Comment.Math.Univ.Carolin. 37,1 (1996)99–116

On the asymmetric divisor problem with congruence conditions

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Abstract. A certain generalized divisor function $d^*(n)$ is studied which counts the number of factorizations of a natural number n into integer powers with prescribed exponents under certain congruence restrictions. An Ω-estimate is established for the remainder term in the asymptotic for its Dirichlet summatory function.

Keywords: multidimensional asymmetric divisor problems

Classification: 11N37, 11P21, 11N69

Introduction

For $N = p + q \geq 2$ (where p and q are positive integers), and fixed natural numbers $a_1, \dots, a_p, a_{p+1} = b_1, \dots, a_{p+q} = b_q$, let $d^*(n)$ denote the number of ways to write the positive integer n as a product of different powers of N factors, of which p satisfy certain congruence conditions,

$$d^*(n) = d(a_1, \dots, a_N; m_1, \dots m_p; n) =$$
$$\#\{(u_1, \dots, u_N) \in \mathbb{N}^N : u_1^{a_1} \dots u_N^{a_N} = n, u_j \equiv l_j \pmod{m_j} \quad (j = 1, \dots, p)\},$$

where l_j and m_j are given natural numbers, with $l_j < m_j$.

For a large real variable x, we consider the remainder term $E(x)$ in the asymptotic formula

$$D^*(x) = \sum_{n \leq x} d^*(n) = H(x) + E(x)$$

where

$$H(x) = \sum_{s_0 = 0, \frac{1}{b_1}, \dots, \frac{1}{b_q}} \operatorname{Res}_{s=s_0} \left(F(s) \frac{x^s}{M^s s} \right)$$

where $M = m_1^{a_1} \dots m_p^{a_p}$ and $F(s)$ is the generating function

$$F(s) = M^s \sum_{n=1}^{\infty} d^*(n) n^{-s} = \prod_{j=1}^{p} \zeta(a_j s, \lambda_j) \prod_{i=1}^{q} \zeta(b_i s) \qquad (\operatorname{Re} s > 1),$$

This article is part of a research project supported by the *Austrian Science Foundation* (Nr. P 9892-PHY).

$\lambda_j = \frac{l_j}{m_j}$ for $j = 1, \ldots, p$ and $\zeta(s), \zeta(s, .)$ denote the Riemann and Hurwitz zeta-functions, respectively.

Upper bounds for the error term $E(x)$ can be readily established as a trivial generalization of the corresponding results for the asymmetric divisor problem. For a historical survey see e.g. the textbooks of Ivić [7], Krätzel [8], Titchmarsh [16].

As in Nowak [10], [11] we generalize the asymmetric divisor problem with respect to arithmetic progressions. In the present paper, we shall be concerned with a lower bound for this remainder term. We therefore use a classical method of Szegö and Walfisz [14] with a more recent technique due to Hafner [5].

Remark. Throughout the paper we denote by $C(\lambda, \mu)$, λ, μ real numbers, the oriented polygonal line which joins the points $\lambda - i\infty$, $\lambda - i$, $\mu - i$, $\mu + i$, $\lambda + i\infty$ in this order.

Statement of results

Theorem 1. *For each integer $m > \frac{1}{2}(N-1)$, the Liouville-Riemann integral of order m of the error term $E(x)$ possesses an absolutely convergent series representation*

$$E_m(x) \stackrel{\text{def}}{=} \frac{1}{\Gamma(m)} \int_0^x (x-u)^{m-1} E(u)\, du =$$

$$= \pi^{\frac{N}{2} - \Sigma(1+m)} M^m \sum_{h=1}^{\infty} h^{-m} \sum_{\substack{(l_1, \ldots, l_p) \\ (l_i = 0,1)}} \beta(l_1, \ldots, l_p; h) I^*_{l_1, \ldots, l_p; m}\left(\frac{x}{M} \pi^{\Sigma} h\right) \tag{1}$$

where $\Sigma = a_1 + \ldots + a_N$ for short, and

$$\beta(l_1, \ldots, l_p; h) =$$

$$= \sum_{\substack{j_1, \ldots, j_p, i_1, \ldots, i_q \\ {j_1}^{a_1} \ldots {j_p}^{a_p} {i_1}^{b_1} \ldots {i_q}^{a_q} = h}} \frac{1}{j_1 \cdots j_p i_1 \cdots i_q} \prod_{k=1}^{p} (\sin(2\pi j_k \lambda_k))^{l_k} (\cos(2\pi j_k \lambda_k))^{1-l_k}. \tag{2}$$

The functions $I^*_{l_1, \ldots, l_p; m}(y)$ are defined, for every integer $m \geq 0$, by

$$I^*_{l_1, \ldots, l_p; m}(y) =$$

$$= \sum_{k=-1, \ldots, -m} \operatorname{Res}_{s=k} \left(G_{l_1, \ldots, l_p}(s) \frac{\Gamma(s)}{\Gamma(s+m+1)} y^{s+m} \right) + I_{l_1, \ldots, l_p; m}(y)$$

where $I_{l_1 \ldots, l_p; m}(y)$ is given by an absolutely convergent integral representation

$$I_{l_1 \ldots, l_p; m}(y) = \frac{1}{2\pi i} \int_{C(\lambda, \mu)} G_{l_1, \ldots, l_p}(s) \frac{\Gamma(s)}{\Gamma(s+m+1)} y^{s+m}\, ds.$$

Here $\lambda, \mu,$ are real numbers satisfying

$$\lambda > \frac{N}{2\Sigma}, \qquad \mu < -m,$$

and

$$G_{l_1,\dots,l_p}(s) = \prod_{i=1}^{q} \frac{\Gamma(\frac{1}{2} - \frac{b_i s}{2})}{\Gamma(\frac{b_i s}{2})} \prod_{k=1}^{p} \left(\frac{\Gamma(\frac{1}{2} - \frac{a_k s}{2})}{\Gamma(\frac{a_k s}{2})} \right)^{1-l_k} \left(\frac{\Gamma(1 - \frac{a_k s}{2})}{\Gamma(\frac{1}{2} + \frac{a_k s}{2})} \right)^{l_k}. \tag{3}$$

The functions $I_{l_1,\dots,l_p;m}(y)$ possess an asymptotic expansion

$$\begin{aligned} I_{l_1,\dots,l_p;m}(y) &= \\ &= \sum_{j=0}^{L} C_{m,j} y^{m+\frac{1}{\Sigma}(-\frac{1}{2}+\frac{N}{2}+m-j)} \\ &\cos(e^{\frac{K}{\Sigma}} y^{\frac{1}{\Sigma}} + \frac{\pi}{4}(N-3) - \frac{\pi}{2}(l_1 + \ldots + l_p) + \frac{\pi}{2} j - \pi m) + \\ &+ O(y^{m+\frac{N}{2\Sigma} - \frac{M+m+\frac{3}{2}}{\Sigma}}) \end{aligned} \tag{4}$$

where L is an arbitrary positive integer and the coefficients $C_{m,j}$ are computable. In particular, the leading coefficient is given by

$$C_{0,0} = \pi \sqrt{\frac{\pi}{2}} \Sigma^{1-\frac{N}{2}} \prod_{i=1}^{N} \sqrt{a_i}.$$

Theorem 2. *Let a^* be the minimum value of the numbers $a_1, \dots, a_N$ and $\theta = \frac{1}{\Sigma}(-\frac{1}{2} + \frac{N}{2})$.*
For $N \geq 4$, and $x \to \infty$,

$$E(a_1,.,a_N; m_1,.,m_p; x) = \Omega_{\pm}(x^{\theta} (\log x)^{a^* \theta} (\log\log x)^{q-1} (\log\log\log x)^{-(\frac{\Sigma}{2}+a^*)\theta}).$$

For $N \geq 2$ and $x \to \infty$,

$$E(a_1,.,a_N; m_1,.,m_p; x) = \Omega(x^{\theta} (\log x)^{a^* \theta} (\log\log x)^{q-1} (\log\log\log x)^{-(\frac{\Sigma}{2}+a^*)\theta}).$$

For the case of $N = 2$, this can be refined to

$$E(x) = \Omega_{\pm}((x(\log x)^{a*})^{\theta} (\log\log\log x)^{-(\frac{\Sigma}{2}+a^*)\theta})$$

if

$$0 < \frac{l}{m} < \frac{1}{6} \quad or \quad \frac{1}{2} < \frac{l}{m} < \frac{5}{6}.$$

For the case of $N = 3$, the remainder term $E(x)$ satisfies

$$E(x) = \Omega_{\pm}((x(\log x)^{a^*})^{\theta}(\log\log\log x)^{-(\frac{\Sigma}{2}+a^*)\theta}),$$

if we induce only on one factor a congruence condition, and this satisfies

$$\frac{l}{m} \neq \frac{1}{2},$$

whereas if we induce congruence conditions on two factors, the remainder term $E(x)$ satisfies

$$E(x) = \Omega_{\pm}((x(\log x)^{a^*})^{\theta}(\log\log\log x)^{-(\frac{\Sigma}{2}+a^*)\theta},$$

if

$$\log(2\sin(\pi\frac{l_1}{m_1}))(\frac{1}{2} - \frac{l_2}{m_2}) + \log(2\sin(\pi\frac{l_2}{m_2}))(\frac{1}{2} - \frac{l_1}{m_1}) \neq 0.$$

Proof of Theorem 1

A version of Perron's formula yields

$$\begin{aligned} D^*_m(x) &\overset{\text{def}}{=} \frac{1}{\Gamma(m)}\int_0^{\infty}(x-u)^{m-1}D^*(u)\,du = \\ &= \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{F(s)}{M^s}\frac{\Gamma(s)}{\Gamma(s+m+1)}x^{s+m}\,ds \end{aligned} \tag{5}$$

where m is an integer greater than $\frac{N}{2}$. Now we shift the line of integration left to zero, observing that for δ be a suitable small positive constant, then for each $\varepsilon > 0$

$$\zeta(\sigma + it, \lambda) \ll (1 + |t|)^{\frac{1}{2}+\varepsilon}$$

in $|t| \geq 1$, $\sigma \geq -\delta$ (this is a consequence of the Phragmén-Lindelöf principle). For the Gamma-functions involved, we recall Stirling's formula in the weak form

$$|\Gamma(\sigma + it)| \asymp |t|^{\sigma-\frac{1}{2}}\exp(-\frac{\pi}{2}|t|)$$

uniformly in $|t| \geq 1$, $\sigma_1 \leq \sigma \leq \sigma_2$, ($\sigma_1, \sigma_2$ arbitrary). From this it is an immediate consequence that the integrand in (5) is $\ll |t|^{-m-1+\frac{N}{2}+\varepsilon'}$ where ε' can be made arbitrarily small by the choice of δ. The sum of the residues at $s = 0, \frac{1}{b_1}, \dots, \frac{1}{b_q}$ is obviously just the order term $H(x)$, thus we obtain

$$E_m(x) = \frac{1}{2\pi i}\int_{-\delta-i\infty}^{-\delta+i\infty}F(s)\frac{\Gamma(s)}{\Gamma(s+m+1)}\frac{x^{m+s}}{M^s}\,ds$$

for the new integral is absolutely convergent, since $m > \frac{N}{2}$.

By the functional equations of the Riemann and the Hurwitz zeta-function (see e.g. [1], pp. 257–259)

$$\zeta(s) = \frac{1}{(2\pi)^{1-s}} 2\Gamma(1-s)\zeta(1-s)\sin(\frac{\pi}{2}s),$$

$$\zeta(s,\lambda) = \frac{1}{(2\pi)^{1-s}} 2\Gamma(1-s)\sum_{h=1}^{\infty}\frac{1}{h^{1-s}}\sin(2\pi h\lambda + \frac{\pi}{2}s) \quad (\operatorname{Re} s < 0),$$

we conclude that, for $\operatorname{Re} s < 0$,

$$F(s) = \frac{2^{\Sigma}}{(2\pi)^{\Sigma(1-s)}}\prod_{i=1}^{N}\Gamma(1-a_i s)\prod_{i=1}^{q}\zeta(1-b_i s)\sin(\frac{\pi}{2}b_i s)\times$$
$$\times\prod_{j=1}^{p}\sum_{h=1}^{\infty}\frac{1}{h^{1-a_j s}}\sin(2\pi h\lambda_j + \frac{\pi}{2}a_j s).$$

Inserting the Dirichlet series for all of the factors $\zeta(1-a_i s)$ gives,

$$F(s) = \frac{2^{\Sigma s}}{\pi^{\Sigma(1-s)}}\prod_{i=1}^{N}\Gamma(1-a_i s)\sum_{h=1}^{\infty}h^{s}\sum_{\substack{(l_1,\dots,l_p)\\(l_i=0,1)}}\beta(l_1,\dots,l_p;h)\times$$
$$\underbrace{\times\prod_{i=1}^{q}\sin(\frac{\pi}{2}b_i s)\prod_{k=1}^{p}(\cos(\frac{\pi}{2}a_k s))^{l_k}(\sin(\frac{\pi}{2}a_k s))^{1-l_k}}_{G_{l_1,\dots,l_p(s)}},$$

with $\beta(l_1,\dots,l_p;h)$ defined in (2).

By well known properties of the Gamma function,

$$\Gamma(1-us)\sin(\frac{\pi}{2}us) = \sqrt{\pi}2^{-us}\frac{\Gamma(\frac{1}{2}-\frac{us}{2})}{\Gamma(\frac{us}{2})}$$

$$\Gamma(1-us)(\cos(\frac{\pi}{2}us))^{l}(\sin(\frac{\pi}{2}us))^{1-l} = \begin{cases} \sqrt{\pi}2^{-us}\frac{\Gamma(\frac{1}{2}-\frac{us}{2})}{\Gamma(\frac{us}{2})}, & \text{for } l=0 \\ \sqrt{\pi}2^{-us}\frac{\Gamma(1-\frac{us}{2})}{\Gamma(\frac{1}{2}+\frac{us}{2})}, & \text{for } l=1 \end{cases}$$

we obtain

$$E_m(x) = \pi^{\frac{N}{2}-\Sigma(1+m)}M^m\sum_{h=1}^{\infty}h^{-m}\sum_{\substack{(l_1,\dots,l_p)\\(l_i=0,1)}}\beta(l_1,\dots,l_p;h)I^{**}_{l_1,\dots,l_p;m}(\frac{x}{M}\pi^{\Sigma}h)$$

with
$$I^{**}_{l_1,\dots,l_p}(y) = \frac{1}{2\pi i}\int_{-\delta-i\infty}^{-\delta+i\infty} G_{l_1,\dots,l_p}(s)\frac{\Gamma(s)}{\Gamma(s+m+1)}y^{s+m}\,ds.$$

It is evident from the functional equation that all the singularities of $G_{l_1,\dots,l_p}(s)$ are on the positive real axis. Observing this, we can deform the line of integration such that $I^{**}_{l_1,\dots,l_p;m}(y) = I^{*}_{l_1,\dots,l_p;m}(y)$, provided that $\lambda \geq 0$ and $\mu < -m$. In order to get absolutely convergent integrals $I_{l_1,\dots,l_p;m}(y)$ for $m \geq 0$ we choose λ greater than $\frac{N}{2\Sigma}$. Therefore

$$\frac{d}{dy}(I^{*}_{l_1,\dots,l_p;m}(y)) = I^{*}_{l_1,\dots,l_p;m}(y). \tag{6}$$

(Notice that this is also valid for $I_{l_1,\dots,l_p;m}(y)$ for this differs from $I^{*}_{l_1,\dots,l_p}(y)$ only by a finite sum of differentiable functions.)

To complete the proof of Theorem 1, it remains to establish the asymptotic expansion of

$$G_{l_1,\dots,l_p}(s)\frac{\Gamma(s)}{\Gamma(s+m+1)}. \tag{7}$$

In what follows we write $R_k(s)$ for expressions of the form

$$R_k(s) = \sum_{j=1}^{L+1} c_{k,j}s^{-j}$$

where $c_{k,j}$ are any complex coefficients. We use Stirling's formula in the form

$$\log\Gamma(s+c) = (s+c-\frac{1}{2})\log s - s + \frac{1}{2}\log 2\pi + R_1(s) + O(|s|^{-L-2})$$

with $c \in \mathbb{C}$ arbitrary, which holds uniformly for $|\arg(s+c)| \leq \beta_0 < \pi$. (The coefficients $c_{1,j}$ and the O-constant may depend on c.) Employing this we compute an asymptotic expansion for the logarithm of (7) and compare it with the asymptotic expansion of the logarithm of

$$\frac{\Gamma(-a's+b')}{\Gamma(\frac{1}{2}-\frac{a'}{2}+c')\Gamma(\frac{1}{2}+\frac{a'}{2}s-c')}\mathrm{e}^{Ks+c}. \tag{8}$$

This yields that the logarithm of (7)

$$F_0(s) = C^{*}_{m}\mathrm{e}^{Ks}\Gamma(-\Sigma s+\frac{N}{2}-m-\frac{1}{2})\cos\pi(\frac{\Sigma s}{2}+1+m-\frac{N}{2}+\frac{1}{2}(l_1+\ldots+l_p))$$

has the same asymptotic expansion as the logarithm of (8), where

$$K = \frac{\Sigma}{2}\log\left(\frac{\Sigma}{2}\right) - \frac{\Sigma}{2} + \sum_{i=1}^{N} a_i\left(1-\log\left(\frac{a_i}{2}\right)\right),$$

$$C_m^* = \exp\left(\frac{1}{2}\log(2\pi) + \log\pi + \left(1+m-\frac{N}{2}\right)\log\left(\frac{\Sigma}{2}\right) + \sum_{i=1}^{N}\frac{1}{2}\log\left(\frac{a_i}{2}\right)\right).$$

Thus, on any set avoiding the poles of the terms involved,

$$G_{l_1,\ldots,l_p}(s)\frac{\Gamma(s)}{\Gamma(s+m+1)} = F_0(s)(1+R_2(s)+O(|s|^{-L-2})) =$$

$$= F_0(s)\Big(1+\sum_{j=1}^{L+1} c_j^*\prod_{i=1}^{j}\Big(-\Sigma s+\frac{N}{2}-m-\frac{1}{2}-i\Big)+O((1+|s|^{-L-2})\Big) =$$

$$= F_0(s) + \sum_{j=1}^{L+1} c_j^* F_j(s) + \Delta(s)$$

with

$$F_j(s) = C_m^* e^{Ks}\Gamma\Big(-\Sigma s+\frac{N}{2}-m-\frac{1}{2}-j\Big)\cos\pi\Big(\frac{\Sigma s}{2}+1+m-\frac{N}{2}+\frac{1}{2}(l_1+\ldots+l_p)\Big)$$

by the functional equation for the Γ-function, and

$$\Delta(s) \ll |t|^{-L-2}|F_0(s)| \ll |t|^{-L-m-3+\frac{N}{2}-\Sigma\sigma}$$

uniformly in $|t|\geq 1$, $\sigma_1\leq\sigma\leq\sigma_2$ (σ_1,σ_2 arbitrary). We can therefore bound the contribution of $\Delta(s)$ to the integral $I_{l_1,\ldots,l_p;m}(y)$,

$$\int_{C(\Lambda,\mu)}\Delta(s)y^{s+m}\,ds \ll y^{\mu+m}+y^{\Lambda+m} \ll y^{m-\frac{L+m+\frac{3}{2}}{\Sigma}+\frac{N}{2}}$$

by the choice of $\Lambda = -\frac{L+m+\frac{3}{2}}{\Sigma}+\frac{N}{2}$ (notice that μ is only restricted by $\mu\leq -m$ and may therefore be assumed to be less than Λ). Consequently,

$$I_{l_1,\ldots,l_p;m}(y) = J_{l_1,\ldots,l_p;0}(y) + \sum_{j=1}^{L+1} c_j^* J_{l_1,\ldots,l_p;j}(y) + O\Big(y^{m+\frac{N}{2\Sigma}-\frac{L+m+\frac{3}{2}}{\Sigma}}\Big)$$

where, for $j=0,1,\ldots,L+1$,

$$J_{l_1,\ldots,l_p;j}(y) = \frac{1}{2\pi i}\int_{C(\lambda,\mu)} F_j(s)y^{s+m}\,ds.$$

To evaluate the remaining integrals, we use the following identity (valid for $\lambda_1 > \frac{1}{2}$, $\mu_1 < 0$, $z \in \mathbb{R}^+$),

$$\frac{1}{2\pi i}\int_{C(\lambda_1,\mu_1)} \Gamma(-s_1)\cos\left(\frac{\pi}{2}s_1+\gamma\right)z^{s_1}\,ds_1 = \cos(z-\gamma)$$

(see e.g. [12]). Recalling the definition of $F_j(s)$, we substitute

$$s_1 = \Sigma * s - \frac{N}{2} + m + \frac{1}{2} + j, \quad \gamma = \frac{\pi}{2} * \left(\frac{3}{2} + m - N - j + (l_1 + \ldots + l_p)\right), \quad z = (\mathrm{e}^K * y)^{\frac{1}{\Sigma}}$$

in this last identity. After a few simple calculations the assertion of Theorem 1 follows, at least for $m \geq \frac{1}{2}Np$. But since $\sum_{h=1}^{\infty}\beta(l_1,\ldots,l_p;h)h^{-\varepsilon} < \infty$ for each $\varepsilon > 0$, it is evident from (4) that the series in (1) converges absolutely for every $m > \frac{1}{2}(Np-1)$. Appealing to (6), we complete the proof for this slightly larger range of m.

Proof of Theorem 2

We employ a classic method of Szegö and Walfisz [14] involving the Borel mean-value with more recent technique due to Hafner [5]. For a large real parameter t, we put

$$X = X(t) = K_1(\log t)^{-a^*}(\log\log\log t)^{\frac{\Sigma}{2}+a^*} \tag{9}$$

and

$$k = k(t) = K_2(\zeta + tX^{-\frac{1}{\Sigma}})^2, \tag{10}$$

with positive constants K_1, K_2 and real ζ to be specified later. We consider

$$B(t) = \frac{1}{\Gamma(k+1)}\int_0^\infty \mathrm{e}^{-u}u^k E(u^{\frac{\Sigma}{2}}X)\,du.$$

We substitute $v = u^{\frac{\Sigma}{2}}$ and put $h(v) = \frac{2}{\Sigma}\exp(-v^{\frac{2}{\Sigma}})v^{\frac{2(k+1)}{\Sigma}-1}$.

We choose $m = [\frac{1}{2}N]+1$ and observe that $h(v)$ and its first m derivatives vanish at $v=0$ and at $v=\infty$ if t and thus k is sufficiently large. Therefore, an iterated integration by parts gives

$$B(t) = \frac{1}{\Gamma(k+1)}\int_0^\infty h(v)E(Xv)\,dv = \frac{(-1)^m X^{-m}}{\Gamma(k+1)}\int_0^\infty h^{(m)}(v)E_m(Xv)\,dv.$$

We insert the series representation (1), interchange the order of summation and integration and apply iterated integration by parts one more time, keeping (6) in mind. This leads to

$$B(t) = \pi^{\frac{N}{2}-\Sigma}\sum_{h=1}^{\infty}\sum_{\substack{(l_1,\ldots,l_p)\\(l_i=0,1)}}\beta(l_1,\ldots,l_p;h)\frac{1}{\Gamma(k+1)}\times$$

$$\times\int_0^\infty \mathrm{e}^{-u}u^k I^*_{l_1,\ldots,l_p;0}\left(u^{\frac{\Sigma}{2}}\frac{X}{M}\pi^{\Sigma}h\right)du.$$

Now we insert the asymptotic expansion (4) for the integrals $I^*_{l_1,\dots,l_p;0}(y) = I_{l_1,\dots,l_p;0}(y)$ and remark that $\beta(l_1,\dots,l_p;h) \ll h^\varepsilon$ for each $\varepsilon > 0$. We choose L so that the exponent of n in the error term of (4) be less than -1. This is achieved for

$$L = [\frac{1}{2}(N-3)+\Sigma]+1.$$

The contribution of the O-term to the asymptotic expansion of $B(t)$ is then bounded by

$$\begin{aligned}&\ll \frac{k^\varepsilon}{\Gamma(k+1)}\int_0^\infty \mathrm{e}^{-u}u^{k+\frac{N}{4}-\frac{1}{2}(L+\frac{3}{2})}\,du \ll\\ &\ll k^{\varepsilon+\frac{N}{4}-\frac{1}{2}(L+\frac{3}{2})} \ll k^{\varepsilon-\frac{1}{2}\Sigma} \ll k^{-\frac{1}{2}},\end{aligned}$$

in view of Stirling's formula.

To deal with the main terms of (4), we make use of a result from classic analysis going back to Szegö [14], and Szegö and Walfisz [15].

Lemma 1. *Let α, c, c', be real constants. Then for $k\to\infty$,*

$$J(k,T) = \frac{1}{\Gamma(k+1)}\int_0^\infty \mathrm{e}^{-u}u^{k+\alpha}\exp(iT\sqrt{u})\,du =$$

$$= \begin{cases} k^\alpha \exp(-\frac{1}{8}T^2)\exp(iT\sqrt{k}) \quad + O(k^{\alpha-\frac{1}{2}+\varepsilon}) & \text{if } ck^{-\varepsilon}\le T\le ck^\varepsilon\\ \ll T^{-C} & \text{for every real constant } C, \text{ if } T\ge c'k^\varepsilon\end{cases}$$

Proof: This is an immediate consequence of a result of Szegö [14, pp. 100–102], and Szegö-Walfisz [15]. Applying this Lemma to the integrals which arise if we insert the significant terms of (4), we conclude that the main term, with $j=0$ is of the form

$$\begin{aligned}&c^*\frac{(hX)^\theta}{\Gamma(k+1)}\int_0^\infty \mathrm{e}^{-u}u^{k+\frac{\Sigma}{2}\theta}\cos\left(c_1(hX)^{\frac{1}{\Sigma}}\sqrt{u}+\frac{\pi}{4}(N-3)-\frac{\pi}{2}(l_1+..+l_p)\right)du =\\ &= c^*(hX)^\theta k^{\frac{N-1}{4}}\mathrm{e}^{-c_2(hX)^{\frac{2}{\Sigma}}}\cos\left(c_1(hX)^{\frac{1}{\Sigma}}\sqrt{k}+\frac{\pi}{4}(N-3)+\frac{\pi}{2}(l_1+..+l_p)\right)+\\ &+\begin{cases} O((hX)^\theta k^{\frac{N-3}{4}+\varepsilon}), & \text{for } ck^{-\varepsilon}\le c_1(hX)^{\frac{1}{\Sigma}}\le c'k^\varepsilon\\ \ll (hX)^{-C}, & \text{for every real constant } C, \text{ if } c_1(hX)^{\frac{1}{\Sigma}} > c'k^\varepsilon\end{cases}\end{aligned}$$

where $c^* = c_{0,0}(M^{-1}\pi^\Sigma)^\theta$ and $c_1 = (\mathrm{e}^K\pi^\Sigma M^{-1})^{\frac{1}{\Sigma}}$.
The contribution of the other terms is

$$\ll (hX)^{\theta-\frac{1}{2}}k^{\frac{\Sigma}{2}(\theta-\frac{1}{2})}\mathrm{e}^{-c_2(hX)^{\frac{2}{\Sigma}}} \ll (hX)^\theta k^{\frac{N-2}{4}},$$

for $c_1 k^{-\varepsilon} \le c_1(hX)^{\frac{1}{\Sigma}} \le c' k^{\varepsilon}$ and $j = 1, \dots, L$.

We estimate the contribution of the error term to the asymptotic expansion of $B(t)$. The terms corresponding to h which satisfy $c_1(hX)^{\frac{1}{\Sigma}} \le c' k^{\varepsilon}$, contribute

$$\ll \sum_{h \le c_3 X^{-1} k^{\varepsilon\Sigma}} \sum_{\substack{(l_1,\dots,l_p) \\ (l_i=0,1)}} \beta(l_1,\dots,l_p;h)(hX)^{\theta} k^{\frac{N-2}{4}} \ll$$

$$\ll X^{\theta} k^{\frac{N-2}{2}} (X^{-1} k^{\varepsilon\Sigma})^{1+\varepsilon+\theta} \ll k^{\frac{N-2}{4}+\varepsilon'} \ll k^{\frac{N}{4}-\frac{3}{8}},$$

whereas the terms corresponding to h which satisfy $c_1(hX)^{\frac{1}{\Sigma}} \ge c' k^{\varepsilon}$, contribute only

$$\ll \sum_{h \ge c_3 X^{-1} k^{\varepsilon\Sigma}} \sum_{\substack{(l_1,\dots,l_p) \\ (l_i=0,1)}} \beta(l_1,\dots,l_p;h)(hX)^{-C} \ll X^{-C} (X^{-1} k^{\varepsilon\Sigma})^{-1} = o(1)$$

as $t \to \infty$ by the choice of $C = 1 + \theta + 2$.
Altogether, we deduce that

$$B(t) = C^{**} X^{\theta} k^{\frac{N-1}{4}} \sum_{h \le c_3 X^{-1} k^{\varepsilon\Sigma}} \sum_{\substack{(l_1,\dots,l_p) \\ (l_i=0,1)}} \beta(l_1,\dots,l_p;h) \times$$

$$\times h^{\theta} \mathrm{e}^{-c_2 (hX)^{\frac{2}{\Sigma}}} \cos\left(c_1 (hX)^{\frac{1}{\Sigma}} \sqrt{k} + \frac{\pi}{4}(N-3) - \frac{\pi}{2}(l_1 + .. + l_p)\right) + O(k^{\frac{N}{4}-\frac{3}{8}})$$

where

$$C^{**} = \pi^{\frac{N}{2}+\Sigma(\theta-1)+1} M^{-\theta} \sqrt{\frac{\pi}{2}} \Sigma^{1-\frac{N}{2}} \prod_{i=1}^{N} \sqrt{a_i}.$$

In order to extend the range of summation in this series to $1 \le h < \infty$, it suffices to observe that

$$X^{\theta} k^{\frac{N-1}{4}} \sum_{h > c_3 X^{-1} k^{\varepsilon\Sigma}} \sum_{\substack{(l_1,\dots,l_p) \\ (l_i=0,1)}} \beta(l_1,\dots,l_p;h) h^{\theta} \exp\left(-c_2 (hX)^{\frac{2}{\Sigma}}\right) \ll$$

$$\ll k^{\frac{N-1}{4}} \sum_{h > c_3 X^{-1} k^{\varepsilon\Sigma}} \exp\left(-c_4 (hX)^{\frac{2}{\Sigma}}\right) \ll$$

$$\ll k^{\frac{N-1}{4}} \left(\exp(-c_5 k^{2\varepsilon}) + \int_{c_3 X^{-1} k^{\varepsilon\Sigma}}^{\infty} \exp\left(-c_4 (uX)^{\frac{2}{\Sigma}}\right) du \right) \ll$$

$$\ll \exp(-c_6 k^{2\varepsilon}) \ll k^{-1}$$

Consequently,

$$B(t) = C^{**} X^{\theta} k^{\frac{N-1}{4}} \sum_{h=1}^{\infty} \sum_{\substack{(l_1,\dots,l_p)\\(l_i=0,1)}} \beta(l_1,\dots,l_p;h) h^{\theta} \exp(-c_2(hX)^{\frac{2}{\Sigma}}) \times$$

(11)

$$\times (\cos(c_1(hX)^{\frac{1}{\Sigma}}\sqrt{k} + \frac{\pi}{4}(N-3) - \frac{\pi}{2}(l_1+\ldots+l_p))) + O(k^{\frac{N}{4}-\frac{3}{8}}).$$

We recall the definition of $\beta(l_1,\dots,l_p;h)$, keep $h = j_1^{a_1}\dots j_p^{a_p} i_1^{b_1}\dots i_q^{b_q}$ fixed for the moment and compute (with $Z = c_1(hX)^{\frac{1}{\Sigma}}\sqrt{k} + \frac{\pi}{4}(N-3)$ for short)

$$\sum_{\substack{(l_1,\dots,l_p)\\(l_i=0,1)}} \beta(l,\dots,l_p;h)\cos(Z - \frac{\pi}{4}(l_1+\ldots+l_p)) =$$

$$= \sum_{\substack{(j_1,\dots,j_p,i_1,\dots,i_q)\\ j_1^{a_1}\dots j_p^{a_p} i_1^{b_1}\dots i_q^{b_q}=h}} \frac{1}{j_1\dots j_p i_1\dots i_q} \times$$

$$\times \sum_{\substack{(l_1,\dots,l_p)\\(l_i=0,1)}} \prod_{k=1}^{p} (\sin(2\pi j_k\lambda_k))^{l_k} (\cos(2\pi j_k\lambda_k))^{1-l_k} \cos(Z - \frac{\pi}{2}(l_1+\ldots+l_p)) =$$

$$= \sum_{\substack{(j_1,\dots,j_p,i_1,\dots,i_q)\\ j_1^{a_1}\dots j_p^{a_p} i_1^{a_1}\dots i_q^{a_q}=h}} \frac{1}{j_1\dots j_p i_1\dots i_q} \cos(Z - 2\pi\sum_{j=1}^{p} h_j\lambda_j)$$

by the general addition theorems for the cosine and sine functions.
We conclude that

$$B(t) = C^{**} k^{\frac{N-1}{4}} \{X^{\theta} \sum_{h=1}^{\infty} h^{\theta} \exp(-c_2(hX)^{\frac{2}{\Sigma}}) \times$$

$$\times (a_h \cos(c_1(hX)^{\frac{1}{\Sigma}}\sqrt{k} + \frac{\pi}{4}(N-3)) + b_h \sin(c_1(hX)^{\frac{1}{\Sigma}}\sqrt{k} + \frac{\pi}{4}(N-3))) +$$

$$+ O(k^{-\frac{1}{8}})\}$$

where

$$a_h = \sum_{\substack{j_1,\dots,j_p,i_1,\dots,i_p\\ j_1^{a_1}\dots j_p^{a_p} i_1^{b_1}\dots i_q^{b_q}=h}} \frac{\cos(2\pi\sum_{k=1}^{p} j_k\lambda_k)}{j_1\dots j_p i_1\dots i_q},$$

$$b_h = \sum_{\substack{j_1,\dots,j_p,i_1,\dots,i_p\\ j_1^{a_1}\dots j_p^{a_p} i_1^{b_1}\dots i_q^{b_q}=h}} \frac{\sin(2\pi\sum_{k=1}^{p} j_k\lambda_k)}{j_1\dots j_p i_1\dots i_q}.$$

The next step is to approximate a finite partial sum of the infinite series in (11) by an expression of the form

$$f(X,\zeta)=\sum_{h\le B_0}(a_h g_1(X,h,\zeta)+b_h g_2(X,h,\zeta)),$$

where, for short,

$$g_1(X,u,\zeta)=\exp(-c_2(Xu)^{\frac{2}{\Sigma}})u^{\theta}\cos(\zeta(Xu)^{\frac{1}{N}}+\frac{\pi}{4}(N-3)),$$
$$g_2(X,u,\zeta)=\exp(-c_2(Xu)^{\frac{2}{\Sigma}})u^{\theta}\sin(\zeta(Xu)^{\frac{1}{N}}+\frac{\pi}{4}(N-3)).$$

Let a^* be the minimum value of $a_1,\dots,a_p$, $b_1,\dots,b_q$, then it is clear that if either of a_h, b_h is $\neq 0$, then h must be a^*-full. It is known that the number of a^*-full numbers $h\le B_1$ is $\le c_8 B_1^{\frac{1}{a^*}}$ (see e.g. Krätzel [8]). We now apply Dirichlet's approximation principle (see e.g. [8]): Let B_1 be a large positive integer and $q=[(\log B_1)^N]$. Then there exists a value of t in the interval

$$B_1\le t\le B_1 q^{c_8 B_1^{\frac{1}{a^*}}} \tag{12}$$

such that $\|\frac{1}{2\pi}h^{\frac{1}{\Sigma}}t\|\le\frac{1}{q}$ for the a^*-full $h\le B_1$, where $\|\,.\,\|$ denotes the distance from the nearest integer. It is an easy consequence of (12) that

$$B_1\gg(\log t)^{a^*}(\log q)^{-a^*}.$$

Let us define

$$B_0=c_8(\log t)^{a^*}(\log q)^{-a^*}$$

with c_8 so small that $B_0\le B_1$ for $q\ge 2$ and sufficiently large t.

Choosing in (10) $K_2=c_1^{-2}$, we thus may conclude that

$$|\cos(c_1(hX)^{\frac{1}{\Sigma}}\sqrt{k}+\frac{\pi}{4}(N-3))-\cos(\zeta(hX)^{\frac{1}{\Sigma}}+\frac{\pi}{4}(N-3))|\le$$
$$\le\|\frac{1}{2\pi}h^{\frac{1}{\Sigma}}t\|\le\frac{1}{q},$$

for all $h\in\mathbb{N}$ with $h\le B_0$ and $\beta(l_1,\dots,l_p;h)\neq 0$.

We consider the contribution from those h with $h \le B_0$. This is bounded by

$$X^\theta | \sum_{h \le B_0} h^\theta \exp(-c_2 (hX)^{\frac{2}{\Sigma}}) \times ((a_h \cos(c_1 (hX)^{\frac{1}{\Sigma}} \sqrt{k} + \frac{\pi}{4}(N-3)))+$$

$$+ b_h \sin(c_1 (hX)^{\frac{1}{\Sigma}} \sqrt{k} + \frac{\pi}{4}(N-3))) - f(X,\zeta)| \ll$$

$$\ll \frac{1}{q} X^\theta \sum_{h \le B_0} h^\theta \exp(-c_2 (hX)^{\frac{2}{\Sigma}}) \sum_{\substack{h_1,\dots,h_N \\ h_1^{a_1}\dots h_N^{a_N}=h}} \frac{1}{h_1 \dots h_N} \ll$$

$$\ll \frac{1}{q} X^\theta \int_{1-}^{B_0} \exp(-c_2 (hX)^{\frac{2}{\Sigma}})\, dS(u) \ll$$

$$\ll \frac{1}{q} X^\theta B_0^\theta (\log B_0)^{N-1}$$

where

$$S(u) = \sum_{h \le u} h^\theta \sum_{\substack{h_1,\dots,h_N \\ h_1^{a_1}\dots h_N^{a_N}=h}} \frac{1}{h_1 \dots h_N} \asymp u^\theta (\log u)^{N-1}.$$

Those h with $h \ge B_0$ contribute,

$$\ll X^\theta \sum_{h \ge B_0} h^\theta \exp(-c_2 (hX)^{\frac{2}{\Sigma}}) \sum_{\substack{h_1,\dots,h_N \\ h_1^{a_1}\dots h_N^{a_N}=h}} \frac{1}{h_1 \dots h_N} \ll$$

$$\ll X^\theta \left\{ B_0^\theta (\log B_0)^{N-1} \mathrm{e}^{(-c_2 (B_0 X)^{\frac{2}{\Sigma}})} + \int_{B_0}^{\infty} \mathrm{e}^{(-c_2 (uX)^{\frac{2}{\Sigma}})} (uX)^{\frac{2}{\Sigma}-1} X S(u)\, du \right\}.$$

We split up the last integral in $\int_{B_0}^{B_0^2} + \int_{B_0^2}^{\infty}$. The first integral contributes,

$$\ll (\log B_0)^{N-1} \int_{B_0}^{B_0^2} \exp(-c_2 (uX)^{\frac{2}{\Sigma}}) (uX)^{\frac{2}{\Sigma}} u^{\theta-1}\, du \ll$$

$$\ll B_0^\theta (\log B_0)^{N-1} \exp(-c_2 (B_0 X)^{\frac{2}{\Sigma}}).$$

In a similar way one verifies that the contribution of the second integral is $o(1)$ (as $t \to \infty$). In exactly the same way the infinite "tail" of the series in (11) can be estimated.

Combining this, we arrive at

$$B(t) = C^{**} k^{\frac{N-1}{4}} \{ X^\theta \sum_{h=1}^{\infty} h^\theta \exp(-c_2 (hX)^{\frac{2}{\Sigma}}) \times$$

$$\times (a_h \cos(\zeta (hX)^{\frac{1}{\Sigma}} + \frac{\pi}{4}(N-3)) + b_h \sin(\zeta (hX)^{\frac{1}{\Sigma}} + \frac{\pi}{4}(N-3)))+$$

$$+ O(\frac{1}{q} X^\theta B_0^\theta (\log B_0)^{N-1}) + O(X^\theta B_0^\theta (\log B_0)^{N-1} \exp(-c_2 (B_0 X)^{\frac{2}{\Sigma}})) + o(1) \}.$$

We conclude that

$$B(t) = C^{**}k^{\frac{N-1}{4}}\{X^\theta \sum_{h=1}^{\infty} h^\theta \exp(-c_2(hX)^{\frac{2}{\Sigma}})\times$$
$$\times(a_h \cos(\zeta(hX)^{\frac{1}{\Sigma}} + \frac{\pi}{4}(N-3)) + b_h \sin(\zeta(hX)^{\frac{1}{\Sigma}} + \frac{\pi}{4}(N-3))) + o(1)\}.$$

Our next step is an asymptotic formula for this last series, as $X \to 0^+$, ζ some real constant, in the spirit of Berndt [2]. To this end, we need an asymptotic formula for $S_1(u) = \sum_{h \le u} h^\theta a_h$, $S_2(u) = \sum_{h \le u} h^\theta b_h$. This can be done in one step.

For $\mathrm{Re}\, s > 1$, consider the generating function of $a_h + ib_h$,

$$Z(s) \stackrel{\text{def}}{=} \sum_{h=1}^{\infty} \frac{a_h + ib_h}{h^s} = \prod_{i=1}^{q} \zeta(b_i s + 1 - b_i\theta) \prod_{k=1}^{p} \sum_{n=1}^{\infty} \frac{\exp(2\pi i n\lambda_k)}{n^{a_k s + 1 - a_k\theta}}, \quad \mathrm{Re}\, s > 0.$$

By standard techniques it follows that

$$S_1(u) + iS_2(u) = \mathrm{Res}_{s=\theta}(Z(S)\frac{u^s}{s}) + o(u^\rho) = B_q u^\theta (\log u)^{q-1} + O(u^\theta(\log u)^{q-2})$$

where $\rho < 1$ and

$$B_q = C(q) \prod_{k=1}^{p} \sum_{n=1}^{\infty} \frac{\exp(2\pi i n\lambda_k)}{n} = C(q) \prod_{k=1}^{p} (-\log(2\sin(\pi\lambda_k) + i(\frac{\pi}{2} - \pi\lambda_k)).$$

Let $B_q = |B_q| \mathrm{e}^{2\pi i\beta}$ with $0 \le \beta \le 1$, then

$$S_1(u) + iS_2(u) = (|B_q|\cos(2\pi\beta) + i|B_q|\sin(2\pi\beta))u^\theta(\log u)^{q-1} + \\ + O(u^\theta(\log u)^{q-2}). \tag{13}$$

$\square$

Lemma 2. *For* $X \to 0^+$,

$$F(X,\zeta) \stackrel{\text{def}}{=} \sum_{h=1}^{\infty} h^\theta \exp(-c_2(hX)^{\frac{2}{\Sigma}})\times$$
$$\times((a_h \cos(\zeta(hX)^{\frac{1}{\Sigma}} + \frac{\pi}{4}(N-3))) + (b_h \sin(\zeta(hX)^{\frac{1}{\Sigma}} + \frac{\pi}{4}(N-3)))) =$$
$$= c_6 |B_q| X^{-\theta} |\log X|^{q-1}(G(\zeta) + o(1)),$$

where

$$G(\zeta) = \int_0^\infty \mathrm{e}^{-v^2} v^{-\frac{N-3}{4}} \cos(c_2^{-\frac{1}{2}}\zeta v - \frac{\pi}{4}(N-3) - 2\pi\beta)\, dv.$$

PROOF: With our previous notation, put $S(u) = S_1(u) + iS_2(u)$ and write $H_1(u) + iH_2(u)$ for the main term on the right hand side of the assertion of Lemma 2. Using Stieltjes integral notation

$$F(X, \zeta) = \int_0^\infty \exp(-c_2(uX)^{\frac{2}{\Sigma}}) \cos(\zeta(uX)^{\frac{1}{\Sigma}} + \frac{\pi}{4}(N-3))\, dS_1(u) + \\ + \int_0^\infty \exp(-c_2(uX)^{\frac{2}{\Sigma}}) \sin(\zeta(uX)^{\frac{1}{\Sigma}} + \frac{\pi}{4}(N-3))\, dS_2(u).$$

Integration by parts and inserting the asymptotic expansion given in (1), we estimate the contribution of the error to be less than

$$\ll \exp(-c_2(uX)^{\frac{2}{\Sigma}}) u^\theta (1 + \log u)^{q-1}|_{u=0}^\infty + \\ + \int_0^\infty \exp(-c_2(uX)^{\frac{2}{\Sigma}})((uX)^{\frac{2}{\Sigma}-1}X + (uX)^{\frac{1}{\Sigma}-1}X) u^\theta (1 + \log u)^{q-1} \ll \\ \ll X^{-\theta} |\log X|^{q-2} \int_0^\infty (v^{\frac{2}{\Sigma}} + v^{\frac{1}{\Sigma}}) v^{\theta-1} |\log v|^{q-2}\, dv \ll X^{-\theta} |\log X|^{q-2}.$$

We obtain the order term by a quite similar reasoning and a change of variable $v = \sqrt{c_2}(nX)^{\frac{1}{\Sigma}}$.

Using this Lemma, we arrive at our desired asymptotic expansion,

$$B(t) = c_{10} k^{\frac{N-1}{4}} |\log X|^{q-1} (G(\zeta) + o(1)) + o(k^{\frac{N-1}{4}}))$$

with a positive constant c_{10}. □

We now make use of a deep result due to Steinig [13] which provides necessary and sufficient conditions for functions like our $G(\zeta)$ to have a change of sign.

Lemma 3. *For $\zeta, B, \gamma \in \mathbb{R}$, $\gamma > -1$, let*

$$G_{\gamma,B}(\zeta) \stackrel{\text{def}}{=} \int_0^\infty e^{-u^2} u^\gamma \cos(au + B\gamma)\, du.$$

Then $G_{\gamma,B}(\zeta)$ as a function of ζ has a sign change if and only if

$$\gamma > -2|B - [B + \frac{1}{2}]|. \tag{14}$$

Otherwise, $G_{\gamma,B}(\zeta) \neq 0$ for all real values of ζ.

For $N \geq 4$, (14) is satisfied for any choice of the λ_j. Thus there exist real numbers ζ_1 and ζ_2 and a positive constant c_{11} such that $G(\zeta_1) \leq -c_{11}$, $G(\zeta_2) \geq c_{11}$. We take once $\zeta = \zeta_1$, then $\zeta = \zeta_2$ in the definition (9), i.e. we put

$$k_i = k_i(t) = K_2(\zeta_i + tX(t)^{-\frac{1}{\Sigma}})^2 \quad (i = 1, 2),$$

define $B_i(t)$ like $B(t)$ before, with k replaced by k_i, and infer from the above argument that there exists an unbounded sequence of reals t with

$$B_1(t) \le -c_{12} k_1^{\frac{N-1}{4}} (\log\log t)^{q-1}$$
$$B_2(t) \le -c_{12} k_2^{\frac{N-1}{4}} (\log\log t)^{q-1}.$$

To complete the proof, let us suppose that, for some small positive constant K_3,

$$\pm E(x) \le K_3 (x(\log x)^{a^*})^{\theta} (\log\log x)^{q-1} (\log\log\log x)^{-(\frac{\Sigma}{2}+a^*)\theta}$$

for all sufficiently large x. By the definition of $B_i(t)$, this would imply that, for every large real t,

$$(-1)^i B_i(t) \le \frac{K_3}{\Gamma(k_i(t)+1)} \int_0^\infty \mathrm{e}^{-u} u^{k_i(t)} (X(t)u^{\frac{\Sigma}{2}})^{\theta} L(X(t)u^{\frac{\Sigma}{2}})\, du$$

where $L(w) = (\log w)^{a^*\theta} (\log\log w)^{q-1} (\log\log\log w)^{-(\frac{\Sigma}{2}+a^*)\theta}$ for $w \ge 10$ and $L(w) = L(10)$ else. Estimating this integral by Hafner's Lemma 2.3.6 in [5, p. 51], we obtain

$$(-1)^i B_i(t) \le c_{13} (k_i(t))^{\frac{N-1}{4}} (\log\log t)^{q-1}.$$

Together this yields a positive lower bound for K_3 (for both $i = 1, 2$) and thus completes the proof of Theorem 2. □

It remains to deal with the case that $N = 2, 3$.

Case $N = 2$. We have to check under which conditions (14) is satisfied. Comparing our asymptotic expansion with the Lemma of Steinig we have

$$\gamma = \frac{1}{2}(N-3), \quad B = \frac{1}{4}(N-3) - 2\beta.$$

Here $\gamma = \frac{1}{2}, B = -\frac{1}{4} - 2\beta$. Hence (14) becomes

$$\frac{1}{4} < |\frac{1}{4} + 2\beta + [\frac{1}{4} - 2\beta]|,$$

which is easily seen to be satisfied if and only if

$$0 < \beta < \frac{1}{4} \quad \text{or} \quad \frac{1}{2} < \beta < \frac{3}{4}.$$

Now β depends on $\lambda = \frac{l}{m}$ by the equation

$$B_1 = C_1(-\log(2\sin(\pi\lambda)) + i(\frac{\pi}{2} - \pi\lambda)) \quad (\beta \in \mathbb{R}, 0 \le \beta \le 1).$$

This implies for the values λ,

$$0 < \lambda < \frac{1}{6} \quad \text{or} \quad \frac{1}{2} < \lambda < \frac{5}{6},$$

which completes the proof of Theorem 2.

Case $N = 3$. Here we have $\gamma = 0$ and $B = -2\beta$, hence (14) is true if and only if $B \notin \mathbb{Z}$ or equivalently $\beta \notin \{0, \frac{1}{2}, 1\}$. Now there are two possibilities. In the case where $p = 1$, $q = 2$ the above arguments holds, and we simply get $\lambda \neq \frac{1}{2}$. In the second case, where $p = 2$, $q = 1$, we have that $\beta \notin \{0, \frac{1}{2}, 1\}$ is equivalent that

$$B_2 = C_2 \prod_{j=1,2} (-\log(2\sin(\pi\lambda_j)) + i(\frac{\pi}{2} - \pi\lambda_j)).$$

We simplify this equation by writing u, v for $\frac{l_1}{m_1}, \frac{l_2}{m_2}$, which yields

$$\log(2\sin(\pi u))(v - \frac{1}{2}) + \log(2\sin(\pi v))(u - \frac{1}{2}) = 0. \tag{15}$$

Writing

$$w = 1 - v\,,$$

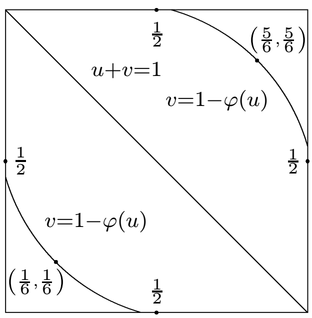
this equation simplifies to

$$\Phi(u) = \Phi(w)\,, \tag{$*$}$$

with

$$\Phi(t) = \frac{\log(2\sin(\pi t))}{t - \frac{1}{2}}\,.$$

Now $\Phi(t)$ decreases monotonically from $+\infty$ to $-\infty$ on each of the subintervals $]0, \frac{1}{2}[$ and $]\frac{1}{2}, 1[$. It follows that $(*)$ possesses two solutions for w: the trivial one $w = u$ and a second function $w = \phi(u)$ which is smooth on $]0, \frac{1}{2}[$ and on $]\frac{1}{2}, 1[$. Consequently, (15) is satisfied for $u + v = 1$ and for $v = 1 - \phi(u)$. Both curves are shown in the picture below. Note that the second one contains the rational points $(\frac{1}{6}, \frac{1}{6})$ and $(\frac{5}{6}, \frac{5}{6})$.



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(*Received* February 16, 1995)