Notes on slender prime rings

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Abstract. If R is a prime ring such that R is not completely reducible and the additive group $R(+)$ is not complete, then R is slender.

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The purpose of this short note is to discuss a few sufficient conditions for a prime ring to be slender. As concerns the concept of slenderness (various results, references, historical remarks, etc.), a reader is fully referred to [4, Chapter III].

1. Introduction

In the sequel, R is a non-zero associative ring with unit and modules are unitary left R-modules. The ring R is said to be prime (resp. a domain) if $aRb \neq 0$ (resp. $ab \neq 0$) for all $a, b \in R$, $a \neq 0 \neq b$. Commutative domains are also called integral domains.

Let M be a module. By a filtration $\mathcal{F}$ of M we mean any sequence M_i, $i < \omega$, of submodules of M such that $M_i \supseteq M_{i+1}$. The filtration $\mathcal{F}$ is said to be separating if $\bigcap_{\mathcal{F}} M_i = 0$ and it is said to be discrete if $0 \in \mathcal{F}$. The filtration $\mathcal{F}$ determines a linear closure operator on M and the module M is said to be $\mathcal{F}$-complete if every Cauchy $\mathcal{F}$-sequence of elements from M is convergent.

A module M is said to be complete if it is $\mathcal{F}$-complete for a non-discrete separating filtration $\mathcal{F}$ of M.

A left (right) ideal I of R is said to be l. s. $\cup$-compact (r. s. $\cup$-compact) if every countable subset S of I is contained in a finitely generated left (right) ideal $K \subseteq I$.

The ring R is said to be left (right) $\cap$-compact if the left (right) module ${}_RR$ (R_R) possesses no non-discrete separating filtration.

We denote by $\mathcal{T}_R$ the set of ideals I of R such that the factor R/I is completely reducible. For a module M, let $Soct(M)$ be the set of all $x \in M$ such that $(0 : x)$ contains an ideal from $\mathcal{T}_R$. Finally, let $V = R^{\omega}$, $U = R^{(\omega)}$ and $W = V/U$. If $i < \omega$, then $V[i] = \{a \in V; a(j) = 0 \text{ for every } j < i\}$.

For further basic terminology concerning rings and modules, we refer to [1].

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2. Slender modules

A module M is said to be slender if, for every homomorphism $\varphi : V \to M$, $\varphi(e_i) = 0$ for almost all $i < \omega$. The following result is implicitly contained in [5] and is proved in [3] for torsionfree modules over integral domains:

2.1 Proposition. *A module M is slender if and only if $Hom_R(W, M) = 0$ and M is not complete.*

2.2 Proposition. *Let M be a module such that there exists a filtration I_i, $i < \omega$, of R satisfying the following conditions:*

(1) *I_i is a r. s. $\cup$-compact ideal for every $i < \omega$.*
(2) *If $i < \omega$ and $0 \neq u \in M$, then $I_i u \neq 0$.*
(3) $\bigcap_{\omega} I_i M = 0$.

Then the module M is slender if and only if it is not complete.

Proof: The result is an immediate consequence of the following observation:
□

2.3 Observation. Let I_i, $i < \omega$, be a filtration of R_R such that all the right ideals I_i are r. s. $\cup$-compact. Put $\mathcal{E} = \{I_i V[i]; i < \omega\}$. Then $\mathcal{E}$ is a separating filtration of $V(+)$ and $V(+)$ is $\mathcal{E}$-complete.

Now, let $\varphi : V \to M$ be a (module) homomorphism. Put $\mathcal{G} = \varphi(\mathcal{E}) = \{I_i \varphi(V[i])\}$. Then $\mathcal{G}$ is a filtration of $M(+)$, φ is continuous and $M(+)$ is $\mathcal{G}$-complete.

Assume $\bigcap \mathcal{G} = 0$. Then $Ker(\varphi)$ is $\mathcal{E}$-closed in V. If $Ker(\varphi)$ is $\mathcal{E}$-open, then $I_m \varphi(V[m]) = 0$ for some $m < \omega$. If $Ker(\varphi)$ is not $\mathcal{E}$-open, then $\mathcal{G}$ is not discrete. Now, $\mathcal{G}$ is a non-discrete separating filtration of $M(+)$ and $M(+)$ is $\mathcal{G}$-complete. In particular, if the right ideals I_i are two-sided, then M is a complete module.

2.4 Corollary. *Suppose that there exists a separating filtration I_i, $i < \omega$, of R such that I_i is an r. s. $\cup$-compact ideal and $(0 : I_i)_r = 0$ for every $i < \omega$. Then the ring R is left slender if and only if it is not left complete.*

2.5 Corollary. *Let M be a module such that there exists a countable non-empty set $\mathcal{M}$ of submodules of M satisfying the following properties:*

(1) $\bigcap \mathcal{M} = 0$.
(2) *$(0 : M/N)$ is r. s. $\cup$-compact for every $N \in \mathcal{M}$.*
(3) *$(0 : M/N) \in \mathcal{T}_R$ for every $N \in \mathcal{M}$.*

Then M is slender if and only if $Soct(M) = 0$ and M is not complete.

2.6 Corollary. *Suppose that R possesses a countable non-empty set $\mathcal{M}$ of maximal left ideals such that $\bigcap \mathcal{M} = 0$, $(0 : R/I) \in \mathcal{T}_R$ and $(0 : R/I)$ is r. s. $\cup$-compact for every $I \in \mathcal{M}$. Then R is left slender if and only if $Soct_l(R) = 0$ and R is not left complete.*

3. Prime rings and slenderness

3.1 Theorem. *Let R be a prime ring.*

(i) *If every right ideal is an ideal and R is not right $\cap$-compact, then R is a domain and R is left slender if and only if R is not left complete.*

(ii) *If the additive group $R(+)$ is not complete, then R is slender if and only if R is not isomorphic to a (full) matrix ring over a division ring.*

(iii) *If $card(R) \geq 2^\omega$ and the additive group $R(+)$ is not complete, then R is slender.*

Proof: (i) Clearly, R is a right uniform domain, and hence there is a separating filtration $r_i R$, $i < \omega$, of non-zero principal right ideals and it remains to apply 2.4.

(ii) Let p denote the characteristic of R. If $p > 0$, then $card(R) < 2^\omega$ (since $R(+)$ is not complete) and we can use [2, Theorem 4.1]. If $p = 0$ and $R(+)$ is reduced, then $R(+)$ is slender (see [6]) and consequently R is also slender. Assume finally that $p = 0$ and the divisible part $Q(+)$ of $R(+)$ is non-zero.

Obviously, Q is an ideal of R and the factorgroup $R(+)/Q(+)$ is slender ([6]), and hence the factormodule ${}_R R/Q$ is slender, too. Now, it remains to show that the module ${}_R Q$ is slender. However, since $Q(+)$ is not complete, we have $card(Q) < 2^\omega$ and then we can proceed similarly as in the proof of [2, Theorem 4.1].

(iii) This assertion follows easily from (ii). □

3.2 Proposition. *Let R be a domain satisfying maximal condition on principal left ideals and such that R is not a division ring and that every right ideal of R is an ideal. Then R is left slender if and only if R is not left complete.*

Proof: Clearly, R is not right $\cap$-compact and the result follows from 3.1 (i). □

3.3 Proposition. *Let R be an integral domain, not a field, satisfying at least one of the following conditions:*

(1) *R is noetherian.*

(2) *R is a unique factorization domain.*

(3) *The quotient field of R is a countably generated R-module (see* [3, Theorem 20]*).*

(4) *R is not $\cap$-compact.*

Then R is slender if and only if it is not complete.

Proof: The first two cases follow from 3.2, the condition (3) implies (4) and, when (4) is true, the result follows from 3.1 (i). □

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