About steady transport equation I — L^p-approach in domains with smooth boundaries

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Abstract. We investigate the steady transport equation

$$\lambda z + w \cdot \nabla z + az = f, \quad \lambda > 0$$

in various domains (bounded or unbounded) with smooth noncompact boundaries. The functions w, a are supposed to be small in appropriate norms. The solution is studied in spaces of Sobolev type (classical Sobolev spaces, Sobolev spaces with weights, homogeneous Sobolev spaces, dual spaces to Sobolev spaces). The particular stress is put onto the problem to extend the results to as less regular vector fields w, a, as possible (conserving the requirement of smallness). The theory presented here is well adapted for applications in various problems of compressible fluid dynamics.

Keywords: steady transport equation, bounded, unbounded, exterior domains, existence of solutions, estimates

Classification: 35Q35, 35L, 76N

1. Introduction

In this paper we investigate the solvability of steady transport equation

$$\begin{aligned} &\lambda z + w \cdot \nabla z + az = f \quad \text{in } \Omega, \\ &\lambda > 0, \quad w \cdot \nu\,|_{\partial\Omega} = 0 \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ $(n = 2, 3, \dots)$ is a domain (not necessarily bounded) with sufficiently smooth boundary $\partial\Omega$ (with outer normal ν) and $w = (w_1, w_2, \dots, w_n)$, a, f are given functions on Ω.

Sometimes, when (1.1) seems to be too general in order to obtain good results, we consider its special form when $a = \operatorname{div} w$, namely

$$\begin{aligned} &\lambda z + \operatorname{div}(wz) = f \quad \text{in } \Omega, \\ &\lambda > 0, \quad w \cdot \nu\,|_{\partial\Omega} = 0. \end{aligned} \tag{1.2}$$

We restrict ourselves only to the case when w and a are small in appropriate norms and thus, one can expect a global sufficiently regular solution (provided a, w and f are smooth enough).

We propose an efficient technique for studying steady transport equation in general classes of domains with sufficiently smooth boundaries (which contain, in particular, bounded and exterior domains, the whole space $\mathbb{R}^n$ or the half space $\mathbb{R}^n_+$, infinite pipes, etc.).

All results of the paper can be extended, practically without changes, to systems

$$\lambda z + W \cdot \nabla z + A \cdot z = f \quad \text{in } \Omega, \quad \lambda > 0 \tag{1.3}$$

or

$$\lambda z + \operatorname{div}(W \cdot z) = f \quad \text{in } \Omega, \quad \lambda > 0 \tag{1.4}$$

where $z = (z_1, \ldots, z_m)$ is an unknown function while $W = (w_{ij})_{\substack{i=1,\ldots,m\\ j=1,\ldots,m}}$, $A = (A_{ij})_{\substack{i=1,\ldots,m\\ j=1,\ldots,m}}$, $f = (f_1, \ldots, f_m)$ are known functions on Ω. The details are left to the reader.

The steady transport equation was already studied by many authors, namely in Ω bounded or $\Omega = \mathbb{R}^n$. Recall the pioneer papers of Lax and Philips [LP], Fridrichs [F], Kohn, Nirenberg [KN] and various articles studying (1.1) in more general context as e.g. Fichera [Fi1], [Fi2], Oleinik [O], Oleinik, Radekevic [OR]. It is usually not very difficult to prove existence theorems when the coefficients a, w are sufficiently smooth and small. It has been a permanent question to extend any part of the theory to less regular vector fields w and a, and to various types of domains. Such questions are pertinent in many applications from compressible fluid dynamics to kinetic theory.

For nonstationary equations various extensions and applications were done by Di Perna and Lions [DL], and B. da Veiga [BV1]. As far as steady equations are concerned, there are the important contributions by B. da Veiga [BV1], [BV2], handling (1.1) in bounded domains, with successive applications to compressible Navier-Stokes and Euler equations (see [BV1], [BV3]).

Here we use B. da Veiga's results for bounded domains as a staring point and extend them in the following sense (see Theorem 2.1 and 2.1* in [BV1] and Theorem 1.1, 2.1, 2.2, 2.3, 2.6 in [BV2]).

(a) For Ω bounded, we need less regularity of the boundary (see Theorem 5.2), and moreover, in Theorem 5.3, even slightly less assumptions on w, a. Namely this (slight) modification is important for several applications in compressible fluids, see Novotný [N1]. As a consequence of presented results we get, similarly as B. da Veiga [BV2], only by duality arguments, existence and estimates for weak solutions in negative Sobolev spaces, see Theorem 6.4. (The latter results were applied to compressible fluids by B. da Veiga [BV3].)

(b) For Ω being of certain (general) class (which contains in particular $\mathbb{R}^n$, $\mathbb{R}^n_+$, bounded and exterior domains in $\mathbb{R}^n$, infinite pipes with bounded cross sections) we prove existence (and uniqueness) of solutions in Sobolev spaces (Theorems 5.1–5.3). Moreover, we also prove, by duality method, existence of weak solutions in negative Sobolev spaces, see Theorem 6.4. Existence of weak solutions in Lebesgue spaces is given in Theorem 5.7. For applications of such results, see Novotný, Padula [NP1], Novotný [N1], [N2], [N3], Padula [P1], Padula, Pileckas [PP], Novotný, Penel [NPe].

(c) In particular for Ω exterior domain, $\Omega = \mathbb{R}^n$, $\Omega = \mathbb{R}^n_+$, we show existence (and uniqueness) in homogeneous Sobolev spaces (Theorems 6.1, 6.2) and in their duals (Theorems 6.4, 6.5). For possible applications see Novotný [N3].

(d) In some particular cases, we investigate a special regularity. This is usually motivated by applications in compressible fluids. Thus, Theorem 7.1 gives estimates for Δ of solutions in Sobolev spaces and eventually in their duals; Theorem 7.2 investigates estimates of Δ of solutions in duals to homogeneous Sobolev spaces for Ω exterior or $\Omega = \mathbb{R}^n$ or $\Omega = \mathbb{R}^n_+$.

(e) Some applications require estimates and existence results in intersections of various Sobolev and/or homogeneous Sobolev spaces. Such results, in general domains, require uniqueness arguments; see Theorem 5.6 for intersections of Sobolev spaces and Theorem 6.3 for intersection of Sobolev and homogeneous Sobolev spaces; see [NP1], [NPe] and [GNP].

(f) Some particular results in weighted Sobolev spaces are given in Theorems 5.4, 5.5 and 5.7. They are useful both as auxiliary results for proving (e) and, in applications, as an important tool for studying decay properties of solutions to compressible Navier-Stokes equations; see Novotný, Padula [NP3], Novotný, Penel [NPe], Novotný [N2], Padula, Pileckas [PP]. The decay of solutions, for arbitrary size of coefficients w, a is investigated in Theorem 5.8.

The technique of proofs is standard. The most novelty (and the main goal) of the paper is to give results fully conform with the requirements of the theory of compressible fluids, especially in unbounded domains. These achievements are very often of a rather subtle nature, and although they seem almost obvious a lot of work is needed to prove them. As far as the author knows, such results have been missing in the mathematical literature about the subject. The various applications justify their importance.

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2. Notation and basic considerations

Denote by B_R the sphere in $\mathbb{R}^n$ with center in 0 and radius $R > 0$; let $B^R = \mathbb{R}^n - B^R$. Let Ω be a domain in $\mathbb{R}^n$, $\Omega_R = \Omega \cap B_R$ and $\Omega^R = \Omega \cap B^R$. We use the following functional spaces:

□ $\mathcal{C}_0^\infty(\Omega)$ is a set of smooth functions with compact support in Ω; $\mathcal{C}_0^\infty(\overline{\Omega})$ is the set of smooth functions with compact support in $\overline{\Omega}$; $\mathcal{C}^s(\overline{\Omega})$ $(s = 0, 1, \dots)$ is a Banach space of bounded continuous functions with bounded and continuous (up to the boundary) derivatives up to the order s. The corresponding norm in

$$|u|_{\mathcal{C}^s} = \sum_{0 \le \alpha \le s} \max_{x \in \overline{\Omega}} |\nabla^\alpha u|,$$

while $\mathcal{C}^s(\Omega)$ is a set of continuously differentiable functions (up to the order s) in Ω.

□ $W^{k,p}(\Omega)$ $(k = 0, 1, \dots, 1 \le p \le +\infty)$ are usual Sobolev spaces of distributions with finite norms

$$\|u\|_{k,p} = \left(\sum_{0 \le \alpha \le k} \int_\Omega |\nabla^\alpha u|^p \, dx \right)^{1/p} \quad (1 \le p < +\infty), \ \| \cdot \|_{0,\infty} = \operatorname*{ess\,sup}_{x \in \overline{\Omega}} |u|;$$

in particular $W^{0,p}(\Omega)$ is usual Lebesgue space $L^p(\Omega)$; $W_0^{k,p}(\Omega)$ is completion of $\mathcal{C}_0^\infty(\Omega)$ in $\| \, \|_{k,p}$ norm. For $\Omega = \mathbb{R}^n$, $W_0^{k,p}(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n)$. The dual space to $W_0^{1,p'}(\Omega)$ $(1 < p' < +\infty,\ 1/p + 1/p' = 1)$ is denoted $W^{-1,p}(\Omega)$ and equipped with standard duality norm $\| \cdot \|_{-1,p}$.

□ In this paper, we also use, in some particular situations, weighted Sobolev spaces and homogeneous Sobolev spaces together with their duals. They will be defined on corresponding places in the text.

□ Further introduce for $0 \le s \le k$, $1 \le p_1, \dots, p_s < +\infty$ auxiliary Banach spaces

$$\begin{aligned} X^k_{p_1,\dots,p_s}(\Omega) = \mathcal{C}^{k-s}(\overline{\Omega}) \cap \Big\{ & v : v \in W^{k,p_i}_{\mathrm{loc}}(\Omega) \quad (i = 1, \dots, s), \\ & \nabla^{k-s+1} v \in L^{p_1}(\Omega),\ \nabla^{k-s+2} v \in L^{p_2}(\Omega), \dots, \nabla^k v \in L^{p_s}(\Omega) \Big\} \end{aligned}$$

equipped with norm

$$\|u; X^k_{p_1,\dots,p_s}\| = |u|_{\mathcal{C}^{k-s}} + \sum_{i=1}^s \|\nabla^{k-s+1} v\|_{0,p_i} \, .$$

(If $s = 0$, then we have $\mathcal{C}^k(\overline{\Omega})$.)

Remark 2.1. In the estimates, we use generic positive constants α_0, α_1, α_2, c, c', c_i $(i = 1, 2, \dots)$. If not stated explicitly, they depend only of k, q, n (and they do not depend of w, a, f, λ, and on the domain). The only dependence which can occur is the one of coefficients c'_s of imbedding $|b|_{\mathcal{C}^s} \leq c'_s \|b\|_{r,q}$, $(r-s)q > n$.

The coefficients in estimates that can depend on the size of the domain are always denoted by k, k_i.

If not stated explicitly, the norms refer always to domain Ω. Otherwise we use the domain as a further index; e.g. $\|\cdot\|_{k,p}$ means a norm in $W^{k,p}(\Omega)$ while $\|\cdot\|_{k,p,G}$ a norm in $W^{k,p}(G)$, $G \in \mathbb{R}^n$. We consider the following class of domains in $\mathbb{R}^n$ $(n \geq 2)$.

Definition 2.1. Let Ω be a domain in $\mathbb{R}^n$. We say that it is of class $\mathcal{B}^{(k)}$, $k = 1, 2, \dots$, if and only if

(i) $\partial\Omega \in \mathcal{C}^k$ (if $\Omega \neq \mathbb{R}^n$);

(ii) for any i $(1 \leq i \leq k)$ and p_i $(1 < p_i \leq p_{i-1} \leq \dots \leq p_2 \leq p_1 < +\infty)$ there exists a continuous extension

$$\mathcal{E} : X^k_{p_1,\dots,p_i}(\Omega) \to X^k_{p_1,\dots,p_i}(\mathbb{R}^n). \tag{2.1}$$

Example 2.1.

(i) $\Omega = \mathbb{R}^n \in \mathcal{B}^{(k)}$, $1 \leq k < +\infty$.

(ii) $\Omega = \mathbb{R}^n_+ \in \mathcal{B}^{(k)}$, $1 \leq k < +\infty$.

(iii) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in \mathcal{C}^k$, $1 \leq k < +\infty$, then $\Omega \in \mathcal{B}^{(k)}$.

(iv) Let $\Omega \subset \mathbb{R}^n$ be an exterior domain to a compact region Ω_c (suppose without loss of generality that $B_1 \subset \Omega_c$), $\partial\Omega \in \mathcal{C}^k$ $(1 \leq k < +\infty)$, then $\Omega \in \mathcal{B}^{(k)}$.

(v) Let

$$\begin{aligned} \Omega = \Omega' = \big\{ & x = (x', x_n) : x' = (x_1, \dots, x_{n-1}), x_n \in \mathbb{R}^1, \\ & 0 < \delta < |x'| \leq \varphi(x_n), \varphi \in \mathcal{C}^k(\mathbb{R}^1), |\varphi|_{\mathcal{C}^k,\mathbb{R}^1} < +\infty \big\}, \\ & 1 \leq k < +\infty \end{aligned} \tag{2.2}$$

be a pipe with bounded cross section. Then it belongs to $\mathcal{B}^{(k)}$.

(vi) Let $\Omega = \mathbb{R}^n - \Omega'$, where Ω' is the set from (v), then it belongs to $\mathcal{B}^{(k)}$.

Proof: Statement (i) is obvious.

Proof of (ii) (see Galdi [G]). Let

$$\Omega = \Omega_n = \big\{ x = (x', x_n) : x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}, x_n \geq 0 \big\},$$

then put

$$\mathcal{E}u(x) = \begin{cases} u(x) & \text{if } x_n \geq 0 \\ \sum_{s=1}^{k+1} \lambda_s u(x', -s x_n) & \text{if } x_n < 0 \end{cases}$$

where $\lambda_s \in \mathbb{R}^1$ are such that

$$\sum_{s=1}^{k+1} \lambda_s(-s)^\ell = 1 \quad \text{for any} \quad \ell = 0, \dots, k.$$

We find

$$\nabla_{x'}^\alpha \nabla_{x_n}^\beta (\mathcal{E}u)(x) = \begin{cases} \nabla_{x'}^\alpha \nabla_{x_n}^\beta u(x) & x_n \geq 0 \\ \sum_{s=1}^{k+1} (-1)^\beta \lambda_s s^\beta (\nabla_{x'}^\alpha \nabla_{x_n}^\beta u)(x', -s x_n). \end{cases}$$

Hence $\nabla_{x'}^\alpha \nabla_{x_n}^\beta \mathcal{E}u \in \mathcal{C}^0(\overline{\mathbb{R}}^n)$ ($\in L^q(\mathbb{R}^n)$) if and only if $\nabla_{x'}^\alpha \nabla_{x_n}^\beta u \in \mathcal{C}^0(\overline{\mathbb{R}}_+^n)$ ($\in L^q(\overline{\mathbb{R}}_+^n)$). It is easily seen that the extension is continuous from $\mathcal{C}^\ell(\overline{\Omega}) \to \mathcal{C}^\ell(\overline{\mathbb{R}}^n)$ ($\ell = 0, \dots, k$) and moreover

$$\left\| \nabla^i (\mathcal{E}u) \right\|_{0,q,\mathbb{R}^n} \leq c \left\| \nabla^i u \right\|_{0,q,\Omega}$$

with c dependent of i, q (provided $\nabla^i u \in L^q(\Omega)$).

Proof of (iii) and (iv). We prove only (iv), the statement (iii) (Ω bounded) is even easier. Let $\Delta_\varepsilon = (-\varepsilon, \varepsilon)^{n-1}$ (cartesian product), $\varepsilon > 0$ sufficiently small. Let $\{\mathcal{U}_r, \varphi_r\}_{r=0}^{m(\varepsilon)}$ be such that

$\mathcal{U}_0 = B^{R_0}(0)$ ($R_0 > 0, \Omega_c \subset B_{R_0/2}$), $\varphi_0 = \begin{cases} 1 & \text{in } B^{2R_0} \\ 0 & \text{in } B_{R_0} \end{cases}$

$\varphi_r = (1 - \varphi_0)\psi_r$ ($r = 1, \dots, m$) where $\{\mathcal{U}_r, \psi_r\}_{r=1}^m$ is a partition of unity of Ω_{2R_0} such that $\bigcup_{r=r_0}^m \mathcal{U}_r \supset \partial\Omega$ and $\mathcal{U}_r \cap \partial\Omega \neq \emptyset$ ($r = r_0, \dots, m$), r_0 being fixed, $2 \leq r_0 < m$.

There exist orthogonal maps $\mathbb{A}_r : \mathbb{R}^n \to \mathbb{R}^n$ ($r = r_0, \dots, m$) and functions

$$a_r : \Delta_\varepsilon \to \mathbb{R}^1, \ a_r \in \mathcal{C}^k(\Delta_\varepsilon) \ (r = r_0, \dots, m), \ \varepsilon \in (0, \varepsilon_0), \ \varepsilon_0 > 0,$$

such that

$$\partial\Omega \cap \mathcal{U}_r = \left\{ Z : Z = \mathbb{A}_r^{-1}(y', a_r(y')), y' \in \Delta_\varepsilon \right\}.$$

Moreover, the maps

$$\begin{aligned} &m_r : \mathcal{U}_r \to B_\varepsilon, \quad B_\varepsilon = \Delta_\varepsilon \times (-\varepsilon, \varepsilon), \quad r_0 \leq r \leq m, \\ &y' = (y_1, \dots, y_{n-1}) = (\mathbb{A}_r x)', \ y_n = (\mathbb{A}_r x)_n - a_r((\mathbb{A}_r x)') \end{aligned}$$

are one to one and map $\mathcal{U}_r$ onto B_ε, $\mathcal{U}_r \cap \Omega$ onto $B_{\varepsilon,+}$ and $\mathcal{U}_r \cap (\mathbb{R}^n - \Omega)$ onto $B_{\varepsilon,-}$, where $B_{\varepsilon,+} = \Delta_\varepsilon \times (0, +\varepsilon)$, $B_{\varepsilon,-} = \Delta_\varepsilon \times (-\varepsilon, 0)$. The determinant of Jacobi matrix $\mathbb{J} = (\partial y_i / \partial x_k)$ of such map reads $J = \det \mathbb{J} = 1$. Clearly $m_r \in \mathcal{C}^k(\overline{\mathcal{U}}_r)$ and therefore $m_r^{-1} \in \mathcal{C}^k(\overline{B}_\varepsilon)$.

Let $u \in X^k_{p_1 \ldots p_s}(\Omega)$, $1 \le s \le k$. Put $u_{(r)} = u\varphi_r$ $(r = 0, \ldots, m)$. We define $(r = r_0, \ldots, m)$ $\tilde{u}_{(r)}(y) = u_{(r)}(m_r^{-1}(y))$, hence $\tilde{u}_{(r)} \in X^k_{p_i \ldots p_s}(\mathbb{R}^n_+)$ (since $p_s \le p_{s-1} \le \cdots \le p_1$ and $u_{(r)}$ has compact support in $\mathcal{U}_r$) $\overline{\operatorname{supp} u_{(r)}} \subset \mathcal{U}_r$ (hence $\overline{\operatorname{supp} \tilde{u}_{(r)}} \subset B_\varepsilon$). According to (ii) there exists a continuous extension (say $\overline{v}_{(r)}$) $\overline{v}_{(r)} \in X^k_{p_1 \ldots p_s}(\mathbb{R}^n)$. Let $\eta_r \in \mathcal{C}^\infty_0(\mathbb{R}^n)$ such that $\eta_r(y) = 1$ for $y \in \operatorname{supp} \tilde{u}_{(r)}$, $\eta_r(y) = 0$ for $y \in \mathbb{R}^n - B_{2\varepsilon_0}$. Then obviously $\tilde{v}_{(r)} = \overline{v}_{(r)}\eta_r$ is also a continuous extension $\overline{v}_{(r)} \in X^k_{p_1 \ldots p_s}(\mathbb{R}^n)$. It is worth noting that

$$v_{(r)}(x) = \begin{cases} \tilde{v}_{(r)}(m_r(x)) & \text{for } \ x \in \mathcal{U}_r \\ & (\in X^k_{p_1 \ldots p_s}(\mathbb{R}^n)) \\ 0 & \text{otherwise} \end{cases}$$

is a continuous extension of $u_{(r)} \in X^k_{p_1 \ldots p_s}(\mathbb{R}^n_+)$ $(r = r_0, \ldots, m)$. For $r = 0, 1, \ldots, r_0 - 1$, define

$$v_{(r)}(x) = \begin{cases} u_r(x) \ \text{ in } \ \mathcal{U}_r & \\ & (\in X^k_{p_1 \ldots p_s}(\mathbb{R}^n)) \\ 0 & \text{otherwise.} \end{cases}$$

Since $u = \sum_{r=0}^m u_r$, one easily verifies that

$$\mathcal{E} : \mathcal{E}u = \sum_{r=0}^m \mathcal{E}u_{(r)}$$

where

$$\mathcal{E}u_{(r)} = \begin{cases} v_{(r)}(x) & \text{if } \ r = 0, 1, \ldots, r_0 - 1 \\ v_{(r)}(m_r(x)) & \text{if } \ r = r_0, \ldots, m \end{cases}$$

is continuous extension $X^k_{p_1 \ldots p_s}(\Omega) \to X^k_{p_1 \ldots p_s}(\mathbb{R}^n)$.

Proof of (v), (vi). For clarity, we restrict ourselves to the case $n \le 3$, letting the general case to interested reader. The set Ω' in cylindrical coordinates reads

$$\Omega' = \big\{(\theta, r, z) : r = |x'|, z = x_n, \ \theta \in [0, 2\pi), 0 \le r \le \varphi(x_n), x_n \in (-\infty, +\infty)\big\}.$$

The map $m = (\theta, r, x_n) \to (\psi, R, z)$

$$\psi = \theta, \ \ R = r/\varphi(x_n), \ \ z = x_n$$

maps Ω' onto a cylinder with cross section $\sum$ a circle with unit radius. The determinant of Jacobi matrix for m reads $J = \varphi(x_n) > \delta > 0$. Now, we apply the method of (iii), (iv) on each cross section $\sum$. Since $\frac{d^i\varphi}{dx_n^i}$ $(i = 1, \ldots, k)$ is bounded, we get easily the desired result. The other is obvious. The statements (i)–(vi) of Example 2.1 are thus proved. □

An important technical tool in our investigations are cut-off functions and mollifiers. In order to have a control of functions at large distance, we use Sobolev cut-off function

$$\begin{aligned} &\psi_R(x) = \psi\left(\frac{\ln\,\ln|x|}{\ln\,\ln R}\right),\ \psi \in \mathcal{C}_0^\infty(\mathbb{R}^n),\ 0 \le \psi \le 1 \\ &\psi(x)\begin{cases} 1 & x \in B_1 \\ 0 & x \in B^2; \end{cases} \end{aligned} \tag{2.4}$$

we easily find

$$\begin{aligned} &\sup_{x\in\mathbb{R}^n} \psi_R \subset B_{\kappa(R)},\ \text{where}\ \ \kappa(R) = e^{(\ln R)^2}, \\ &\sup_{x\in\mathbb{R}^n} \nabla^\beta \psi_R \subset \widetilde{\Omega}_R = B_{\kappa(R)} - B_{\ln R}\ (\beta \ge 1), \\ &\psi_R(x) = 1\ \text{ in }\ B_R, \\ &\left\|\nabla^\beta \psi_R\right\| \le \frac{c}{(\ln\,\ln R)^\beta}\frac{1}{|x|\ln|x|}\,. \end{aligned} \tag{2.5}$$

Due to Galdi, Simader [GS], we have the following statement:

Let $u \in L^q_{\rm loc}(\mathbb{R}^n)$, $\nabla u \in L^q(\mathbb{R}^n)$ $(n \le q < +\infty)$ or $u \in L^q_{\rm loc}(\mathbb{R}^n) \cap L^s(\mathbb{R}^n)$ for some $1 < s < +\infty$ and $\nabla u \in L^q(\mathbb{R}^n)$ $(1 < q < n)$. Then

$$\left\|u\nabla^k\psi_r\right\|_{0,q,\widetilde{\Omega}_R} \to 0 \ \text{ as }\ R \to +\infty,\ k = 1, 2, \dots\ . \tag{2.6}$$

Last but not least recall a definition of a mollifier

$$\varrho_\varepsilon(x) = \frac{1}{\varepsilon^n}\varrho\left(\frac{x}{\varepsilon}\right),\ \varrho \in \mathcal{C}_0^\infty(\mathbb{R}^n),\ \text{supp}\ \varrho(x) \subset B_1,\ \int_{\mathbb{R}^n} \varrho(x)\,dx = 1. \tag{2.7}$$

For a function f, we denote shortly by f_ε the convolution

$$f_\varepsilon(x) = \varrho_\varepsilon * f = \int_{\mathbb{R}^n} \varrho_\varepsilon(x-y)f(y)\,dy. \tag{2.8}$$

It is worth noting that

$$\begin{aligned} &f_\varepsilon \in \mathcal{C}_0^\infty(\mathbb{R}^n)\ \text{ provided }\ f \in L^q_{\rm loc}(\mathbb{R}^n)\ \ (1 \le q < +\infty), \\ &\nabla^i f_\varepsilon(x) = (\nabla^i f)_\varepsilon(x),\ i = 0, \dots, k\ \text{ provided }\ f \in W^{k,p}_{\rm loc}(\mathbb{R}^n), \\ &f_\varepsilon \in \mathcal{C}^k(\overline{\mathbb{R}^n})\ \text{ provided }\ f \in W^{k,q}(\mathbb{R}^n) \end{aligned} \tag{2.9}$$

and

$$
\begin{aligned}
&\|f_\varepsilon - f\|_{k,p} \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad \text{provided} \quad f \in W^{k,p}(\mathbb{R}^n),\\
&|f_\varepsilon - f|_{\mathcal{C}^s} \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad \text{provided} \quad f \in \mathcal{C}^s(\overline{\mathbb{R}^n}).
\end{aligned} \tag{2.10}
$$

In order to avoid cumbersome expressions in theorems, we denote ($k = 0, 1, 2, \dots,\ 1 < q < \infty$)

$$
\begin{cases}
\vartheta_0^{(k,q)}(w,a) = |w|_{\mathcal{C}^k} + |a|_{\mathcal{C}^k}\\
\vartheta_1^{(k,q)}(w,a) = |w|_{\mathcal{C}^k} + |a|_{\mathcal{C}^{k-1}} + \|\nabla^k a\|_{0,q}\\
\vartheta_1'^{(k,q)}(w,a) = |w|_{\mathcal{C}^k}\\
\vartheta_2^{(k,q)}(w,a) = |w|_{\mathcal{C}^k} + |a|_{\mathcal{C}^{k-1}} + \|\nabla^k a\|_{0,n}\\
\vartheta_2'^{(k,q)}(w,a) = |w|_{\mathcal{C}^k} + |a|_{\mathcal{C}^{k-1}} + \|\nabla a\|_{k-1,n}\\
\vartheta_3^{(k,q)}(w,a) = |w|_{\mathcal{C}^{k-1}} + |a|_{\mathcal{C}^{k-2}} + \|\nabla^k w\|_{0,n} + \|\nabla^{k-1} a\|_{0,n} +\\
\qquad\qquad\qquad + \|\nabla^k a\|_{0,q}\\
\vartheta_4^{(k,q)}(w,a) = |w|_{\mathcal{C}^{k-1}} + |a|_{\mathcal{C}^{k-2}} + \|\nabla^k w\|_{0,n} + \|\nabla^{k-1} a\|_{1,n}\\
\vartheta_5^{(k,q)}(w,a) = |w|_{\mathcal{C}^{k-1}} + |a|_{\mathcal{C}^{k-2}} + \|\nabla^k w\|_{0,n} + \|\nabla^{k-1} a\|_{1,q}\\
\vartheta_6^{(k,q)}(w,a) = |w|_{\mathcal{C}^{k-1}} + |a|_{\mathcal{C}^{k-2}} + \|\nabla^k w\|_{0,q} + \|\nabla^{k-1} a\|_{1,q}\\
\vartheta_7^{(k,q)}(w,a) = |w|_{\mathcal{C}^{k-1}} + |a|_{\mathcal{C}^{k-2}} + \|\nabla^k w\|_{0,q} + \|\nabla^{k-1} a\|_{1,n}\\
\vartheta_8^{(k,q)}(w,a) = |w|_{\mathcal{C}^1} + |a|_{\mathcal{C}^0} + \|\nabla(a - \operatorname{div} w)\|_{0,q'} \left(\frac{1}{q} + \frac{1}{q'} = 1\right)\\
\vartheta_9^{(k,q)}(w,a) = |w|_{\mathcal{C}^1} + |a|_{\mathcal{C}^0} + \|\nabla(a - \operatorname{div} w)\|_{0,n}\\
\vartheta_{10}^{(k,q)}(w,a) = |w|_{\mathcal{C}^1} + |a|_{\mathcal{C}^0}
\end{cases} \tag{2.11}
$$

If not confusing, the variables w, a (or even index (k,q)) are omitted in the notation and $\vartheta^{(k,q)}$ (or even ϑ_i) means $\vartheta_i^{(k,q)}(w,a)$.

Next important auxiliary result is due to Lax and Philips [LP], see also Misohata [Mi, VI.6.1].

Corollary 2.1. *Let $1 < q < +\infty$, $w \in \mathcal{C}^1(\overline{\mathbb{R}}^n)$ ($|w|_{\mathcal{C}^1} < +\infty$), $z \in L^q(\mathbb{R}^n)$, $w \cdot \nabla z \in L^q(\mathbb{R}^n)$. Then*

$$\|(w \cdot \nabla z)_\varepsilon - w \cdot \nabla z_\varepsilon\|_{0,q,\mathbb{R}^n} \to 0 \quad \textit{as} \quad \varepsilon \to 0$$

and

$$w \cdot \nabla z_\varepsilon \to w \cdot \nabla z \quad \textit{as} \quad \varepsilon \to 0 \quad \textit{in} \quad L^q(\mathbb{R}^n).$$

An easy consequence of this fundamental statement reads.

Corollary 2.2. *Let $1 < q < +\infty$, $k = 1, 2, \ldots$, $\Omega \in \mathcal{B}^{(k)}$, $w \in \mathcal{C}^1(\overline{\Omega})$, $w \cdot \nu \,|\, _{\partial\Omega} = 0$, $z \in L^q(\Omega)$, $w \cdot \nabla z \in L^q(\Omega)$. Then*

$$\left\| (\tilde{w} \cdot \nabla \overline{z})_\varepsilon - \tilde{w} \cdot \nabla \overline{z}_\varepsilon \right\|_{0,q,\Omega} \to 0$$

and

$$w \cdot \nabla \overline{z}_\varepsilon \to w \cdot \nabla z \quad \text{in} \quad L^q(\Omega).$$

Here $\tilde{w}$ is a continuous extension of w (i.e. $\tilde{w} \in \mathcal{C}^1(\overline{\mathbb{R}}^n)$) and $\overline{z}(x) = \begin{cases} z(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \notin \Omega \end{cases}$.

Proof: First, we define the distribution $\tilde{w} \cdot \nabla \overline{z}$:

$$(\tilde{w} \cdot \nabla \overline{z}, \varphi) = \int_{\mathbb{R}^n} \overline{z} \operatorname{div}(\tilde{w}\varphi)\, dx, \quad \forall\, \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n).$$

We have

$$\int_{\mathbb{R}^n} \overline{z} \operatorname{div}(\tilde{w}\varphi)\, dx = \int_\Omega z \operatorname{div}(w\varphi)\, dx = \int_\Omega w \cdot \nabla z \varphi\, dx, \quad \forall\, \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n).$$

From the last identity we conclude that

$$\tilde{w} \cdot \nabla \overline{z} \in L^q(\mathbb{R}^n).$$

Corollary 2.2 thus follows directly from Corollary 2.1. □

3. Some estimates independent of the domain and auxiliary theorems

Lemma 3.1. *Let $k = 1, 2, \ldots$, $s = 1, \ldots, k$, $1 < q < +\infty$, $\Omega \in \mathcal{B}^{(k)}$ and*

$$a, w \in \mathcal{C}^k(\overline{\Omega}),\ w \cdot \nu \,|\, _{\partial\Omega} = 0,\ f \in W^{k,q}(\Omega). \tag{3.1}$$

Then there exists a constant $\alpha_0 > 0$ (see Remark 2.1) such that we have: Let $z \in W^{k,q}(\Omega)$ be a solution of problem (1.1); then

$$\lambda \|z\|_{s,q} \le \|f\|_{s,q} + \alpha_0 \vartheta_0 \|z\|_{s,q} \tag{3.2}$$

(for definition of ϑ_0 see (2.11)).

Lemma 3.2. *Let $k = 1, 2, \ldots$, $s = 1, \ldots, k$, $1 < q < +\infty$, $\Omega \in \mathcal{B}^{(k)}$ and*

$$w \in \mathcal{C}^k(\overline{\Omega}),\ w \cdot \nu \,|\, _{\partial\Omega} = 0,\ a \in \mathcal{C}^{k-1}(\overline{\Omega}),\ f \in W^{k,q}(\Omega). \tag{3.3}$$

There exists a constant $\alpha_0 > 0$ (see Remark 2.1) such that we have:

(a) *Let* $z \in W^{s,q}(\Omega)$ *be a solution of problem* (1.1). *If*

$$kq > n, \quad \nabla^k a \in L^q(\Omega) \tag{3.4$_1$}$$

or if

$$1 < q < n, \quad \nabla^k a \in L^n(\Omega), \tag{3.4$_2$}$$

then

$$\lambda \|z\|_{s,q} \leq \|f\|_{s,q} + \alpha\vartheta_i \|z\|_{s,q} \tag{3.5}$$

where $i = 1, 2$ *corresponds to* $(3.4)_i$ *and* ϑ_i *is defined by* (2.11).

(b) *Let* $z \in W^{s,q}_{\mathrm{loc}}(\Omega)$, $\nabla z \in W^{s-1,q}(\Omega)$ *be a solution of problem* (1.1). *If*

$$a = 0 \tag{3.6$_1$}$$

or if

$$1 < q < n, \quad \nabla a \in W^{k-1,n}(\Omega), \tag{3.6$_2$}$$

then

$$\lambda \|\nabla z\|_{s-1,q} \leq \|\nabla f\|_{s-1,q} + \alpha\vartheta'_i \|\nabla z\|_{s-1,q} \tag{3.7}$$

where $i = 1, 2$ *corresponds to* $(3.6)_i$. *For definition of* ϑ'_i *see* (2.11).

Lemma 3.3. *Let* $k = 2, 3, \ldots$, $s = 1, \ldots, k$, $1 < q < +\infty$, $kq > n$, $\Omega \in \mathcal{B}^{(k)}$. *Let*

$$w \in \mathcal{C}^{k-1}(\overline{\Omega}),\ w \cdot \nu\,|_{\partial\Omega} = 0,\ a \in \mathcal{C}^{k-2}(\overline{\Omega}),\ f \in W^{k,q}(\Omega). \tag{3.8}$$

There exists a constant $\alpha_0 > 0$ (*see Remark* 2.1) *such that we have: Let* $z \in W^{k,q}(\Omega)$ *be a solution of problem* (1.1). *If*

$$1 < q < n,\ \nabla^k w \in L^n(\Omega),\ \nabla^{k-1} a \in L^n(\Omega),\ \nabla^k a \in L^q(\Omega) \tag{3.9$_1$}$$

or

$$1 < q < n,\ \nabla^k w \in L^n(\Omega),\ \nabla^{k-1} a \in W^{1,n}(\Omega), \tag{3.9$_2$}$$

or

$$1 < q < n,\ \nabla^k w \in L^n(\Omega),\ \nabla^{k-1} a \in W^{1,q}(\Omega), \tag{3.9$_3$}$$

or

$$(k-1)q > n,\ \nabla^k w \in L^q(\Omega),\ \nabla^{k-1} a \in W^{1,q}(\Omega), \tag{3.9$_4$}$$

or

$$(k-1)q > n,\ \nabla^k w \in L^q(\Omega),\ \nabla^{k-1} a \in W^{1,n}(\Omega), \tag{3.9$_5$}$$

then

$$\lambda \|z\|_{s,q} \leq \|f\|_{s,q} + \alpha\vartheta_{i+2} \|z\|_{s,q} \tag{3.10}$$

(*here index* i *corresponds to* $(3.9)_i$, $i = 1, \ldots 5$ *and* ϑ_{i+2} *are defined in* (2.11)).

Proof of Lemma 3.1, 3.2 and 3.3: Multiply (1.1) by $|z|^{q-2}z$ and integrate over Ω. We have, only using obvious integration by parts,

$$\lambda\|z\|_{0,q}^q = \int_\Omega |z|^{q-2}zf\,dx - \frac{1}{q}\int_\Omega |z|^q \operatorname{div} w\,dx + \int_\Omega |z|^q a\,dx$$

which yields, by the Hölder and Young inequalities applied to the first integral

$$\lambda\|z\|_{0,q}^q \le \|f\|_{0,q}^q + c\big(|w|_{\mathcal{C}^1} + |a|_{\mathcal{C}^0}\big)\|z\|_{0,q}^q. \tag{3.11}$$

Differentiate (1.1) by taking ∇^r, $r = 1, 2, \dots k$, to obtain

$$\lambda\nabla^r z = -w\cdot\nabla\nabla^r z - \sum_{\substack{i+j=r\\ 0\le j\le r-1}} \nabla^i w\cdot\nabla\nabla^j z - \nabla^r a z - \sum_{\substack{i+j=r\\ 0\le i\le r-1}} \nabla^i a\nabla^j z + \nabla^r g. \tag{3.12}$$

Multiplying $(3.12)_r$ scalarly by $|\nabla^r z|^{q-2}\nabla z$ and integrating over Ω, we obtain

$$\lambda\|\nabla^r z\|_{0,q} = \sum_{m=1}^{5} \mathcal{I}_m^r \tag{3.13}$$

where the integrals $\mathcal{I}_m^r$ are defined and estimated as follows:

$$\begin{aligned} \mathcal{I}_1^r &= -\int_r \Big[w\cdot\nabla\nabla^r z\Big] : \Big[|\nabla^r z|^{q-2}\nabla^r z\Big]\,dx = \\ &= -\frac{1}{q}\int_\Omega w\cdot\nabla\big(|\nabla^r z|^q\big)\,dx = \frac{1}{q}\int_\Omega \operatorname{div} w|\nabla^r z|^q\,dx \le c|w|_{\mathcal{C}^1}\|\nabla^r z\|_{0,q}^q\,. \end{aligned} \tag{3.14}$$

(The process above needs some explanation, especially for $r = k$, see the last part of this proof.)

(3.15)

$$\begin{aligned} \mathcal{I}_2^r &= \sum_{\substack{i+j=r\\ 0\le j\le r-1}} \int_\Omega \Big[\nabla^i w\cdot\nabla\nabla^j z\Big] : \Big[|\nabla^r z|^{q-2}\nabla^r z\Big]\,dx \le \\ &\le \begin{cases} c|w|_{\mathcal{C}^k}\|\nabla z\|_{r-1,q}^q & (1\le r\le k)\\ c|w|_{\mathcal{C}^{k-1}}\|\nabla z\|_{r-1,q}^q & (1\le r\le k-1)\\ \sum_{\substack{i+j=k\\ 1\le j\le k-1\\ i<k}} \int_\Omega |\nabla^i w|\;|\nabla\nabla^j z|\;|\nabla^k z|^{q-1}\,dx + \int_\Omega |\nabla^k w|\;|\nabla z|\;\nabla^k z|^{q-1}\,dx \end{cases}\\ &\le c|w|_{\mathcal{C}^{k-1}}\|\nabla z\|_{k-1,q}^q + \begin{Bmatrix} \|\nabla^k w\|_{0,n}\|\nabla z\|_{0,nq/(n-q)}\|\nabla z\|_{k-1,q}^{q-1}\\ \|\nabla^k w\|_{0,q}|\nabla z|_{\mathcal{C}^0}\|\nabla z\|_{k-1,q}^{q-1} \end{Bmatrix}\\ &\le c|w|_{\mathcal{C}^{k-1}}\|\nabla z\|_{k-1,q}^q + \begin{cases} \|\nabla^k w\|_{0,n}\|\nabla^2 z\|_{0,q}\|\nabla z\|_{k-1,q}^{q-1} & (r{=}k,\ 1{<}q{<}n)\\ \|\nabla^k w\|_{0,q}\|\nabla z\|_{k-1,q}^q & (r{=}k,\ (k{-}1)q{>}n) \end{cases} \end{aligned}$$

(3.16)

$$\mathcal{I}_3^r = -\int_\Omega (\nabla^r z) : (|\nabla^r z|^{q-2}\nabla^r z)\,dx \le$$

$$\le \begin{cases} |a|_{\mathcal{C}^k}\|z\|_{r,q}^q & (r \le k) \\ |a|_{\mathcal{C}^{k-1}}\|z\|_{0,q}\|\nabla^r z\|_{0,q}^{q-1} \le |a|_{\mathcal{C}^{k-1}}\|z\|_{r,q}^q & (r \le k-1) \\ |a|_{\mathcal{C}^{k-2}}\|z\|_{0,q}\|\nabla^r z\|_{0,q}^{q-1} \le |a|_{\mathcal{C}^{k-2}}\|z\|_{r,q}^q & (r \le k-2) \\ \|\nabla^r a\|_{0,n}\|z\|_{0,nq/(n-q)}\|\nabla^r z\|_{0,q}^{q-1} \le \|\nabla^r a\|_{0,n}\|\nabla z\|_{0,q}\|\nabla^r z\|_{0,q}^{q-1} & \\ & (1 < q < n,\ r \le k) \\ \|\nabla^r a\|_{0,q}|z|_{\mathcal{C}^0}\|\nabla^r z\|_{0,q}^{q-1} \le \|\nabla^r a\|_{0,q}\|z\|_{r,q}^q & (r = k, k-1,\ kq > n) \end{cases}$$

$$\mathcal{I}_4^r = \sum_{\substack{i+j=r \\ 0\le i\le r-1}} \int_\Omega (\nabla^i a \nabla^j z) : (|\nabla^r z|^{q-2}\nabla^r z)\,dx \le$$

$$\le \begin{cases} |a|_{\mathcal{C}^{k-1}}\|z\|_{r,q}^q \\ |a|_{\mathcal{C}^{k-2}}\|z\|_{r,q}^q \qquad (r \le k-1) \\ \sum\limits_{\substack{i+j=k-1 \\ 0\le i\le k-2}} \int_\Omega |\nabla^i a||\nabla^j z|\nabla^r z|^{q-1}\,dx \end{cases} \tag{3.17}$$

$$\le \left\{ |a|_{\mathcal{C}^{k-2}}\|\nabla z\|_{k-1,q}^q + \begin{cases} \|\nabla^{k-1} a\|_{0,n}\|\nabla z\|_{0,nq/(n-q)}\|\nabla z\|_{k-1,q}^{q-1} \\ \|\nabla^{k-1} a\|_{0,q}|\nabla z|_{\mathcal{C}^0}\|\nabla^r z\|_{k-1,q}^{q-1} \end{cases} \right\}$$

$$\le \begin{cases} \|\nabla^{k-1} a\|_{0,n}\|\nabla^2 z\|_{0,q}\|\nabla z\|_{k-1,q}^{q-1} & (r = k,\ 1 < q < n) \\ \|\nabla^{k-1} a\|_{0,q}\|\nabla z\|_{k-1,q}^q & (r = k,\ (k-1)q > n) \end{cases}$$

$$\mathcal{I}_5^r = \int_\Omega \nabla^r f : |\nabla^r z|^{q-2}\nabla^r z\,dx \le \|\nabla^r f\|_{0,q}\|\nabla^r z\|_{0,q}^{q-1} \tag{3.18}$$

Taking into account (3.11), (3.13) and (3.14)–(3.18), we verify the statements of Lemmas 3.1–3.3.

The only thing desiring an explanation is the calculation in (3.14) for $r = k$. Put

$$y = \begin{cases} \nabla^k z & \text{if } x \in \Omega \\ 0 & \text{if } x \notin \Omega \end{cases}$$

and extend w continuously to $\mathbb{R}^n$ (hence $w \in \mathcal{C}^1(\overline{\mathbb{R}}^n)$). By Corollary 2.2

$$w \cdot \nabla y_\varepsilon \to w \cdot \nabla y \ \text{ in } \ L^q(\Omega) \tag{3.19}$$

(for definition of y_ε see (2.8)). We have

$$-\int_{\Omega\cap B_R}(w\cdot\nabla y_\varepsilon)(|y_\varepsilon|^{q-2}y_\varepsilon)\,dx = \tag{3.20}$$
$$=\frac{1}{q}\int_{\Omega\cap B_R}\operatorname{div} w|y_\varepsilon|^q\,dx-\frac{1}{q}\int_{\partial B_R}w\cdot\nu|y_\varepsilon|^q\,dS.$$

Since $R^2\int_{\partial B_1}|y_\varepsilon|^q\,d\omega\in L^1(0,+\infty)$ uniformly with respect to ε ($d\omega$ is an infinitesimal element on the unit sphere), there exists a sequence $\{R_i\}_{i=1}^{+\infty}$, $R_i\to+\infty$ such that $R_i^2\int_{\partial B}|y_\varepsilon|^q\,d\omega\to 0$. Writing (3.20) with $R=R_i$ and passing to the limit $i\to+\infty$, we get

$$-\int_\Omega(w\cdot\nabla y_\varepsilon):(|y_\varepsilon|^{q-2}y_\varepsilon)\,dx=\frac{1}{q}\int_\Omega\operatorname{div} w|y_\varepsilon|^q\,dx. \tag{3.21}$$

By $\varepsilon\to 0$, we get, due to (3.19) and (2.10), estimate (3.14). The proofs of Lemmas 3.1–3.3 are thus complete. □

Remark 3.1. The reader easily sees that the constant α_0 in Lemma 3.1 is, in fact, independent of q (this is not the case in Lemmas 3.2 and 3.3). The above fact is seen from the proofs; we find from (3.11), (3.14), (3.15), (3.16), (3.17) and (3.18) that

$$\lambda\|z\|_{k,q}\le c\left\{\left[\left(1+\frac{1}{q}\right)|w|_{\varepsilon^k}+|a|_{\varepsilon^k}\right]\|z\|_{k,q}+\|f\|_{k,q}\right\}$$

with $c>0$ independent of q. This remark is very important in the part II of the paper (forthcoming [N4]), for deriving estimates in Hölder spaces.

4. Auxiliary existence theorems in $\mathbb{R}^n$

We begin this section by recalling one well known existence result of B. da Veiga [BV1, Theorem 2.1], which holds for bounded domains.

Lemma 4.1. *Let $k=1,2,\dots$, $\ell=1,\dots,k$, $1<q<+\infty$, $\lambda>0$, G be a bounded domain in $\mathbb{R}^n$ with $\partial G\in\mathcal{C}^{k+2}$. Let*

$$a,w\in\mathcal{C}^k(\overline{G}),\ f\in W^{k,q}(G)\cap W_0^{\ell,q}(G),\ w\cdot\nu\,|_{\partial G}=0.$$

Then there exists a constant α_G (depending of k, q, G and independent of λ) such that if $\alpha_G\vartheta_0<\lambda$, then there exists just one solution

$$z\in W^{k,q}(G)\cap W_0^{\ell,q}(G)$$

of problem $(1.1)_{\lambda>0}$ which satisfies estimate

$$\|z\|_{k,q}\le\frac{1}{\lambda-\alpha_G\vartheta_0}\|f\|_{k,q}. \tag{4.1}$$

(For definition of ϑ_0 see (2.11).)

In the next step, we extend this lemma to the whole space $\mathbb{R}^n$. The following statement is the starting point of all proofs of existence theorems (see the following sections).

Lemma 4.2. *Let* $k = 1, 2, \dots$, $\Omega = \mathbb{R}^n$ *and*

$$a, w \in \mathcal{C}^k(\overline{\mathbb{R}}^n),\ f \in W^{k,q}(\mathbb{R}^n).$$

Then there exists a constant $\alpha_2 > 0$ *(see Remark* 2.1*)*[(1)]*, such that if* $\alpha_2\vartheta_0 < 1$*, then there exists just one solution of problem* (1.1) $z \in W^{k,q}(\mathbb{R}^n)$ *satisfying estimate*

$$\|z\|_{k,q} \le \frac{1}{\lambda - \alpha_2\vartheta_0}\|f\|_{k,q} \tag{4.2}$$

(for definition of ϑ_0 *see* (2.11)*).*

Proof: Consider in $\Omega_{(R)} = B_{\kappa(R)}$ (see (2.5)) the following auxiliary problem for unknown function z_R:

$$\lambda z_R + (w\psi_R)\cdot\nabla z_R + a z_R = f. \tag{4.3}$$

(The cut off function ψ_R is defined in (2.4).) In virtue of Lemma 4.1, there exists $\alpha_R > 0$ (dependent possibly of R) such that if

$$\alpha_R\vartheta_{0R} < \lambda,\ \vartheta_{0R} = |a|_{\mathcal{C}^k} + |w\psi_R|_{\mathcal{C}^k},$$

then there exists a (unique) solution of (4.3) $z_R \in W^{k,q}(\Omega_{(R)})$. In virtue of Lemma 3.1, there exists $\alpha_0 > 0$ (independent of R and λ (see Remark 2.1)) such that

$$\lambda\|z_R\|_{k,q,\Omega_{(R)}} \le \|f\|_{k,q,\Omega_{(R)}} + \alpha_0\vartheta_{0R}\|z_R\|_{k,q,\Omega_{(R)}}.$$

Let w, a be such that $\alpha_0\vartheta_0 < \lambda$; hence $\alpha_0\vartheta_{0R} < \lambda$ for $R > R_0$, R_0 sufficiently great (recall that in virtue of (2.5), $\vartheta_{0R} \to \vartheta_0$ as $R \to +\infty$). Suppose that $\lambda < \alpha_R\vartheta_{0R} < \overline{\lambda}$ (in this case, Lemma 4.1 does not guarantee the existence of a solution). Nevertheless, it guarantees existence of a solution $z'_R \in W^{k,q}(\Omega_{(R)})$ of the problem

$$\overline{\lambda} z'_R + (w\psi_R)\cdot\nabla z'_R + a z'_R = f + (\overline{\lambda} - \lambda)\xi,$$

where ξ is an arbitrary element of $W^{k,q}(\Omega)$. This solution satisfies estimate

$$(\overline{\lambda} - \alpha_R\vartheta_{0R})\|z'_R\|_{k,q,\Omega_{(R)}} \le \|f\|_{k,q,\Omega_{(R)}} + (\overline{\lambda} - \lambda)\|\xi\|_{k,q,\Omega_{(R)}}.$$

One easily verifies that the (linear) map $T\xi = z'_R$ is, in virtue of the last inequality, a contraction in $W^{k,q}(\Omega_{(R)})$. As a consequence, it possesses a (unique) fixed point (say z_R) which obviously satisfies equation (4.3) and estimate

$$\|z_R\|_{k,q,\Omega_{(R)}} \le \frac{1}{\lambda - \alpha_0\vartheta_{0R}}\|f\|_{k,q,\Omega_{(R)}}. \tag{4.4}$$

---

(1) The constant α_2 is independent of q, see Remark 3.1.

Further, we proceed by the method of invading domains (cf. Leray [L], or Heywood [H]). We start with some $R > 0$ "sufficiently large" and denote $R_i = R + i$ $(i = 1, 2, \dots)$. Consider a sequence of solutions $\{z_{R_i} \equiv z_i\}_{i=1}^{+\infty}$ of the problem (4.3) in $\Omega_{(R_s)}$. For any fixed $\ell > 0$ there exists a subsequence $\{z_{i(\ell)}\}_{\ell=1}^{+\infty}$ and $z_{(\ell)} \in W^{k,q}(\Omega_{(R_s)})$ such that

$$\begin{aligned} z_{i(\ell)} \to z_{(\ell)} &\ \text{ weakly in }\ W^{k,q}(\Omega_{(R_s)}), \\ z_{i(\ell)} \to z_{(\ell)} &\ \text{ strongly in }\ W^{k-1,q}(\Omega_{(R_s)}). \end{aligned}$$

If $s > \ell$, one can choose a subsequence of $\{z_{i(\ell)}\}_{\ell=1}^{+\infty}$ which converges strongly in $W^{k-1,q}(\Omega_{(R_s)})$ and weakly in $W^{k,q}(\Omega_{(R_s)})$ to $z_{(s)} \in W^{k,q}(\Omega_{(R_s)})$. Clearly $z_{(s)}(x) = z_{(\ell)}(x)$ for $x \in \Omega_{(R_s)}$. We can thus define a function z in Ω

$$z(x) = z_{(s)}(x) \ \text{ provided }\ x \in \Omega_{(R_s)}.$$

We see that

$$z \in W^{k,q}_{\text{loc}}(\mathbb{R}^n);$$

it satisfies equation

$$\int_{\mathbb{R}^n} (z\varphi - z\nabla\cdot v\varphi + az\varphi)\,dx = \int g\varphi\,dx$$

for any $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. Due to (4.4)

$$\|z_{(s)}\|_{k,q,\Omega_{(R_s)}} = \|z\|_{k,q,\Omega_{(R_s)}} \le \frac{1}{1-\alpha_2\vartheta_0}\|f\|_{k,q,\mathbb{R}^n}$$

for a suitable $\alpha_2 > 0$. This yields, when $s \to +\infty$,

$$z \in W^{k,q}(\mathbb{R}^n)$$

and estimate (4.2). Moreover, equation (1.1) is satisfied a.e. in $\mathbb{R}^n$. Uniqueness is obvious. The proof of Lemma 4.2 is thus complete. □

Lemma 4.3. *Let*

(i) $k = 1, 2, \dots,\ 1 < q < +\infty,\ \Omega = \mathbb{R}^n,\ f \in W^{k,q}(\mathbb{R}^n)$,
$a, w \in (3.1) \cap (3.4)_j$, $j = 1$ *or* 2 (*see Lemma* 3.2)

or

(ii) $k = 2, 3, \dots,\ 1 < q < +\infty,\ \Omega = \mathbb{R}^n,\ f \in W^{k,q}(\mathbb{R}^n)$,
$a, w \in (3.1) \cap (3.9)_j$, $j = 1$ *or* 2 *or* 3 *or* 4 *or* 5 (*see Lemma* 3.3).

Then there exists a constant $\alpha_3 > 0$ (*see Remark* 2.1) *such that if* $\alpha_3\vartheta_j < \lambda$ (*case* (i)) *for at least one* j *or if* $\alpha_3\vartheta_{j+2} < \lambda$ (*case* (ii)) *for at least one* j (ϑ_j *is defined in* (2.11)), *then we have:*
There exists just one solution of problem (1.1) $z \in W^{k,q}(\mathbb{R}^n)$ *satisfying estimate*

$$(4.5)_1 \qquad \|z\|_{k,q,\mathbb{R}^n} \leq \frac{1}{\lambda - \alpha_3\vartheta_j}\|f\|_{k,q,\mathbb{R}^n} \qquad (j = 1, 2 \ \text{— case (i)})$$

or

$$(4.5)_2 \qquad \|z\|_{k,q,\mathbb{R}^n} \leq \frac{1}{\lambda - \alpha_3\vartheta_{j+2}}\|f\|_{k,q,\mathbb{R}^n} \qquad (j = 1, \dots, 5 \ \text{— case (ii)}).$$

Proof: We prove only Lemma 4.3 (i). The proof of statement (ii) is similar. Take $\alpha_2 > 0$ from Lemma 4.2 and suppose $\alpha_2\vartheta_0 < \lambda$. Then there exists a solution $z \in W^{k,q}(\mathbb{R}^n)$ of problem (1.1). Take α_0 from Lemma 3.2 and suppose $\alpha_0(\vartheta_0 + \vartheta_i) < \lambda$, $i = 1, \dots, 7$. Then estimate $(4.5)_1$ follows by Lemma 3.2. Suppose that $\lambda < \alpha_0(\vartheta_0 + \vartheta_i) < \overline{\lambda}$ and $\alpha_2\vartheta_0 < \overline{\lambda}$. Then, by the previous reasoning, for any $\xi \in W^{k,q}(\mathbb{R}^n)$ there exists a solution $z' \in W^{k,q}(\mathbb{R}^n)$ of problem

$$\overline{\lambda}z' + w \cdot \nabla z' + az' = f + (\overline{\lambda} - \lambda)\xi$$

which satisfies, by Lemma 3.2, estimate

$$\big(\overline{\lambda} - \alpha_0\vartheta_i\big)\|z'\|_{k,q,\mathbb{R}^n} \leq c\big(\|f\|_{k,q,\mathbb{R}^n} + (\overline{\lambda} - \lambda)\|\xi\|_{k,q,\mathbb{R}^n}\big).$$

The last inequality yields the contraction of the map $T_{\overline{\lambda}}\xi = z'$, in $W^{k,q}(\mathbb{R}^n)$ and existence of a fixed point z. It is easy to verify that z satisfies problem (1.1) and estimate $(4.5)_1$. □

5. Existence of solutions in Sobolev spaces for $\Omega \in \mathcal{B}^{(k)}$.

Lemmas 4.2–4.3 give existence of solutions in Sobolev spaces in $\Omega = \mathbb{R}^n$. Here we prove existence and uniqueness of solutions for domains of class $\mathcal{B}^{(k)}$ (in particular for bounded and exterior domains with sufficiently smooth boundary, for $\Omega = \mathbb{R}^n$ or $\Omega = \mathbb{R}^n_+$) for small a, w (in appropriate norms) under two different sets of assumptions on the regularity of a, w. Theorem 5.1 is an easy consequence of Lemma 4.2. Theorems 5.2, 5.3, for bounded domains, give practically the same results as B. da Veiga's Theorem 2.1* in [BV1], however, under less assumptions on the regularity of the boundary. For another domains of class $\mathcal{B}^{(k)}$ (e.g. exterior, etc.), as far as the author knows, the results are new. In the second part of this section we investigate solutions in weighted Sobolev spaces (see Theorems 5.4 and 5.5). Third part of this section is devoted to the investigation of the regularity of solutions (see Theorem 5.6). Finally, we investigate existence of weak solutions in Lebesgue spaces (Theorem 5.7) and the decay of continuous solutions (Theorem 5.8). All presented results are important in applications in the theory of compressible fluids.

5.1 Existence of solutions in Sobolev spaces

Theorem 5.1. *Let* $k = 1, 2, \ldots$, $1 < q < +\infty$, $\ell = 1, \ldots k$, $\Omega \in \mathcal{B}^{(k)}$. *Let*

$$w \in \mathcal{C}^k(\overline{\Omega}),\ w \cdot \nu\,|_{\partial\Omega} = 0,\ a \in \mathcal{C}^k(\overline{\Omega}),\qquad (5.1)$$
$$f \in W^{k,q}(\Omega) \cap W_0^{\ell,q}(\Omega).$$

Then there exists a constant $\alpha_1 > 0$ *(see Remark* 2.1*)*[(1)] *such that if*

$$\alpha_1 \vartheta_0 < \lambda,$$

then there exists just one solution

$$z \in W^{k,q}(\Omega) \cap W_0^{\ell,q}(\Omega)$$

satisfying estimate

$$\|z\|_{k,q} \le \frac{1}{\lambda - \alpha_1 \vartheta_0} \|f\|_{k,q}.$$

For definition of ϑ_0 *see* (2.11).

Theorem 5.2. *Let* $k = 1, 2, \ldots$, $1 < q < +\infty$, $\ell = 1, \ldots, k$, $\Omega \in \mathcal{B}^{(k)}$. *Let*

$$w \in \mathcal{C}^k(\overline{\Omega}),\ w \cdot \nu\,|_{\partial\Omega} = 0,\ a \in \mathcal{C}^{k-1}(\overline{\Omega}),\qquad (5.2)$$
$$f \in W^{k,q}(\Omega) \cap W_0^{\ell,q}(\Omega).$$

Then there exists a constant $\alpha_1 > 0$ *(see Remark* 2.1*) such that:*

(a) *If*

$$kq > n,\ \nabla^k a \in L^q(\Omega), \qquad (5.3)_1$$

and

$$\alpha_1 \vartheta_1 < \lambda$$

or if

$$1 < q < n,\ \nabla^k a \in L^n(\Omega), \qquad (5.3)_2$$

and

$$\alpha_1 \vartheta_2 < \lambda,$$

then there exists just one solution of problem (1.1)

$$z \in W^{k,q}(\Omega) \cap W_0^{\ell,q}\Omega$$

such that

$$\|z\|_{k,q} \le \frac{1}{\lambda - \alpha_1 \vartheta_i} \|f\|_{k,q} \qquad (5.4)$$

(where $i = 1, 2$, *refers to* $(5.3)_i$ *and* ϑ_i *is defined in* (2.11)*).*

---

(1) The constant α_1 in Theorem 5.1 is, in fact, independent of q, see Remark 3.1.

(b) *If*

$$a = 0$$

and

$$(5.5)_1 \qquad \alpha_1 \vartheta_1' < 1$$

or

$$(5.5)_2 \qquad 1 < q < n,\ \nabla a \in W^{k-1,n}(\Omega)$$

and

$$\alpha_1 \vartheta_2' < \lambda$$

then the solution satisfies estimate

$$(5.6) \qquad \|\nabla z\|_{k-1,q} \le \frac{1}{\lambda - \alpha_1 \vartheta_i'} \|\nabla f\|_{k-1,q}.$$

(*Here* $i = 1, 2$ *refers to* $(5.5)_i$ *and* ϑ_i' *is defined in* (2.11).)

Theorem 5.3. *Let* $k = 2, 3, \dots,$ $1 < q < +\infty,$ $kq > n,$ $\ell = 1, \dots, k,$ *and* $\Omega \in \mathcal{B}^{(k)}$. *Let*

$$(5.7) \qquad w \in \mathcal{C}^{k-1}(\overline{\Omega}),\ w \cdot \nu\,|_{\partial\Omega} = 0,\ a \in \mathcal{C}^{k-2}(\overline{\Omega}),$$

$$(5.8) \qquad f \in W^{k,q}(\Omega) \cap W_0^{\ell,q}(\Omega).$$

Then there exists a constant α_1 (*see Remark* 2.1) *such that we have:*
If

$$(5.9)_1 \qquad 1 < q < n,\ \nabla^k w \in L^n(\Omega),\ \nabla^{k-1} a \in L^n(\Omega),\ \nabla^k a \in L^q(\Omega)$$

and

$$\alpha_1 \vartheta_3 < \lambda$$

or

$$(5.9)_2 \qquad 1 < q < n,\ \nabla^k w \in L^n(\Omega),\ \nabla^{k-1} a \in W^{1,n}(\Omega)$$

and

$$\alpha_1 \vartheta_4 < \lambda$$

or

$$(5.9)_3 \qquad 1 < q < n,\ \nabla^k w \in L^n(\Omega),\ \nabla^{k-1} a \in W^{1,q}(\Omega)$$

and

$$\alpha_1 \vartheta_5 < \lambda$$

or

$$(5.9)_4 \qquad (k-1)q > n,\ \nabla^k w \in L^q(\Omega),\ \nabla^{k-1} a \in W^{1,q}(\Omega)$$

and

$$\alpha_1 \vartheta_6 < \lambda$$

or

$$(5.9)_5 \qquad (k-1)q > n,\ \nabla^k w \in L^q(\Omega),\ \nabla^{k-1} a \in W^{1,n}(\Omega)$$

and

$$\alpha_1 \vartheta_7 < \lambda$$

then there exists just one solution of problem (1.1)

$$z \in W^{k,q}(\Omega) \cap W_0^{\ell,q}(\Omega)$$

satisfying estimate

$$(5.10) \qquad \|z\|_{k,q} \le \frac{1}{\lambda - \alpha_1 \vartheta_{i+2}} \|f\|_{k,q}.$$

(*Here* $i = 1, \dots, 5$ *refers to* $(5.9)_i$. *For definition of* ϑ_j *see* (2.11).)

Proof of Theorems 5.1, 5.2 and 5.3: We prove only Theorem 5.3 under assumption $(5.9)_1$. The other cases and Theorems 5.1, 5.2 follow by the same (even technically easier) arguments, and therefore are left to the reader.

By Definition 2.1 (since $\Omega \in \mathcal{B}^{(k)}$), there exists a continuous extension of a, w, f (denoted again a, w, f)

$$\begin{aligned}
&a \in \mathcal{C}^{k-2}(\overline{\mathbb{R}}^n),\ \nabla^{k-1} a \in L^n(\mathbb{R}^n),\ \nabla^k a \in L^q(\mathbb{R}^n),\\
&w \in \mathcal{C}^{k-1}(\overline{\mathbb{R}}^n),\ \nabla^k w \in L^n(\mathbb{R}^n),\ f \in W^{k,q}(\mathbb{R}^n).
\end{aligned}$$

For the sequences of mollified functions (see (2.7)–(2.9))

$$\left\{a_{1/s}\right\}_{s=1}^{+\infty},\ \left\{w_{1/s}\right\}_{s=1}^{+\infty} \left(a_{1/s} \in \mathcal{C}^k(\overline{\mathbb{R}}^n),\ w_{1/s} \in \mathcal{C}^k(\overline{\mathbb{R}}^n)\right)$$

we have, in virtue of (2.10),

$$\begin{aligned}
&w_{1/s} \to w \ \text{ in } \ \mathcal{C}^{k-1}(\overline{\mathbb{R}}^n),\ \nabla^k w_{1/s} \to \nabla^k w \ \text{ in } \ L^n(\mathbb{R}^n),\\
&a_{1/s} \to a \ \text{ in } \ \mathcal{C}^{k-2}(\overline{\mathbb{R}}^n),\ \nabla^{k-1} a_{1/s} \to \nabla^{k-1} a \ \text{ in } \ L^n(\mathbb{R}^n),\\
&\nabla^k a_{1/s} \to \nabla^k a \ \text{ in } \ L^q(\mathbb{R}^n).
\end{aligned}$$

Put $\vartheta_{3,s} = \vartheta_3(w_{1/s}, a_{1/s})$. In virtue of Lemma 4.3, there exists a constant $\alpha' > 0$ (independent of λ, s) such that if $\alpha'\vartheta_{3,s} < \lambda$, then we have a solution

$$z_s \in W^{k,q}(\mathbb{R}^n)$$

of problem

$$\lambda z_s + w_{1/s} \cdot \nabla z_s + a_{1/s} z_s = f \quad \text{in} \quad \mathbb{R}^n \tag{5.11}$$

satisfying estimate

$$\|z_s\|_{k,q,\mathbb{R}^n} \le \frac{1}{\lambda - \alpha'\vartheta_{3,s}} \|f\|_{k,q,\mathbb{R}^n}.$$

Since $\vartheta_{3,s} \to \vartheta_3$ as $s \to +\infty$, we have also for s sufficiently large

$$\|z_s\|_{k,q,\mathbb{R}^n} \le \frac{1}{\lambda - \alpha'\vartheta_3} \|f\|_{k,q,\mathbb{R}^n}. \tag{5.12}$$

On the other hand, the difference $z_s - z_{s'}$ satisfies equation

$$\begin{aligned}(z_s - z_{s'}) + w_{1/s} \cdot \nabla(z_s - z_{s'}) + a_{1/s}(z_s - z_{s'}) = \\ = z_{s'}(a_{1/s'} - a_{1/s}) + \nabla z_{s'} \cdot (w_{1/s'} - w_{1/s}).\end{aligned}$$

We estimate

$$\begin{aligned}&\big\|z_{s'}(a_{1/s'} - a_{1/s})\big\|_{k-1,q} \le \big\|z_{s'}\big\|_{0,\frac{nq}{n-q}} \big\|\nabla^{k-1}(a_{1/s'} - a_{1/s})\big\|_{0,n} \\ &+ \big\|z_{s'}\big\|_{k-1,q} \big|a_{1/s'} - a_{1/s}\big|_{\mathcal{C}^{k-2}} \\ &\le \big\|z_{s'}\big\|_{k-1,q}\big(\big|a_{1/s'} - a_{1/s}\big|_{\mathcal{C}^{k-2}} + \big\|\nabla^{k-1}(a_{1/s'} - a_{1/s})\big\|_{0,n}\big), \\ &\big\|\nabla z_{s'} \cdot (w_{1/s'} - w_{1/s})\big\|_{k-1,q} \le \big\|\nabla z_{s'}\big\|_{k-1,q} \big|w_{1/s'} - w_{1/s}\big|_{\mathcal{C}^{k-1}}.\end{aligned}$$

This yields, by Lemma 3.3, that the sequence $\{\|z_s\|\}_{s=1}^{+\infty}$ is a Cauchy sequence in $W^{k-1,q}(\mathbb{R}^n)$. Moreover, by (5.12), it is also bounded in $W^{k,q}(\mathbb{R}^n)$. Therefore

$$\begin{aligned}z_s &\to z \quad \text{strongly in} \quad W^{k-1,q}(\mathbb{R}^n), \\ z_s &\to z \quad \text{weakly in} \quad W^{k,q}(\mathbb{R}^n)\end{aligned}$$

at least for a chosen subsequence. It is straightforward to show (by passing to the limit $s \to +\infty$ in (5.11)) that z solves problem (1.1) in $\mathbb{R}^n$. The restriction $z\,|\,_\Omega$ on Ω (denoted again z) obviously satisfies (1.1) in Ω. Estimate (5.10) follows directly from (5.12).

The only thing left to be shown is $z \in W_0^{\ell,q}(\Omega)$. It is enough to do it in the case $\Omega = \mathbb{R}^n_+$. The general case of the "curved" boundary $\partial\Omega$ can be transformed to the previous one by the localisation technique explained in Example 2.1.

First take $\ell = 1$. Suppose, without loss of generality, that $\partial\mathbb{R}^n_+ = \{x' : x' = (x_1,\dots,x_{n-1}) \in \mathbb{R}^{n-1}\} = \mathbb{R}^{n-1}$. Multiply the equation (1.1) by $|z|^{q-2}z$ and integrate over $\mathbb{R}^{n-1}$. For the most complicated integral, we have, in particular, $\int_{\mathbb{R}^{n-1}} w\cdot(\nabla z)|z|^{q-2}z\,dx = \int_{\mathbb{R}^n} w\cdot(\nabla' z)|z|^{q-2}z\,dx$, where $\nabla' = (\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n})$; the term containing the expression $w_n\cdot\frac{\partial z}{\partial x_n}$ vanishes due to the condition $w\cdot\nu\,|_{\,\partial\Omega} = 0$. Therefore, we get, after some calculation

$$\|z\|_{0,q,\mathbb{R}^{n-1}} \le \frac{1}{\lambda - \alpha_1\vartheta_3}\|f\|_{0,q,\mathbb{R}^{n-1}}.$$

This yields the proof for $\ell = 1$. If $\ell > 1$, we proceed by induction. □

Remark 5.1. Here we give several sufficient conditions on a, w, to satisfy assumptions of Theorems 5.2 or 5.3.

(i) Let Ω be a bounded domain ($\partial\Omega \in \mathcal{C}^k$) and

$$w \in W^{k+1,q}(\Omega) \cap W^{1,q}_0(\Omega),\ \ a \in W^{k,q}(\Omega). \tag{5.13}$$

(i)$_1$ If $n \ge 2$, $q > n$, $k = 1,2,\dots$, then $a, w \in$ (5.13) satisfy assumptions (5.2) and (5.3)$_1$.

(i)$_2$ If $\frac{n}{n-1} < q < n$, $k = (n-1),\dots$, then $a, w \in$ (5.13) satisfy assumptions (5.7) and (5.9)$_1$, (5.9)$_3$, (5.9)$_4$.

For applications of (i) to compressible fluids, see B. da Veiga [BV3] ($n = 3$, $k = 1,2,\dots$, $q > 3$), Novotný [N1] ($n = 3$, k=2,3, … , $q = 2$).

(ii) Let Ω be an exterior domain to some compact region Ω_c ($\partial\Omega \in \mathcal{C}^k$).

(ii)$_1$ Let a, w be such that

$$\textit{there exists}\ \ w_\infty \in \mathbb{R}^n\ \ \textit{such that}\ \ w - w_\infty \in L^{\frac{nt}{n-t}}(\Omega), \tag{5.14}$$

and

$$\begin{gathered} w\,|_{\,\partial\Omega},\ w \in W^{2,t}_{\mathrm{loc}}(\Omega) \cap W^{k+1,q}_{\mathrm{loc}}(\Omega),\ \nabla w \in W^{1,t}(\Omega) \cap W^{k,q}(\Omega) \\ a \in W^{1,t}(\Omega) \cap W^{k,q}(\Omega) \end{gathered} \tag{5.15}$$

where

$$1 < t < n,\ q > n,\ k = 1,2,\dots\ .$$

Then a, w satisfy assumptions (5.2) and (5.3)$_1$.

(ii)$_2$ Let a, w be such that

$$\textit{there exists}\ \ w_\infty \in \mathbb{R}^n\ \ \textit{such that}\ \ w - w_\infty \in L^{\frac{nq}{n-q}}(\Omega), \tag{5.16}$$

and

$$w\,|_{\,\partial\Omega} = 0,\ w \in W^{k+1,q}_{\mathrm{loc}}(\Omega),\ \nabla w \in W^{k,q}(\Omega),\ a \in W^{k,q}(\Omega) \tag{5.17}$$

where

$$\frac{n}{n-1} < q < n,\ k = (n-1), n, \dots\ .$$

Then a, w satisfy assumptions (5.7), (5.9)$_4$.

For applications of (ii) see Novotný, Padula [NP1] ($n = 3$, $k = 1, 2, \ldots$, $3/2 < t < 3$, $q < 3$), Novotný [N1] ($n = 3$, $k = 2, \ldots$, $q = 2$), Novotný, Penel [NPe]. For another applications for 2-D exterior domains see Galdi, Novotný, Padula [GNP].

Remark 5.2. We have the following consequence of the former proof. Let z be a solution of problem (1.1) guaranteed by Theorem 5.1 or 5.2 or 5.3 (i.e. a, w, f satisfy all assumptions of at least one of these theorems). Denote by $\tilde{a}$, $\tilde{w}$, $\tilde{f}$ a continuous extension of a, w, f to $\mathbb{R}^n$ (i.e. in the case of assumptions (5.7), (5.8), $(5.9)_1$ $\tilde{a} \in \mathcal{C}^{k-1}(\mathbb{R}^n)$, $\nabla^{k-1}\tilde{a} \in L^n(\mathbb{R}^n)$, $\nabla^k\tilde{a} \in L^q(\mathbb{R}^n)$, $\tilde{w} \in \mathcal{C}^{k-1}(\overline{\mathbb{R}}^n)$, $\nabla^k\tilde{w} \in L^n(\mathbb{R}^n)$, etc.). If a, w are sufficiently small in corresponding norms, then (due to continuity of extension) $\tilde{a}$, $\tilde{w}$ are also sufficiently small in corresponding norms, such that there exists (in virtue of Theorem 5.1 or 5.2 or 5.3 with $\Omega = \mathbb{R}^n$) a unique solution $\tilde{z}$ of the problem

$$\lambda\tilde{z} + \tilde{w} \cdot \nabla\tilde{z} + \tilde{a}\tilde{z} = \tilde{f} \ \text{ in } \ \mathbb{R}^n.$$

Then obviously

$$z = \tilde{z}\,|_{\Omega}.$$

5.2 Existence of solutions in weighted Sobolev spaces

Let $\Omega \subset \mathbb{R}^n$ be an unbounded domain in at least one direction. We introduce the weights

$$g \in \mathcal{C}^k(\Omega),\ g(x) > 0,\ g(x) \to +\infty \ \text{ as } \ x = t\mathbf{e} \to +\infty, \tag{5.18}$$

at least for one direction $\mathbf{e} \in \mathbb{R}^n$, such that $t\mathbf{e} \in \Omega$, $t > t_0$. We define weighted Sobolev spaces $W^{k,q}_{(g)}(\Omega)$, $k = 1, 2, \ldots$, $1 < q < +\infty$, as follows

$$u \in W^{k,q}_{(g)}(\Omega) \ \textit{ if and only if } \ ug \in W^{k,q}(\Omega). \tag{5.19}$$

The corresponding norm reads

$$\|u\|_{(g),k,q} \le \|ug\|_{k,q,\Omega}. \tag{5.20}$$

Last but not least introduce

$$\begin{cases} \vartheta^{k,q}_{(g)i} = \vartheta^{k,q}_i + \sum_{s=0}^{k}\sum_{\alpha=0}^{s} \left|\nabla^{1+s-\alpha}\ln g\nabla^\alpha w\right|_{\mathcal{C}^0} \\ (i = 0, 1, 2) \\ \vartheta^{k,q}_{(g)j} = \vartheta^{k,q}_j + \sum_{s=0}^{k-1}\sum_{\alpha=0}^{s} \left|\nabla^{1+s-\alpha}\ln g\nabla^\alpha w\right|_{\mathcal{C}^0} \\ + \sum_{\alpha=0}^{k-1} \left|\nabla^{1+k-\alpha}\ln g\nabla^\alpha w\right|_{\mathcal{C}^0} + \left\|\nabla\ln g\nabla^k w\right\|_{0,n} \\ (j = 3, 4, 5) \\ \vartheta^{k,q}_{(g)i} = \vartheta^{k,q}_i + \sum_{s=0}^{k-1}\sum_{\alpha=0}^{s} \left|\nabla^{1+s-\alpha}\ln g\nabla^\alpha w\right|_{\mathcal{C}^0} \\ + \sum_{\alpha=0}^{k-1} \left|\nabla^{1+k-\alpha}\ln g\nabla^\alpha w\right|_{\mathcal{C}^0} + \left\|\nabla\ln g\nabla^k w\right\|_{0,q} \\ (i = 6, 7). \end{cases} \tag{5.21}$$

The most usual weights in applications are e.g. $g(x)=\mathrm{e}^{\beta|x|}$, $\beta>0$ or $g(x)=(1+|x|)^\beta$, $\beta>0$, in the case of $\mathbb{R}^n$, $\mathbb{R}^n_+$ or Ω exterior and $g(x)=\mathrm{e}^{\beta\sqrt{1+x_n^2}}$ in the case of the pipe (2.2). For application of the latter case to compressible fluids see Padula, Pileckas [PP].

Theorem 5.4. *Let $k=1,2,\dots$, $1<q<+\infty$, $\Omega\in\mathcal{B}^{(k)}$ be a domain unbounded in at least one direction. Let g be a weight* (5.18) *and $\vartheta_{(g)i}$ $(i=0,1,2,)$ be defined in* (5.21). *Suppose that*

$$w\in\mathcal{C}^k(\overline{\Omega}),\ \nabla^{s+1-\alpha}\ln g\nabla^\alpha w\in\mathcal{C}^0(\overline{\Omega})\ \ (\alpha=0,\dots,s;\ s=0,\dots,k)$$
$$w\cdot\nu\,|_{\,\partial\Omega}=0,\ a\in\mathcal{C}^{k-1}(\overline{\Omega}),\ f\in W^{k,q}_{(g)}(\Omega). \tag{5.22}$$

Then there exist a constant $\alpha_1>0$ (see Remark 2.1*) such that:*
If

$$kq>n,\ \nabla^k a\in L^q(\Omega) \tag{5.23$_1$}$$

and

$$\alpha_1\vartheta_{(g)1}<\lambda$$

or if

$$1<q<n,\ \nabla^k a\in L^n(\Omega) \tag{5.23$_2$}$$

and

$$\alpha_1\vartheta_{(g)2}<\lambda,$$

then there exists just one solution of problem (1.1) $z\in W^{k,q}_{(g)}\Omega$ *such that*

$$\|z\|_{(g),k,q}\le\frac{1}{\lambda-\alpha_1\vartheta_{(g)i}}\|f\|_{(g),k,q} \tag{5.24}$$

(where $i=1,2$ refers to $(5.23)_i$ *and $\vartheta_{(g)i}$ are defined by* (5.21)*).*
If we replace in (5.22) *the hypothesis $a\in\mathcal{C}^{k-1}(\overline{\Omega})$ by the hypothesis $a\in\mathcal{C}^k(\overline{\Omega})$, we have: There exists $\alpha_1>0$ (see Remark* 2.1*) such that if*

$$\alpha_1\vartheta_{(g)0}<\lambda,$$

then there exists just one solution $z\in W^{k,q}_{(g)}(\Omega)$ which satisfies estimate (5.24) *with $i=0$.*

Proof of Theorem 5.4: Define a map $T : W^{k,q}(\Omega) \to W^{k,q}(\Omega)$ which maps $\zeta \to \xi$

$$\lambda\xi + w\cdot\nabla\xi + a\xi = gf - w\cdot\frac{\nabla g}{g}\zeta; \tag{5.25}$$

this map exists by Theorem 5.2, provided ϑ_i (see (2.11)) is sufficiently small.

We easily estimate

$$\Big\|\frac{w\cdot\nabla g}{g}\zeta\Big\|_{k,q} \le \sum_{s=0}^{k}\sum_{\alpha=0}^{s}\Big|\nabla^{1+s-\alpha}\ln g\nabla^\alpha w\Big|_{\mathcal{C}^0}\|\zeta\|_{k,q}.$$

Hence ξ satisfies estimate

$$(\lambda - \alpha_1'\vartheta_i)\|\xi\|_{k,q} \le \|gf\|_{k,q} + \vartheta_{(g)i}\|\zeta\|_{k,q}$$

with $\alpha_1' > 0$.

Therefore, if $\vartheta_{(g)i} < \lambda - \alpha_1'\vartheta_i$, T is a contraction and possess a (unique) fixed point (say ξ). Set $z = \frac{\xi}{g}$; then one easily verifies that z ($\in W^{k,q}_{(g)}(\Omega)$) is a solution of problem (1.1) and satisfies estimate (5.24). The uniqueness is obvious. Theorem 5.4 is thus proved. $\square$

Similarly we have (the proof is left to the reader)

Theorem 5.5. *Let $k = 2,3,\dots$, $1 < q < +\infty$ and $\Omega \in \mathcal{B}^{(k)}$ be a domain unbounded in at least one direction. Let g be a weight which satisfies requirements* (5.18) *and let $\vartheta_{(g)i}$ $(i = 3,\dots,7)$ be constants defined in* (5.21)*. Suppose that*

$$\begin{gathered} w \in \mathcal{C}^{k-1}(\overline{\Omega}),\ \nabla^{s+1-\alpha}\ln g\nabla^\alpha w \in \mathcal{C}^0(\overline{\Omega}),\ \nabla^{1+k-\alpha'}\ln g\nabla^{\alpha'} w \in \mathcal{C}^0(\overline{\Omega}) \\ (\alpha' = 0,\dots,k-1,\ \alpha = 0,\dots,s,\ s = 0,\dots,k-1),\ w\cdot\nu\,|_{\partial\Omega} = 0, \\ a \in \mathcal{C}^{k-2}(\overline{\Omega}),\ f \in W^{k,q}_g(\Omega). \end{gathered} \tag{5.26}$$

Then there exists a constant α_1 (see Remark 2.1*) such that we have:*
If

$$1 < q < n,\ \nabla^k w, \nabla\ln g\nabla^k w \in L^n(\Omega),\ \nabla^{k-1}a \in L^n(\Omega),\ \nabla^k a \in L^q(\Omega) \tag{5.27$_1$}$$

and

$$\alpha_1\vartheta_{(g)3} < \lambda$$

or if

$$1 < q < n,\ \nabla^k w, \nabla\ln g\nabla^k w \in L^n(\Omega),\ \nabla^{k-1}a \in W^{1,n}(\Omega) \tag{5.27$_2$}$$

and

$$\alpha_1 \vartheta_{(g)4} < \lambda$$

or if

(5.27)$_3$ $$1 < q < n,\ \nabla^k w, \nabla \ln g \nabla^k w \in L^n(\Omega),\ \nabla^{k-1} a \in W^{1,q}(\Omega)$$

and

$$\alpha_1 \vartheta_{(g)5} < \lambda$$

or if

(5.27)$_4$ $$(k-1)q > n,\ \nabla^k w, \nabla \ln g \nabla^k w \in L^q(\Omega),\ \nabla^{k-1} a \in W^{1,q}(\Omega)$$

and

$$\alpha_1 \vartheta_{(g)6} < \lambda$$

or if

(5.27)$_5$ $$(k-1)q > n,\ \nabla^k w, \nabla \ln g \nabla^k w \in L^q(\Omega),\ \nabla^{k-1} a \in W^{1,n}(\Omega)$$

and

$$\alpha_1 \vartheta_{(g)7} < \lambda.$$

Then there exists just one solution of problem (1.1)

$$z \in W^{k,q}_{(g)}(\Omega)$$

satisfying estimate

(5.28) $$\|z\|_{(g),k,q} \le \frac{1}{\lambda - \vartheta_{(g)i+2}} \|f\|_{(g),k,q}.$$

(*Here* $i = 1, \dots, 5$ *corresponds to* $(5.27)_i$. *For definition of* $\vartheta_{(g)j}$ *see* (5.21)).

5.3 Regularity of solutions

In this subsection we prove a result about the regularity of solutions. We restrict ourselves only to the case $a = \operatorname{div} w$ (since it is most important in applications). The general case can be studied by the same method. Nevertheless, the assumptions would be, due to the technical reasons, much more complicated. Therefore, this case is omitted here.

The following theorem is very important in several applications to compressible fluids, see e.g. Novotný, Padula [NP1], [NP3], Novotný [N1], Galdi, Novotný, Padula [GNP] and Novotný, Penel [NPe].

Theorem 5.6. *Let* $k, m = 1, 2, \dots,\ m \le k,\ 1 < q, p < +\infty,\ \Omega \in \mathcal{B}^{(k)}$. *Let*

$$w \in \mathcal{C}^k(\overline{\Omega}),\ w \cdot \nu\,|_{\partial\Omega} = 0,\ a = \operatorname{div} w,\ f \in W^{k,q}(\Omega) \cap W^{m,p}(\Omega).$$

Then there exists a constant $\alpha_1 > 0$ (*see Remark* 2.1) *such that:*
If

(5.29)$_1$ $$kq > n,\ m < k,\ \nabla^k a \in L^q(\Omega)$$

and

$$\alpha_1\eta_1 < \lambda,\ \eta_1 = \vartheta_1^{(k,q)} + \vartheta_0^{(m,p)}$$

or if

$$(5.29)_2 \qquad kq > n,\ m = k,\ \nabla^k a \in L^q(\Omega) \cap L^p(\Omega)$$

and

$$\alpha_1\eta_2 < \lambda,\ \eta_2 = \vartheta_1^{(k,q)} + \vartheta_1^{(m,p)}$$

or if

$$(5.29)_3 \qquad 1 < q < n,\ m < k,\ \nabla^k a \in L^n(\Omega)$$

and

$$\alpha_1\eta_3 < \lambda,\ \eta_3 = \vartheta_2^{(k,q)} + \vartheta_0^{(m,p)}$$

or if

$$(5.29)_4 \qquad 1 < q, p < n,\ m = k,\ \nabla^k a \in L^n(\Omega)$$

and

$$\alpha_1\eta_4 < \lambda,\ \eta_4 = \vartheta_2^{(k,q)} + \vartheta_2^{(m,p)}$$

or if

$$(5.29)_5 \qquad 1 < q < n,\ mp > n,\ k = m,\ \nabla^k a \in L^q(\Omega) \cap L^n(\Omega)$$

and

$$\alpha_1\eta_5 < \lambda,\ \eta_5 = \vartheta_2^{(k,q)} + \vartheta_1^{(m,p)},$$

then we have: There exists a unique solution of problem (1.2)

$$z \in W^{k,q}(\Omega) \cap W^{m,p}(\Omega)$$

such that

$$(5.30) \qquad \|z\|_{k,q} + \|z\|_{m,p} \leq \frac{1}{\lambda - \alpha_1\eta_i}\left(\|f\|_{k,q} + \|f\|_{m,p}\right)$$

(*here* $i = 1, 2, 3, 4, 5$ *corresponds to* $(5.29)_i$ *and* ϑ_j *are defined in* (2.11)).

Proof: We prove the theorem under assumption $(5.29)_1$. The other cases are similar and are left to the reader. According to Remark 5.2, it is sufficient to investigate only the case $\Omega = \mathbb{R}^n$. According to Theorems 5.1 and 5.2, there exist solutions of problem (1.1), $z_1 \in W^{k,q}(\mathbb{R}^n)$, $z_2 \in W^{m,p}(\mathbb{R}^n)$ (provided η_1 is sufficiently small). Denote by $\tilde z$ the difference $\tilde z = z_1 - z_2$. It obviously satisfies equation

$$\lambda \tilde z + w\cdot\nabla\tilde z + \operatorname{div} w\tilde z = 0 \ \text{ a.e. in } \ \mathbb{R}^n. \tag{5.31}$$

Using mollifier, we get for mollified $\tilde z$

$$\lambda \tilde z_\varepsilon + w\cdot\nabla\tilde z_\varepsilon + \operatorname{div} w\tilde z_\varepsilon = r_\varepsilon \ \text{ in } \ \mathbb{R}^n \tag{5.32}$$

where $r_\varepsilon = w\cdot\nabla\tilde z_\varepsilon - (w\cdot\nabla\tilde z)_\varepsilon$.

Obviously (cf. Corollary 2.1)

$$\begin{aligned} &r_\varepsilon \to 0 \ \text{ in } \ L^{s_0}_{\mathrm{loc}}(\Omega),\ 1\le s_0\le \min(p,q),\\ &r_\varepsilon \to 0 \ \text{ a.e. in } \ \mathbb{R}^n. \end{aligned} \tag{5.33}$$

Multiply (5.32) by $\psi_R\,\varphi$, where ψ_R is cut-off function (2.4) (with R sufficiently great) and $\varphi\in W^{1,t_0}(\mathbb{R}^n)$ $(t_0\ge\max(p,q,p',q'),\ p'=p/(p-1),\ q'=q/(q-1))$. We get, after integration by parts,

$$\int_{\mathbb{R}^n}\tilde z_\varepsilon\psi_R(\lambda\varphi - w\cdot\nabla\varphi)\,dx = \int_{\mathbb{R}^n} r_\varepsilon\psi_R\,\varphi\,dx + \int_{\mathbb{R}^n}\nabla\psi_R\cdot w z_\varepsilon\varphi\,dx. \tag{5.34}$$

Let $\mathcal F\in\mathcal C_0^\infty(\mathbb{R}^n)$. Then Theorem 5.1 and Remark 3.1 guarantee, for $\vartheta_0(w,0)$ $(\le\eta_1)$ sufficiently small, existence of a unique solution

$$\varphi\in W^{1,t_0}(\mathbb{R}^n)$$

of the problem

$$\lambda\varphi - w\cdot\nabla\varphi = \mathcal F$$

which satisfies estimate

$$\|\varphi\|_{1,s}\le\frac{1}{\lambda-\alpha_1\vartheta_0}\|\mathcal F\|_{1,s} \tag{5.36}$$

with α_1 independent of s for any $1<s\le t_0$. We estimate

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^n} r_\varepsilon\psi_R\,\varphi\,dx\Big| \le c(R)\|r_\varepsilon\|_{0,t_0',B_{\chi(R)}}\|\varphi\|_{0,t_0,B_{\chi(R)}}\\
&\Big|\int_{\mathbb{R}^n}\nabla\psi_R\cdot\tilde z_\varepsilon\varphi\,dx\Big| \le |w|_{\mathcal C^0}\Big[\|z_{1\varepsilon}\|_{0,q}\|\nabla\psi_R\,\varphi\|_{0,q'} + \|z_{2\varepsilon}\|_{0,p}\|\nabla\psi_R\,\varphi\|_{0,p'}\Big]\\
&\le |w|_{\mathcal C^0}\Big(\|z_1\|_{0,q}+\|z_2\|_{0,p}+\delta'\Big)\Big[\|\nabla\psi_R\,\varphi\|_{0,q',\widetilde\Omega_R}+\|\nabla\psi_R\,\varphi\|_{0,p',\widetilde\Omega_R}\Big]
\end{aligned}$$

for ε sufficiently small and certain $\delta'>0$ (for definition of $\chi(R)$ and $\widetilde\Omega_R$ see (2.5)). The r.h.s. of the first inequality tends to 0 (for R fixed) as $\varepsilon\to0$ due to (5.33). The r.h.s. of the second inequality tends to 0 by (2.6) and (5.36), as $R\to+\infty$. We therefore have, in virtue of (5.34),

$$\lim_{R\to+\infty}\Big(\lim_{\varepsilon\to0}\Big|\int_{\mathbb{R}^n}\tilde z_\varepsilon\psi_R\mathcal F\,dx\Big|\Big)\le\lim_{R\to+\infty}\Big|\int\tilde z\psi_R\,dx\Big| = \Big|\int_{\mathbb{R}^n}\tilde z\mathcal F\,dx\Big| = 0$$

for every $\mathcal F\in\mathcal C_0^\infty(\Omega)$. This yields $\tilde z=0$. The other is obvious. □

5.4 Existence of weak solutions in L^q-spaces

A function $z \in L^q(\Omega)$ is called a weak solution of problem (1.1) if and only if it satisfies integral identity

$$\int_\Omega z(\lambda\varphi - w \cdot \nabla\varphi + (a - \operatorname{div} w)\varphi)\, dx = \int_\Omega f\varphi\, dx \tag{5.37}$$

for any $\varphi \in \mathcal{C}_0^\infty(\overline{\Omega})$. It is seen that this solution is, in fact, strong. Indeed, if $z \in L^q(\Omega)$, then the distribution $w \cdot \nabla z \in L^q(\Omega)$ as a consequence of identity (1.1), and thus (1.1) holds a.e. in Ω. We have the following statement

Lemma 5.1. *Let $1 < q < \infty$, $\Omega \in \mathcal{B}^{(1)}$ and*

$$a \in \mathcal{C}^0(\overline{\Omega}),\ w \in \mathcal{C}^1(\overline{\Omega}),\ w \cdot \nu\,|_{\partial\Omega} = 0,\ f \in L^q(\Omega). \tag{5.38}$$

Then there exists a positive constant $\alpha_0 > 0$ (see Remark 2.1) such that we have: Let $z \in L^q(\Omega)$ be a weak solution of problem (1.1), then

$$\lambda\|z\|_{0,q} \le \|f\|_{0,q} + \alpha_0\vartheta_{10}\|z\|_{0,q}. \tag{5.39}$$

Recall that (see (2.11))

$$\vartheta_{10} = |a|_{\mathcal{C}^0} + |w|_{\mathcal{C}^1}. \tag{5.40}$$

Proof: is similar as that one of Lemma 3.1. We want to derive estimate (3.11). Since $z \in L^q(\Omega)$, also $w \cdot \nabla z \in L^q(\Omega)$ and equation (1.1) holds a.e. in Ω. We multiply it by $|z|^{q-2}z$, integrate over Ω. For estimating term $\int_\Omega w \cdot \nabla z |z|^{q-2} z\, dx$, we use the reasoning (3.19)–(3.21) with $y = z$. The rest of the proof is obvious. $\square$

Having Lemma 5.1, we can formulate the following theorem, which gives a statement similar to Theorem 5.1, in the case of only L^q-summable r.h.s.

Theorem 5.7. (a) *Let $1 < q < \infty$, $\Omega \in \mathcal{B}^{(1)}$ and*

$$a \in \mathcal{C}^0(\overline{\Omega}),\ w \in \mathcal{C}^1(\overline{\Omega}),\ w \cdot \nu\,|_{\partial\Omega} = 0,\ f \in L^q(\Omega). \tag{5.41}$$

Then there exists a positive constant $\alpha_0 > 0$[1] *(see Remark 2.1) such that we have: If*

$$\alpha_1\vartheta_{10} < \lambda, \tag{5.42}$$

then there exists just one solution $z \in L^q(\Omega)$ of problem (1.1) satisfying estimate

$$\|z\|_{0,q} \le \frac{1}{\lambda - \alpha_1\vartheta_{10}}\|f\|_{0,q}. \tag{5.43}$$

[1] The constant α_1 in Theorem 8.1 is, in fact, independent of q, see Remark 3.1.

(b) *Let* a, w *satisfy* (5.41) *and let* g *be a weight* (5.18). *Suppose moreover that* $w \cdot \nabla \ln g \in \mathcal{C}^0(\overline{\Omega})$. *Then the statement of the theorem holds true if we replace* $L^q(\Omega)$ *by* $L^q_{(g)}(\Omega) = W^{0,q}_{(g)}(\Omega)$, $\|\cdot\|_{k,q}$ *by* $\| \ \|_{(g),k,q}$ *(see* (5.19)–(5.20)*) and* ϑ_{10} *by* $\vartheta_{10(g)} = \vartheta_{10} + |w \cdot \frac{\nabla g}{g}|_{\mathcal{C}^0}$.

PROOF: We perform the proof only for $\Omega = \mathbb{R}^n$; the general case $\Omega \in \mathcal{B}^{(1)}$ can be treated standardly by the extension method.

Firstly, consider equation

$$\lambda z_{(\varepsilon)} + w \cdot \nabla z_{(\varepsilon)} + a_\varepsilon z_{(\varepsilon)} = f_\varepsilon, \quad \varepsilon > 0 \tag{5.44}$$

with a_ε, f_ε being mollified a and f, respectively (cf. (2.8)) and $z_{(\varepsilon)}$ being the unknown function. Let $\overline{\alpha}_0 = \max(\alpha_0, \alpha_1)$, where α_0, α_1 are defined in Lemma 3.1 and Theorem 5.1, respectively. If

$$\overline{\alpha}_0 \vartheta_{10\varepsilon} = \overline{\alpha}_0(|a_\varepsilon|_{\mathcal{C}^1} + |w|_{\mathcal{C}^1}) < \lambda,$$

then, in virtue of Theorem 5.1, there exists just one solution $z_{(\varepsilon)} \in W^{1,q}(\Omega)$, which satisfies estimate

$$\|z_{(\varepsilon)}\|_{0,q} \le \frac{1}{\lambda - \alpha_1 \vartheta_{10\varepsilon}} \|f_\varepsilon\|_{0,q}. \tag{5.45}$$

If

$$\overline{\lambda} > \overline{\alpha}_0\big(|a_\varepsilon|_{\mathcal{C}^1} + |w|_{\mathcal{C}^1}\big) > \overline{\alpha}_0\big(|a_\varepsilon|_{\mathcal{C}^0} + |w|_{\mathcal{C}^1}\big),$$

then for any $\xi \in L^q(\Omega)$, there exists a (unique) solution $\overline{z}_{(\varepsilon)} \in W^{1,q}(\Omega)$ of problem

$$\overline{\lambda}\overline{z}_{(\varepsilon)} + w \cdot \nabla \overline{z}_{(\varepsilon)} + a_\varepsilon \overline{z}_{(\varepsilon)} = f_\varepsilon + (\overline{\lambda} - \lambda)\xi. \tag{5.46}$$

It satisfies estimate

$$(\overline{\lambda} - \overline{\alpha}_o \vartheta_{10\varepsilon})\|\overline{z}_{(\varepsilon)}\|_{0,q} \le \|f_\varepsilon\|_{0,q} + (\overline{\lambda} - \lambda)\|\xi\|_{0,q}. \tag{5.47}$$

Define $\overline{z} \in L^q(\Omega)$ as a limit $\lim_{\varepsilon \to 0} \overline{z}_{(\varepsilon)}$. Then $\overline{z}$ satisfies equation

$$\overline{\lambda}\overline{z} + w \cdot \nabla \overline{z} + a\overline{z} = f + (\overline{\lambda} - \lambda)\xi \tag{5.48}$$

and estimate

$$(\overline{\lambda} - \overline{\alpha}_0 \vartheta_{10\varepsilon})\|\overline{z}\|_{0,q} \le \|f\|_{0,1} + (\overline{\lambda} - \lambda)\|\xi\|_{0,q}. \tag{5.49}$$

The operator

$$\mathcal{T} : L^q(\Omega) \to L^q(\Omega)$$

which maps ξ onto $\overline{z}$ is, in virtue of (5.49), a contraction in $L^q(\Omega)$. It possesses therefore a (unique) fixed point $z \in L^q(\Omega)$, which obviously satisfies the integral identity (5.37) and estimate (5.43). The proof is thus complete. □

5.5 About the decay of continuous solutions

In this section we prove that continuous solutions of equation (1.1) conserve the decay of r.h.s., whatever is the size of the coefficients w, a (the only condition is that w, a have to be small at infinity).

Theorem 5.8. *Let $\Omega \in \mathcal{B}^{(1)}$ be an unbounded domain and let $g \in \mathcal{C}^1(\Omega)$ be defined as in* (5.18). *Suppose that*

$$\operatorname{div} w \in \mathcal{C}^0(\overline{\Omega}), \;\; a \in \mathcal{C}^0(\overline{\Omega}).$$

Let $1 < p < \infty$. Then there exists constant α_0 (see Remark 2.1*) such that if*

$$\alpha_0\big(|w \cdot \nabla g/g\,|_{\mathcal{C}^0,\Omega^R} + \frac{1}{R}|w|_{\mathcal{C}^0,\Omega^R} + |\operatorname{div} w|_{\mathcal{C}^0,\Omega^R} + |a|_{\mathcal{C}^0,\Omega^R}\big) < \lambda \tag{5.50}$$

for all $R > R_0 > 0$, then we have: If $z \in \mathcal{C}^0(\Omega)$ is a solution of problem (1.1) *with*

(a) $gf \in L^p(\Omega)$,

(b) $gf \in L^\infty(\Omega) \cap L^r(\Omega)\ \forall r,\ 1 < r_0 < r < \infty$;

then

(5.51)
(a) $gf \in L^p(\Omega)$,
(b) $gf \in L^\infty(\Omega)$.

Remark 5.3. Condition (5.50) is automatically satisfied, e.g. if $w \cdot \nabla g/g = o(1)$, $\operatorname{div} w, a = o(1)$.

Proof: For $0 < R < \mathcal{R}$, take $\widetilde{\psi}_R(x) = 1 - \psi(x/R)$ and $\psi_{\mathcal{R}}(x) = \psi(x/\mathcal{R})$, see (2.4); here $\psi \in \mathcal{C}_0^\infty(\Omega)$ is taken in such a way that

$$\begin{aligned}
&\psi(x) = 1 - \exp(-1/(|x|-1)^2) \quad (1 \le |x| \le 3/2),\\
&\psi(x) = 0 \quad (|x| \ge 2),\\
&\psi(x) = 1 \quad (|x| \le 1).
\end{aligned}$$

Put $g_{R\mathcal{R}} = g\widetilde{\psi}_R\psi_{\mathcal{R}}$ and multiply (1.1) by $g_{R\mathcal{R}}$

$$zg_{R\mathcal{R}} = -w\widetilde{\psi}_{R/2}\psi_{2\mathcal{R}} \cdot \nabla(g_{R\mathcal{R}}z) - [a - (w \cdot \nabla g_{R\mathcal{R}}/g_{R\mathcal{R}})]g_{R\mathcal{R}}z = g_{R\mathcal{R}}f. \tag{5.52}$$

Notice that $\nabla g_{R\mathcal{R}}/g_{R\mathcal{R}}$ is well defined also outside $\Omega^R \cap \Omega_{2\mathcal{R}}$ (due to the choice of ψ, it can be continuously extended by 0 to $\mathbb{R}^n$). Equation (5.52) can be regarded as a transport equation in $\Omega^{R/2} \cap \Omega_{2\mathcal{R}}$, for unknown function σ $(= g_{R\mathcal{R}}z)$. We realize that under the hypothesis of Theorem 5.8, assumptions of Theorem 5.7 (see also Remark 8.1) are satisfied (for R sufficiently great); we thus get existence of a solution $\sigma \in L^p(\Omega^{R/2} \cap \Omega_{2\mathcal{R}})$ (provided $g_{R\mathcal{R}} f \in L^p(\Omega^{R/2} \cap \Omega_{2\mathcal{R}})$, which, due to uniqueness, is equal to $g_{R\mathcal{R}} z$. Multiplying (5.52) by $|g_{R\mathcal{R}} z|^{p-1} g_{R\mathcal{R}} z$ and integrating over $\Omega^{R/2} \cap \Omega_{2\mathcal{R}}$, we get (after some calculations — see proof of Lemmas 3.1–3.3)

$$\|g_{R\mathcal{R}} z\|_{0,p,\Omega^{R/2}\cap\Omega_{2\mathcal{R}}} \le c\big[\vartheta_{00R\mathcal{R}} \|g_{R\mathcal{R}} z\|_{0,p,\Omega^{R/2}\cap\Omega_{2\mathcal{R}}} + \|g_{R\mathcal{R}} f\|_{0,p,\Omega^{R/2}\cap\Omega_{2\mathcal{R}}}\big]$$

where

$$\begin{aligned}\vartheta_{00R\mathcal{R}} = {} & |\nabla(w\widetilde{\psi}_R \psi_{2\mathcal{R}})|_{\mathcal{C}^0,\Omega^{R/2}\cap\Omega_{2\mathcal{R}}} \\ & + |a|_{\mathcal{C}^0,\Omega^{R/2}\cap\Omega_{2\mathcal{R}}} + |w \cdot \nabla g_{R\mathcal{R}} / g_{R\mathcal{R}}|_{\mathcal{C}^0,\Omega^{R/2}\cap\Omega_{2\mathcal{R}}}\end{aligned}$$

and c is independent of p, R, $\mathcal{R}$. Passing to the limit $\mathcal{R} \to \infty$, we get $(5.51)_1$; the limit $p \to \infty$ in the resulting estimate yields $(5.51)_2$. The proof is thus complete. □

6. Existence of solution in homogenous Sobolev spaces and their duals

Homogenous Sobolev spaces play essential role in studying namely elliptic problems in Ω exterior $\Omega = \mathbb{R}^n, \mathbb{R}^n_+$ (and also in various domains unbounded in all directions). For applications to compressible fluids, it sometimes seems useful to have existence theorems for transport equation in $\widehat{H}_0^{1,q}(\Omega)$ (and its duals $\widehat{H}^{-1,q'}(\Omega)$). For such situation see e.g. B. da Veiga [BV1] and Novotný, Padula [NP1]. In the first part of this section we therefore investigate existence and regularity in $\widehat{H}_0^{1,q}(\Omega)$, respectively $\widehat{H}_\infty^{1,q}(\Omega)$ spaces (see subsections 6.1 and 6.2, Theorems 6.1, 6.2, and 6.2′) and their duals $\widehat{H}^{-1,q}(\Omega)$, $W^{-1,q}(\Omega)$ (see subsection 6.3, Theorem 6.3). In the last part of this section (see subsection 6.4, Theorems 6.4, 6.5) we prove certain regularity of solutions connected with homogenous Sobolev spaces.

6.1 Existence in homogenous Sobolev spaces

Let Ω be an exterior domain to a compact region Ω_c in $\mathbb{R}^n$ (suppose without loss of generality $B_1 \in \Omega_c$). Define the spaces

$$\widehat{H}_\infty^{1,q}(\Omega) = \overline{\mathcal{C}_0^\infty(\overline{\Omega})}^{\|\nabla\cdot\|_{0,q}}, \quad \widehat{H}_0^{1,q}(\Omega) = \overline{\mathcal{C}_0^\infty(\Omega)}^{\|\nabla\cdot\|_{0,q}} \quad (1 < q < +\infty)$$

where the superposed bar with the norm denotes completion with respect to the corresponding norm. These are Banach spaces with norm

$$|\cdot|_{1,q} = \|\nabla\cdot\|_{0,q}.$$

The dual space to $\widehat{H}_0^{1,q'}(\Omega)$ is denoted by $\widehat{H}^{-1,q'}(\Omega)$ and equipped with usual duality norm $|\cdot|_{-1,q}$. Recall fundamental properties of spaces $\widehat{H}_\infty^{1,q}(\Omega)$, $\widehat{H}_0^{1,q}(\Omega)$, see Simader [S], Galdi, Simader [GS], Simader, Sohr [SiSo1], [SiSo2].

(i) $\widehat{H}_\infty^{1,q}(\Omega) = \{u : u \in L^q_{\mathrm{loc}}(\Omega),\ \nabla u \in L^q(\Omega)\}\,|_{\mathbb{R}^1}$
$(n \le q \le +\infty)$ where $|_{\mathbb{R}^1}$ denotes factorization with respect to the addition of a constant.

(ii) $\widehat{H}^{1,q}_\infty(\Omega) = \{u : u \in L^{nq/(n-q)}(\Omega),\ \nabla u \in L^q(\Omega)\}$ $(1 < q < n)$.
For any $u \in \widehat{H}^{1,q}_\infty(\Omega)$, we have $\|u\|_{0,nq/(n-q)} \le c\|\nabla u\|_{0,q}$.

(iii) Let $1 < q < n$ and let $u \in \{u : u \in L^q_{\rm loc}(\Omega),\ \nabla u \in L^q(\Omega)\}$. Then there exists $u_\infty \in \mathbb{R}^1$ such that $u - u_\infty \in \widehat{H}^{1,q}_\infty(\Omega)$. Moreover,

$$\int_{\mathcal{S}_1} |u(R,\omega) - u_\infty|^q \, d\omega \le c\, R^{q-n} \int_{B^R} |\nabla u|^q \, dx$$

(where $\mathcal{S}_1$ is a unit sphere). As an easy consequence of this statement we find

(iv) Let $1 < q < n$, $1 < p < +\infty$ and $u \in \{u : u \in L^q_{\rm loc}(\Omega),\ \nabla u \in L^q(\Omega)\} \cap L^p(\Omega)$. Then $u \in \widehat{H}^{1,q}_\infty(\Omega)$.

Further we have

(v) $\widehat{H}^{1,q}_\infty(\Omega) = \{u :$ *there exists a sequence* $u_m \in \mathcal{C}^\infty_0(\overline{\Omega})$
such that $u_m \to u$ *in* $L^q_{\rm loc}(\Omega)$ *and* $\nabla u_m \to \nabla u$ *in* $L^q(\Omega)\}$.

(vi) $\widehat{H}^{1,q}_0(\Omega) = \{u : u \in L^q_{\rm loc}(\Omega),\ \nabla u \in L^q(\Omega),\ u\,|_{\partial\Omega} = 0\}$ $(n \le q < +\infty)$, $\Omega \ne \mathbb{R}^n$;
$\widehat{H}^{1,q}_0(\Omega) = \{u : u \in L^{nq/(n-q)}(\Omega),\ \nabla u \in L^q(\Omega),\ u\,|_{\partial\Omega} = 0\}$ $(1 < q < n)$, $\Omega \ne \mathbb{R}^n$.
If $\Omega = \mathbb{R}^n$, then $\widehat{H}^{1,q}_0(\mathbb{R}^n) = \widehat{H}^{1,q}_\infty(\mathbb{R}^n)$.

(vii) $\widehat{H}^{1,q}_0(\Omega) = \{u :$ *there exists a sequence* $u_m \in \mathcal{C}^\infty_0(\Omega)$
such that $u_m \to u$ *in* $L^q_{\rm loc}(\Omega)$ *and* $\nabla u_m \to \nabla u$ *in* $L^q(\Omega)\}$.

Proofs of (i)–(iii), (v)–(vii) are in [S], [GS] and [SiSo1], [SiSo2]. We prove only (iv).

Proof of (iv): Let $q \le p < +\infty$. Since $u \in L^p(\Omega)$, there exists a sequence R_i, $i \to +\infty$, such that $R_i^2 \int_{\mathcal{S}_1} |u(R_i,\omega)|^p \, d\omega \to 0$, where $\mathcal{S}_1$ is a unit sphere with infinitesimal element $d\omega$ and $u(R_i,\omega)$ is written in spherical coordinates. This implies $\int_{\mathcal{S}_1} |u(R_i,\omega)|^q \, d\omega \to 0$ (by Hölder inequality) and necessarily $u_\infty = 0$ (see (iii)).

Let $1 < p < q$. Then there exists $u_\infty \in \mathbb{R}^1$ such that $u - u_\infty \in \widehat{H}^{1,q}_\infty(\Omega) \subset L^{\frac{nq}{n-q}}(\Omega)$ (see (iii)). Therefore, by similar arguments as before, $\int_{\mathcal{S}_1} |u(R_i,\omega) - u_\infty|^p \, d\omega \to 0$, and necessarily $u_\infty = 0$. □

First we prove

Theorem 6.1. *Let* Ω *be an exterior domain with* $\partial\Omega \in \mathcal{C}^1$ *or* $\Omega = \mathbb{R}^n_+$ *or* $\Omega = \mathbb{R}^n$ *and*

$$w \in \mathcal{C}^1(\overline{\Omega}),\ w \cdot \nu\,|_{\partial\Omega} = 0,\ a \in \mathcal{C}^0(\overline{\Omega}),\ f \in \widehat{H}^{1,q}_\infty(\Omega) \quad (1 < q < +\infty). \tag{6.1$'$}$$

Then there exists a constant $\alpha_1 > 0$ (*see Remark* 2.1) *such that we have:*
If

$$a = 0 \tag{6.2$'$)$_1$ $\qquad$ (}$$

and

$$\alpha_1 \vartheta_1' < \lambda$$

or if

$$1 < q < n, \ \nabla a \in L^n(\Omega) \tag{6.2$'$)$_2$ $\qquad$ (}$$

and

$$\alpha_1 \vartheta_2' < \lambda,$$

then there exists just one solution of problem (1.1) $z \in \widehat{H}^{1,q}_\infty(\Omega)$ *satisfying estimate*

$$|z|_{1,q} \le \frac{1}{\lambda - \alpha_1 \vartheta_i'} |f|_{1,q} \tag{6.3$'$}$$

(*where* $i = 1, 2$ *corresponds to* $(6.2')_i$; ϑ_i' *are defined in* (2.11)).

Proof: We prove Theorem 6.1 with assumptions $(6.2')_1$. The proof with assumptions $(6.2')_2$ can be established in the same way, and therefore is left to the reader. Let $f \in \widehat{H}^{1,q}_\infty(\Omega)$. Then there exists a sequence $\{f_r\}_{r=1}^\infty$, $f_r \in \mathcal{C}_0^\infty(\overline{\Omega})$ such that

$$f_r \to f \ \text{ in } \ L^q_{\text{loc}}(\Omega), \ \nabla f_r \to \nabla f \ \text{ in } \ L^q(\Omega),$$
$$f_r \to f \ \text{ in } \ L^{\frac{nq}{n-q}}(\Omega) \quad (\text{if } 1 < q < n)$$

(see (v)). By Theorem 5.2, there exists a solution z_r of problem (1.1) with f_r (instead of f) such that

$$z_r \in W^{1,q}(\Omega), \ z_r \in L^{\frac{nq}{n-q}}(\Omega) \ \ (\text{if } 1 < q < n)$$

which satisfies uniform estimates

$$\|\nabla z_r\|_{0,q,\Omega} \le \frac{1}{\lambda - \alpha_1 \vartheta_1'} |f|_{1,q,\Omega},$$
$$\|\nabla (z_r - z_m)\|_{0,q,\Omega} \le \frac{1}{\lambda - \alpha_1 \vartheta_1'} |f_r - f_m|_{1,q,\Omega}.$$

Multiplying (1.1) (with f_r) by $|z_r|^{q-1} z_r$ and integrating over Ω_R ($\Omega_R = \Omega \cap B_R$), with arbitrary R, we get

$$\|z_r\|_{0,q,\Omega_R} \le \frac{1}{\lambda - \alpha_1 \vartheta_1'} \Big(\|f_r\|_{0,q,\Omega_R} + c_1 \vartheta_1' \|\nabla z_r\|_{0,q,\Omega_R} \Big)$$

and similarly for the difference

$$\|z_r - z_m\|_{0,q,\Omega_R} \le \frac{1}{\lambda - \alpha_1 \vartheta_1'} \Big(\|f_r - f_m\|_{0,q,\Omega_R} + c_1 \vartheta_1' \|\nabla z_r - \nabla z_m\|_{0,q,\Omega_R} \Big).$$

Therefore z_r is a Cauchy sequence in $\widehat{H}^{1,q}_\infty(\Omega)$ (see (v)), for ϑ_1' sufficiently small. The other is obvious. □

The proof of the following theorem is an easy consequence of Theorem 6.1, and therefore is left to the reader.

Theorem 6.2. *Let* $1 < q < +\infty$, Ω *be exterior domain with* $\partial\Omega \in \mathcal{C}^1$ *or* $\Omega = \mathbb{R}^n_+$ *or* $\Omega = \mathbb{R}^n$ *and*

$$w \in \mathcal{C}^1(\overline{\Omega}),\ w \cdot \nu \,|_{\partial\Omega} = 0,\ a \in \mathcal{C}^0(\overline{\Omega}),\ f \in \widehat{H}^{1,q}_0(\Omega). \tag{6.1}$$

Then there exists a constant $\alpha_1 > 0$ (*see Remark* 2.1) *such that we have:*
If

$$a = 0$$

and

$$\alpha_1 \vartheta_1' < \lambda \tag{6.2_1}$$

or if

$$1 < q < n,\ \nabla a \in L^n(\Omega) \tag{6.2_2}$$

and

$$\alpha_1 \vartheta_2' < \lambda,$$

then there exists just one solution of problem (1.1) $z \in \widehat{H}^{1,q}_0(\Omega)$ *satisfying estimate*

$$|z|_{1,q} \le \frac{1}{\lambda - \alpha_1 \vartheta_i'} |f|_{1,q} \tag{6.3}$$

(*where* $i = 1, 2$ *corresponds to* $(6.2)_i$ *and* ϑ_i' *are defined in* (2.11)).

□ Define for Ω exterior domain

$$\mathcal{H}^{k,q}_\infty(\Omega) = \overline{\mathcal{C}^\infty_0(\overline{\Omega})}^{\|\nabla.\|_{k-1,q}} \quad (1 < q < +\infty,\ k = 1, 2 \dots)$$

(completion in $\|\nabla.\|_{k-1,q}$-norm). It is useful to have a more regular version of Theorem 6.2:

Theorem 6.2′. *Let Ω be an exterior domain with $\partial\Omega \in \mathcal{C}^k$ or $\Omega = \mathbb{R}^n_+$ (or $\Omega = \mathbb{R}^n$) and*

$$(6.1'') \qquad w \in \mathcal{C}^k(\overline{\Omega}),\ w \cdot \nu\,|_{\partial\Omega} = 0,\ a \in \mathcal{C}^{k-1}(\overline{\Omega}),\ f \in \mathcal{H}^{k,q}_\infty(\Omega).$$

Then there exists a constant $\alpha_1 > 0$ (see Remark 2.1) such that we have:
If

$$(6.2'')_1 \qquad a = 0$$

and

$$\alpha_1 \vartheta_1' < \lambda$$

or if

$$(6.2'')_2 \qquad 1 < q < n,\ \nabla a \in W^{k-1,n}(\Omega)$$

and

$$\alpha_1 \vartheta_2' < \lambda,$$

then there exists just one solution of problem (1.1) $z \in \mathcal{H}^{k,q}_\infty(\Omega)$ satisfying estimate

$$(6.3'') \qquad \|\nabla z\|_{k-1,q} \le \frac{1}{\lambda - \alpha_1 \vartheta_i'} \|\nabla f\|_{k-1,q}$$

(where $i = 1, 2$ corresponds to $(6.2'')_i$, ϑ_i' are defined in (2.11)).

6.2 Regularity of solutions

With Theorems 6.2′ and 5.3 at hand, we can prove (similarly as Theorem 5.6) a theorem about the regularity. It finds application, in particular, in investigating two dimensional exterior compressible flows, see [GNP].

Theorem 6.3. *Let $k, m = 1, 2, \dots$, $1 < q < +\infty$, $1 < p < n$ and Ω be an exterior domain ($\Omega \in \mathcal{C}^{\max(k,m)}$) or $\Omega = \mathbb{R}^n$. Let*

$$(6.4) \quad w \in \mathcal{C}^{\max(k,m)}(\overline{\Omega}),\ w \cdot \nu\,|_{\partial\Omega} = 0,\ a = \operatorname{div} w,\ f \in W^{k,q}(\Omega) \cap \mathcal{H}^{m,p}_\infty(\Omega).$$

Then there exists a constant $\alpha_1 > 0$ such that:
If

$$(6.5)_1 \qquad k \ge m,\ kq > n,\ \nabla^k a \in L^q(\Omega) \cap W^{m-1,n}(\Omega)$$

and

$$\alpha_1 \eta_1' < \lambda,\ \eta_1' = \vartheta_1^{(k,q)} + \vartheta_2^{'(m,p)},$$

or if

$$(6.5)_2 \qquad k < m,\ \nabla a \in W^{m-1,n}(\Omega)$$

and

$$\alpha_1\eta_2' < \lambda,\ \eta_2' = \vartheta_0^{(k,q)} + \vartheta_2'^{(m,p)},$$

or if

$$(6.5)_3 \qquad k \geq m,\ 1 < q < n,\ \nabla^k a \in L^n(\Omega),\ \nabla a \in W^{m-1,n}(\Omega)$$

and

$$\alpha_1\eta_3' < \lambda,\ \eta_3' = \vartheta_1^{(k,q)} + \vartheta_2'^{(m,p)},$$

then there exists just one solution of problem (1.2) $z \in \mathcal{H}_\infty^{m,p}(\Omega) \cap W^{k,q}(\Omega)$ *satisfying estimate*

$$(6.6) \qquad \|\nabla z\|_{m-1,p} + \|z\|_{k,q} \leq \frac{1}{\lambda - \alpha_1\eta_i'}\Big(\|\nabla f\|_{m-1,p} + \|f\|_{k,q}\Big)$$

(*where* $i = 1,2,3$ *corresponds to* $(6.5)_i$ *and* ϑ_j, ϑ_j' *are defined in* (2.11)).

6.3 Existence of solutions in dual spaces to Sobolev spaces and to homogenous Sobolev spaces

For Ω bounded ($\partial\Omega \in \mathcal{C}^2$), the existence of weak solutions was established by B. da Veiga [BV2]. Here we follow step by step his duality argument and use essentially the solvability of the adjoint problem, which was proved in Theorem 5.2 (in order to prove existence of weak solutions in $W^{-1,q}(\Omega)$, $\Omega \in \mathcal{B}^{(k)}$) or in Theorem 6.2 (in order to prove existence of weak solutions in $\widehat{H}^{-1,q}(\Omega)$, Ω exterior domain or $\Omega = \mathbb{R}^n, \mathbb{R}^n_+$).

As far as applications are concerned, the most important case is that one with $a = \operatorname{div} w$. This corresponds to the transport equation (1.2).

It is necessary to recall the definition of weak solution (see e.g. B. da Veiga [BV1]).

Definition 6.1.

(a) Let Ω be exterior domain ($\partial\Omega \in \mathcal{C}^1$) in $\mathbb{R}^n$ or $\Omega = \mathbb{R}^n$ or $\Omega = \mathbb{R}^n_+$ and $f \in \widehat{H}^{-1,q}(\Omega)$, $1 < q < +\infty$. Then $z \in \widehat{H}_0^{-1,q}(\Omega)$ is a weak solution of problem (1.1) if and only if

$$\langle z, \lambda\varphi + w\cdot\nabla\varphi + (a - \operatorname{div} w)\varphi\rangle = \langle f, \varphi\rangle$$

for every $\varphi \in \mathcal{D}_H^{q'}$ where

$$\mathcal{D}_H^{q'} = \Big\{\varphi : \varphi \in \widehat{H}_0^{1,q'}(\Omega),\ w\cdot\nabla\varphi \in \widehat{H}_0^{1,q'}(\Omega)\Big\}.$$

(Here $\langle\cdot,\cdot\rangle$ denotes duality in $\widehat{H}_0^{1,q'}(\Omega)$, $\frac{1}{q} + \frac{1}{q'} = 1$.)

(b) Let $\Omega \in \mathcal{B}^{(1)}$, $f \in W^{-1,q}(\Omega)$, $1 < q < +\infty$. Then $z \in W^{-1,q}(\Omega)$ is a weak solution of problem (1.1) if and only if

$$\langle z, \lambda\varphi + w \cdot \nabla\varphi + (a - \operatorname{div} w)\varphi \rangle = \langle f, \varphi \rangle$$

for every $\varphi \in \mathcal{D}_w^{q'}$ where

$$\mathcal{D}_w^{q'} = \Big\{ \varphi : \varphi \in W_0^{1,q'}(\Omega),\ w \cdot \nabla\varphi \in W_0^{1,q'}(\Omega) \Big\}.$$

(Here $\langle \cdot, \cdot \rangle$ denotes duality in $W_0^{1,q}(\Omega)$ and $\frac{1}{q} + \frac{1}{q'} = 1$.)

Theorem 6.4. *Let $\Omega \in \mathcal{B}^{(1)}$, $1 < q < +\infty$ and*

$$w \in \mathcal{C}^1(\overline{\Omega}),\ w \cdot \nu \,|_{\partial\Omega} = 0,\ a \in \mathcal{C}^0(\overline{\Omega}),\ f \in W^{-1,q}(\Omega). \tag{6.7}$$

Then there exists a constant $\alpha_1 > 0$ (see Remark 2.1) such that we have:
If

$$1 < q < \frac{n}{n-1}\,,\ \nabla(a - \operatorname{div} w) \in L^q(\Omega) \tag*{$(6.8)_1$}$$

and

$$\alpha_1 \vartheta_8 < \lambda$$

or if

$$\frac{n}{n-1} < q < +\infty,\ \nabla(a - \operatorname{div} w) \in L^n(\Omega) \tag*{$(6.8)_2$}$$

and

$$\alpha_1 \vartheta_9 < \lambda$$

or if

$$1 < q < +\infty,\ a = \operatorname{div} w \tag*{$(6.8)_3$}$$

and

$$\alpha_1 \vartheta_{10} < \lambda,$$

then there exists just one weak solution $z \in W^{-1,q}(\Omega)$ of problem (1.1) satisfying estimate

$$\|z\|_{-1,q} \le \frac{1}{1 - \alpha_1 \vartheta_{7+i}} \|f\|_{-1,q}.$$

(Here $i = 1, 2, 3$ corresponds to $(6.8)_i$ and ϑ_8–ϑ_{10} are defined in (2.11).)

Theorem 6.5. *Let $\Omega = \mathbb{R}^n, \mathbb{R}^n_+$ or Ω be an exterior domain, $\partial\Omega \in \mathcal{C}^1$ and*

$$w \in \mathcal{C}^1(\overline{\Omega}),\ w \cdot \nu\,|_{\,\partial\Omega} = 0,\ a \in \mathcal{C}^0(\overline{\Omega}),\ f \in \widehat{H}^{-1,q}(\Omega) \quad (1 < q < +\infty). \tag{6.9}$$

Then there exists a constant $\alpha_1 > 0$ (see Remark 2.1) such that we have:
If

$$\frac{n}{n-1} < q < +\infty,\ \nabla(a - \operatorname{div} w) \in L^n(\Omega) \tag{6.10$_1$}$$

and

$$\alpha_1 \vartheta_9 < \lambda$$

or if

$$1 < q < +\infty,\ a = \operatorname{div} w \tag{6.10$_2$}$$

and

$$\alpha_1 \vartheta_{10} < \lambda,$$

then there exists just one weak solution of problem (1.1) $z \in \widehat{H}^{-1,q}(\Omega)$ *satisfying estimate*

$$|z|_{-1,q} \le \frac{1}{1 - \alpha_1 \vartheta_{8+i}} |f|_{-1,q} \tag{6.11}$$

($i = 1, 2$ *refers to* (6.10)$_i$ *and* ϑ_j *are defined in* (2.11)).

Proof: We prove only Theorem 6.5 with assumption (6.10)$_2$. For the other cases, it is easy to repeat the established argument, which is left to the kind reader. In any case, we only follow step by step B. da Veiga's arguments, proposed in [BV2]. Due to Theorem 6.1 there exists a bounded linear map $B : \widehat{H}^{1,q'}_0(\Omega) \to \widehat{H}^{1,q'}_0(\Omega)$ (i.e. $B \in \mathcal{L}(\widehat{H}^{1,q}_0(\Omega))$) such that

$$\varphi = B\xi$$

is a unique solution of the adjoint problem

$$\lambda\varphi - w \cdot \nabla\varphi + (a - \operatorname{div} w)\varphi = \xi$$

with $\xi \in \widehat{H}^{1,q'}_0(\Omega)$. Due to the uniqueness, B has inverse (say $B^{-1} = A$). Its domain of definitions is

$$\mathcal{D}(A) = \Big\{\varphi : \varphi \in \widehat{H}^{1,q'}_0(\Omega),\ w \cdot \nabla\varphi \in \widehat{H}^{1,q'}_0(\Omega)\Big\};$$

hence it is dense in $\widehat{H}^{1,q'}_0(\Omega)$. The range of A is

$$\mathcal{R}(A) = \widehat{H}^{1,q'}_0(\Omega)$$

and its representation

$$A\varphi = \lambda\varphi - w \cdot \nabla\varphi + (a - \operatorname{div} w)\varphi,\ \ \varphi \in \mathcal{D}(A).$$

Since A is a linear transformation with a bounded inverse, with domain of definition $\mathcal{D}(A)$ dense in $\widehat{H}_0^{1,q}(\Omega)$ and with range $\mathcal{R}(A) = \widehat{H}_0^{1,q'}(\Omega)$, we deduce that there exists an adjoint A^* with dense definition domain $\mathcal{D}(A^*)$ such that

$$(A^*)^{-1} = (A^{-1})^* = B^*$$

(see Hille [Hi, Example 10.4]). Since $B \in \mathcal{L}(\widehat{H}_0^{1,q'}(\Omega))$, we have $B^* \in \mathcal{L}(\widehat{H}^{-1,q}(\Omega))$ (the Banach space of bounded linear operators from $\widehat{H}^{-1,q}(\Omega)$ to $\widehat{H}^{-1,q}(\Omega)$) and therefore

$$(A^*)^{-1} \in \mathcal{L}(\widehat{H}^{-1,q}(\Omega))$$

and

$$\|A^{*-1}\|_{\mathcal{L}(\widehat{H}^{-1,q'})} = \|A^{-1}\|_{\mathcal{L}(\widehat{H}_0^{1,q})}.$$

Let $u = (A^*)^{-1}f$.
Then $\langle A\varphi, u\rangle = \langle \varphi, A^*u\rangle\, \forall\varphi \in \mathcal{D}(A)$, according to the definition of A. Moreover

$$\langle A\varphi, (A^*)^{-1}f\rangle = \langle \varphi, f\rangle$$

(here $\langle\cdot,\cdot\rangle$ denotes duality in $\widehat{H}_0^{1,q}(\Omega)$), hence u is a weak solution of problem (1.1). Obviously, it is unique and

$$|u|_{-1,q} = |(A^*)^{-1}f|_{-1,q} \le \|B\|_{\mathcal{L}(\widehat{H}^{1,q'}(\Omega))}|f|_{-1,q}.$$

The proof is thus complete. □

7. About one particular regularity of solutions

As it is seen from Theorems 5.1–5.8, 6.1–6.5, the general property of transport equation (provided the coefficients are small and sufficiently smooth) is the conservation of regularity and summability. Here we investigate this property in a very particular situation:

(a) Let $\Omega \in \mathcal{B}^{(k)}$ and $z \in W^{k,q}(\Omega)$ be a solution of problem (1.1); then clearly $\Delta z \in W^{k-2,q}(\Omega)$. Does the corresponding estimate

$$\|\Delta z\|_{k-2,q} \le c\|\Delta f\|_{k-2,q}$$

(eventually with other quadratic terms at r.h.s.) hold?

(b) Let Ω be an exterior domain or $\Omega = \mathbb{R}^n$ or $\Omega = \mathbb{R}^n_+$ and $z \in \widehat{H}_0^{1,q}(\Omega)$ be a solution of problem (1.1). Then, of course, $\Delta z \in \widehat{H}^{-1,q}(\Omega)$. Does the estimate

$$|\Delta z|_{-1,q} \leq c|\Delta f|_{-1,q}$$

(eventually with other quadratic terms at r.h.s.) hold?

These two questions are by no means trivial especially when $k = 1$ and their positive answer has nice applications in the theory of compressible fluids, cf. Novotný, Padula [NP1], Novotný [N1], [N3], Novotný, Penel [NPe], Padula, Pileckas [PP].

The results of this section are not restricted only to Δ; the Laplace operator can be replaced by any second order differential operator of the type $a_{ij}\frac{\partial^2}{\partial x_i \partial x_j}$ ($a_{ij} \in \mathbb{R}^1$). This generalization is left to the kind reader.

Again, we restrict ourselves to the problem (1.2) ($a = \operatorname{div} w$), since it is important in applications. The general problem can be treated in a similar way, but the assumptions are more complicated.

Theorem 7.1.

(a) *Let* $k = 1, 2, \ldots$, $1 < q < +\infty$, $\Omega \in \mathcal{B}^{(k)}$ *and*

$$w \in \mathcal{C}^k(\overline{\Omega}),\ w \cdot \nu\,|_{\partial\Omega} = 0,\ a = \operatorname{div} w,\ f \in W^{k,q}(\Omega) \tag{7.1}$$

and

$$kq > n,\ \nabla^k a \in L^q(\Omega) \tag*{(7.2)$_1$}$$

or

$$1 < q < n,\ \nabla^k a \in L^n(\Omega) \tag*{(7.2)$_2$}$$

(b) *Let* $k = 2, 3, \ldots$, $1 < q < +\infty$, $\Omega \in \mathcal{B}^{(k)}$ *and*

$$w \in \mathcal{C}^{k-1}(\overline{\Omega}),\ w \cdot \nu\,|_{\partial\Omega} = 0,\ a = \operatorname{div} w,\ f \in W^{k,q}(\Omega) \tag{7.1$'$}$$

and

$$1 < q < n,\ \nabla^k w \in L^n(\Omega),\ \nabla^{k-1} a \in L^n(\Omega),\ \nabla^k a \in L^q(\Omega) \tag*{(7.2)$_3$}$$

or

$$1 < q < n,\ \nabla^k w \in L^n(\Omega),\ \nabla^{k-1} a \in W^{1,n}(\Omega) \tag*{(7.2)$_4$}$$

or

$$1 < q < n,\ \nabla^k w \in L^n(\Omega),\ \nabla^{k-1} a \in W^{1,q}(\Omega) \tag*{(7.2)$_5$}$$

or

$$(7.2)_6 \qquad (k-1)q > n,\ \nabla^k w \in L^q(\Omega),\ \nabla^{k-1} a \in W^{1,q}(\Omega)$$

or

$$(7.2)_7 \qquad (k-1)q > n,\ \nabla^k w \in L^q(\Omega),\ \nabla^{k-1} a \in W^{1,n}(\Omega).$$

Let $z \in W^{k,q}(\Omega)$ *be a solution of problem* (1.2). *Then* $\Delta z \in W^{k-2,q}(\Omega)$ *and*

$$(7.3) \qquad \|\Delta z\|_{k-2,q} \le \|\Delta f\|_{k-2,q} + c\vartheta_i \|z\|_{k,q}$$

where ϑ_i *corresponds to* $(7.2)_i$ (*it is defined in* (2.11)) *and* c *is a positive constant* (*see Remark* 2.1).

Theorem 7.2. *Let* Ω *be an exterior domain* ($\partial\Omega \in \mathcal{C}^1$) *or* $\Omega = \mathbb{R}^n$ *or* $\Omega = \mathbb{R}^n_+$ *and*

$$(7.4) \qquad w \in \mathcal{C}^1(\overline{\Omega}),\ w \cdot \nu\,|_{\partial\Omega} = 0,\ a = \operatorname{div} w,\ f \in W^{1,q}(\Omega)$$

and

$$(7.5)_1 \qquad kq > n,\ \nabla^k a \in L^q(\Omega)$$

or

$$(7.5)_2 \qquad 1 < q < n,\ \nabla^k a \in L^n(\Omega).$$

Let $z \in W^{1,q}(\Omega)$ *be a solution of problem* (1.2). *Then*

$$\Delta z \in \widehat{H}^{-1,q}(\Omega)$$

and

$$(7.6) \qquad |\Delta z|_{-1,q} \le |\Delta f|_{-1,q} + c\vartheta_i \|z\|_{1,q}$$

where ϑ_i ($i = 1, 2$) *is defined in* (2.11) *and corresponds to* $(7.5)_i$, *and* $c > 0$ (*see Remark* 2.1).

Proof of Theorems 7.1 and 7.2: We prove only Theorem 7.2 under assumptions (7.4) and $(7.5)_1$. The proofs of Theorem 7.1 and the rest of Theorem 7.2 are similar (even easier) and therefore left to the reader. We closely follow [NP1].

Since w, f can be extended continuously to $\mathbb{R}^n$, it is convenient to consider (1.2) in $\mathbb{R}^n$ (instead of Ω, cf. Remark 5.2). Regularize (1.2) by using mollifier and take Δ, we get

$$\lambda \Delta z_\varepsilon + \Delta \operatorname{div}(wz)_\varepsilon = \Delta f_\varepsilon \tag{7.7}$$

Put, for $\varphi \in \widehat{H}_0^{1,q'}(\mathbb{R}^n)$,

$$A_{(\varepsilon)} = -(\Delta \operatorname{div}(wz)_\varepsilon, \varphi),\;\; B_\varepsilon = (\operatorname{div}(w\Delta z_\varepsilon), \varphi)$$

where (,) is a scalar product in $L^2(\mathbb{R}^n)$. We have

$$\begin{aligned}
A_{(\varepsilon)} &= \big(\nabla[(\operatorname{div} w)z + w\cdot\nabla z]_\varepsilon, \nabla z]_\varepsilon \nabla\varphi\big) \\
&= \big([\nabla \operatorname{div} wz + \operatorname{div} w\nabla z + \nabla w\cdot\nabla z + w\cdot\nabla\nabla z]_\varepsilon, \nabla\varphi\big), \\
B_{(\varepsilon)} &= \big(\operatorname{div}(\operatorname{div} w\nabla z_\varepsilon) + \operatorname{div}(w\cdot\nabla\nabla z_\varepsilon) - \operatorname{div}(\nabla z_\varepsilon\cdot\nabla w), \varphi\big) \\
&= -\big(\operatorname{div} w\nabla z_\varepsilon + w\cdot\nabla\nabla z_\varepsilon - \nabla z_\varepsilon\cdot\nabla w, \nabla\varphi\big).
\end{aligned}$$

Coming back to (7.7), we get

$$\lambda(\Delta z_\varepsilon, \varphi) + (\operatorname{div}(w\Delta z_\varepsilon), \varphi) = (\Delta f_\varepsilon, \varphi) + A_\varepsilon + B_\varepsilon = (\Delta f_\varepsilon, \varphi) + C_{(\varepsilon)} + D_{(\varepsilon)}$$

where

$$\begin{aligned}
C_{(\varepsilon)} &= (w\cdot\nabla\nabla z_\varepsilon - (w\cdot\nabla\nabla z)_\varepsilon, \nabla\varphi) = (c_{(\varepsilon)}, \nabla\varphi) \\
D_{(\varepsilon)} &= ([\nabla z \operatorname{div} w + z\nabla \operatorname{div} w + \nabla w\cdot\nabla z]_\varepsilon - \operatorname{div} w\nabla z_\varepsilon + \nabla z_\varepsilon\cdot\nabla w, \nabla\varphi) \\
&= (d_{(\varepsilon)}, \nabla\varphi).
\end{aligned}$$

Hence

$$(\Delta z_\varepsilon, \lambda\varphi - w\cdot\nabla\varphi) = (\Delta f_\varepsilon, \varphi) + (c_{(\varepsilon)}, \nabla\varphi) + (d_{(\varepsilon)}, \nabla\varphi). \tag{7.8}$$

In virtue of Corollary 2.1

$$\|c_{(\varepsilon)}\|_{0,q,\mathbb{R}^n} \to 0.$$

By basic properties of mollifier, cf. (2.10) and Sobolev imbedding theorem,

$$\|d_{(\varepsilon)}\|_{0,q,\mathbb{R}^n} \le c\vartheta_1 \|z\|_{1,q,\mathbb{R}^n}$$

and

$$d_{(\varepsilon)} \to d = [\nabla z \operatorname{div} w + z\nabla \operatorname{div} w + \nabla w\cdot\nabla z] - \operatorname{div} w\cdot\nabla z + \nabla z\cdot\nabla w$$

in $L^q(\mathbb{R}^n)$. Hence $\{c_{(\varepsilon)}\}_{\varepsilon>0}$ $\{d_{(\varepsilon)}\}_{\varepsilon>0}$ are Cauchy sequences in $L^q(\mathbb{R}^n)$. On the other hand, by Theorem 6.2, there exists just one solution $\varphi \in \widehat{H}_0^{1,q'}(\mathbb{R}^n)$ of the problem

$$\lambda\varphi - w\cdot\nabla\varphi = \psi$$

such that

$$\|\nabla\varphi\|_{0,q',\mathbb{R}^n} \le c\|\nabla\psi\|_{1,q',\mathbb{R}^n}$$

(provided ϑ_1 is sufficiently small and $\psi \in \widehat{H}_0^{1,q'}(\mathbb{R}^n)$). Take the difference $(7.8)_\varepsilon$–$(7.8)_{\varepsilon'}$; taking supremum over all $\psi \in \widehat{H}_0^{1,q'}(\Omega)$, $\|\nabla\psi\|_{1,q',\Omega} \le 1$, one gets

$$\begin{aligned}|\Delta z_{\varepsilon'} - \Delta z_{\varepsilon'}|_{-1,q,\mathbb{R}^n} \le c|\Delta f_{\varepsilon'} - \Delta f_{\varepsilon'}|_{-1,q,\mathbb{R}^n} + \|d_{(\varepsilon)} - d_{(\varepsilon')}\|_{0,q,\mathbb{R}^n}\\ + \|c_{(\varepsilon)} - c_{(\varepsilon')}\|_{0,q,\mathbb{R}^n}.\end{aligned}$$

Hence $\{\Delta z_\varepsilon\}_{\varepsilon>0}$ is a Cauchy sequence in $\widehat{H}^{-1,q}(\mathbb{R}^n)$. Clearly

$$\Delta z_\varepsilon \to \Delta z \ \text{ in } \ \widehat{H}^{-1,q}(\mathbb{R}^n).$$

Taking in (7.8) a supremum over all $\psi \in \widehat{H}_0^{1,q'}(\mathbb{R}^n)$, $\|\nabla\psi\|_{1,q',\mathbb{R}^n} \le 1$, one arrives at

$$|\Delta z_\varepsilon|_{-1,q,\mathbb{R}^n} \le c\Big(\|f_\varepsilon\|_{1,q,\mathbb{R}^n} + c\vartheta_1\|z\|_{1,q,\mathbb{R}^n} + \|c_{(\varepsilon)}\|_{0,q,\mathbb{R}^n}\Big)$$

which yields (7.6). The proof is thus complete. $\square$

8. A remark about one possible generalization

In this section we explain how to weaken the assumptions on the smallness of coefficients a, w. This observation is due to M. Padula.

It is easily seen from the proofs of Lemmas 3.1–3.3, which are dealing with estimates of solutions, that we never need the complete norms $|w|_{\mathcal{C}^k}$ $(k = 1, 2, \dots)$ but only $|\nabla w|_{\mathcal{C}^{k-1}}$. Indeed in estimates (3.11) and (3.14) the bound $|w|_{\mathcal{C}^1}$ can be replaced by more precise one, which is $|\operatorname{div} w|_{\mathcal{C}^0}$ and in (3.15), the bounds $|w|_{\mathcal{C}^k}$ and $|w|_{\mathcal{C}^{k-1}}$ by more precise one, $|\nabla w|_{\mathcal{C}^{k-1}}$ and $|\nabla w|_{\mathcal{C}^{k-2}}$, respectively.

The consequences of this observation are formulated in the following remark.

Remark 8.1. Put

$$
(8.1)\qquad
\begin{cases}
\overline{\vartheta}_0^{(k,q)}(w,a) = |\nabla w|_{\mathcal{C}^{k-1}} + |a|_{\mathcal{C}^k} \\
\overline{\vartheta}_1^{(k,q)}(w,a) = |\nabla w|_{\mathcal{C}^{k-1}} + |a|_{\mathcal{C}^{k-1}} + \|\nabla^k a\|_{0,q} \\
\overline{\vartheta}_1'^{(k,q)}(w,a) = |\nabla w|_{\mathcal{C}^{k-1}} \\
\overline{\vartheta}_2^{(k,q)}(w,a) = |\nabla w|_{\mathcal{C}^{k-1}} + |a|_{\mathcal{C}^{k-1}} + \|\nabla^k a\|_{0,n} \\
\overline{\vartheta}_2'^{(k,q)}(w,a) = |\nabla w|_{\mathcal{C}^{k-1}} + |a|_{\mathcal{C}^{k-1}} + \|\nabla a\|_{k-1,n} \\
\overline{\vartheta}_3^{(k,q)}(w,a) = |\nabla w|_{\mathcal{C}^{k-2}} + |a|_{\mathcal{C}^{k-2}} + \|\nabla^k w\|_{0,n} + \|\nabla^{k-1} a\|_{0,n} + \\
\qquad\qquad + \|\nabla^k a\|_{0,q} \\
\overline{\vartheta}_4^{(k,q)}(w,a) = |\nabla w|_{\mathcal{C}^{k-2}} + |a|_{\mathcal{C}^{k-2}} + \|\nabla^k w\|_{0,n} + \|\nabla^{k-1} a\|_{1,n} \\
\overline{\vartheta}_5^{(k,q)}(w,a) = |\nabla w|_{\mathcal{C}^{k-2}} + |a|_{\mathcal{C}^{k-2}} + \|\nabla^k w\|_{0,n} + \|\nabla^{k-1} a\|_{1,q} \\
\overline{\vartheta}_6^{(k,q)}(w,a) = |\nabla w|_{\mathcal{C}^{k-2}} + |a|_{\mathcal{C}^{k-2}} + \|\nabla^k w\|_{0,q} + \|\nabla^{k-1} a\|_{1,q} \\
\overline{\vartheta}_7^{(k,q)}(w,a) = |\nabla w|_{\mathcal{C}^{k-2}} + |a|_{\mathcal{C}^{k-2}} + \|\nabla^k w\|_{0,q} + \|\nabla^{k-1} a\|_{1,n} \\
\overline{\vartheta}_8^{(k,q)}(w,a) = |\nabla w|_{\mathcal{C}^0} + |a|_{\mathcal{C}^0} + \|\nabla(a - \operatorname{div} w)\|_{0,q'} \left(\frac{1}{q} + \frac{1}{q'} = 1\right) \\
\overline{\vartheta}_9^{(k,q)}(w,a) = |\nabla w|_{\mathcal{C}^0} + |a|_{\mathcal{C}^0} + \|\nabla(a - \operatorname{div} w)\|_{0,n} \\
\overline{\vartheta}_{10}^{(k,q)}(w,a) = |\nabla w|_{\mathcal{C}^0} + |a|_{\mathcal{C}^0}
\end{cases}
$$

Then the following statements remain valid if we replace ϑ_j, $j = 0, 1, \ldots, 10$ (see (2.11)) systematically by $\overline{\vartheta}_j$ (see (8.1)).

☐ Lemma 3.1, 3.2 and 3.3 about estimates independent of the domain.

☐ Theorems 4.2 and 4.3 about existence and uniqueness of solutions in $\mathbb{R}^3$.

☐ Theorems 5.1, 5.2, 5.3 about existence and uniqueness of solutions in $\Omega \in \mathcal{B}^{(k)}$.

☐ Theorem 5.6 about the regularity of solutions.

☐ Theorem 5.7 about existence of weak solutions in Lebesgue spaces.

☐ Theorems 6.1 and 6.2 about the solvability in homogenous Sobolev spaces.

☐ Theorem 6.3 about the regularity in homogenous Sobolev spaces.

☐ Theorems 6.4 and 6.5 about the solvability in dual spaces.

☐ Theorems 7.1 and 7.2 about estimates of Laplacian of solutions.

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