recall the definition of the property (α) given in [13].

Definition 4.2. We say that a Banach space E has the property (α) (equivalently, E has the so-called drop property [5], [12], [13]) if for every $f \in S_{E^*}$ we have that

$$\lim_{\varepsilon \to 0} \alpha(F(f, \varepsilon)) = 0.$$

The authors of [7] asked the question if there is any relationship between the properties (α) and (α').

In the paper [9] the following example given below is discussed in connection with this question.

Consider the space $H = l^2(l^2, l^3, l^4, \dots)$ consisting of all sequences $x = (x_1, x_2, \dots)$ such that $x_n \in l^n$, $(n = 2, 3, \dots)$ and

$$\|x\|_H = \|(x_1, x_2, \dots)\|_H = \left(\sum_{n=2}^{\infty} \|x_n\|_{l^n}^2 \right)^{1/2} < \infty.$$

The space H has the property (α) which was established in the paper [12].

On the other hand we know that

$$\Delta_{l^p}(\varepsilon) = 1 - \big(1 - (\varepsilon/2)\big)^{1/p}, \quad \varepsilon \in [0, 2],$$

for every $p \in (1, \infty)$ [8].

Thus we have

$$\Delta_H(\varepsilon) \le \Delta_{l^n}(\varepsilon)$$

for any $n = 2, 3, \dots$ and for any $\varepsilon \in [0, 2]$.

The last inequality implies that

$$\Delta_H(\varepsilon) = 0$$

for $\varepsilon < 2$ but it does not imply that $\Delta_H(2) = 0$, as stated in [9]. Thus we cannot infer (as in [9]) that the space $H = l^2(l^2, l^3, l^4, \dots)$ does not have the property (α').

Observe that keeping in mind that the space H is <u>nearly strictly convex</u> [1] we can expect that $\Delta_H(2) \neq 0$.

Finally, let us raise the following open questions.

1. *Calculate the exact value of* $\Delta_H(2)$ *or at least show that* $\Delta_H(2) \neq 0$, *where* $H = l^2(l^2, l^3, l^4, \dots)$.
2. *Is it true that a Banach space* E *is nearly strictly convex* [1] *if and only if* $\Delta_E(2) \neq 0$?
3. *Prove* (*or disprove*) *that the space* $H = l^2(l^2, l^3, l^4, \dots)$ *has the property* (α').

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