Asymptotic behaviour of the time dependent Norton-Hoff law in plasticity theory and H^1 regularity

A. Bensoussan, J. Frehse

Abstract. We prove H^1_{loc}-regularity for the stresses in the Prandtl-Reuss-law. The proof runs via uniform estimates for the Norton-Hoff-approximation.

Keywords: elasto-plasticity, regularity, variational inequalities

Classification: 35A15, 35D10, 35J45, 35J50, 35K65, 35K85, 35Q72, 73E50

1. Introduction

In this article, we continue the study of the asymptotic behaviour of the Norton-Hoff model initiated in our previous work [1]. This time, we study the time dependent case, which leads to a monotone differential equation instead of a monotone algebraic equation. The monotone operator is a penalty operator. When the penalization coefficient tends to 0, we get a parabolic variational inequality instead of the elliptic variational inequality in the static case, corresponding to the Hencky model of plasticity. The parabolic variational inequality is the Prandtl-Reuss model of perfect plasticity. As in the static case, we provide a H^1_{loc} regularity theory.

Recently G.A. Seregin [5] obtained similar results concerning quasi-static models of plasticity with kinematic and isotropic hardening. Our result, concerning perfect plasticity can be considered as a limit case of the isotropic hardening he considered. The method of proof that we use, relies on the dual theory of elliptic equations, and is of a different nature.

2. The time dependent Norton-Hoff model

2.1 Preliminary notation.

Let Ω be a bounded Lipschitz domain of R^n, whose boundary is denoted by Γ. The boundary will be divided in two parts, $\Gamma_0 \cup \Gamma_1$. Let be $W^{1,p}(\Omega)$, $1 \leq p < \infty$, the Sobolev space of functions which are p integrable on Ω as well as their distributional derivatives, with the norm

$$\|\phi\|_{W^{1,p}(\Omega)} = |\phi|_{L^p(\Omega)} + \sum_{i=1}^{n} |D_i\phi|_{L^p(\Omega)} .$$

For $p = 2$, one writes $H^1(\Omega)$ instead of $W^{1,2}(\Omega)$ and one takes the Hilbert space norm

$$\|\phi\|_{H^1(\Omega)} = (|\phi|_{L^2(\Omega)} + \sum_{i=1}^{n} |D_i\phi|^2_{L^2(\Omega)})^{1/2}.$$

We shall denote by $W^{1,p}_{\Gamma_0}(\Omega)$ and $H^1_{\Gamma_0}(\Omega)$, the closed subspaces of functions which vanish on Γ_0, respectively in $W^{1,p}(\Omega)$, $H^1(\Omega)$. We shall use the spaces of vector functions $(W^{1,p}_{\Gamma_0}(\Omega))^n$, $(H^1_{\Gamma_0}(\Omega))^n$. When $\Gamma_0 = \Gamma$, one writes $W^{1,p}_0(\Omega)$ and $H^1_0(\Omega)$, following the usual notation. When $\Gamma_0 \subset \Gamma$ we assume that the capacity of Γ_0 is positive. We next consider the space of $n \times n$ symmetric matrices whose elements are in L^p, denoted by $\mathcal{L}^p_{\mathrm{sym}}$, with the norm

$$\|\sigma\|_{\mathcal{L}^p_{\mathrm{sym}}} = |\sigma|_{L^p(\Omega)},$$

where the symbol $|\sigma|_{L^p(\Omega)}$ designates the L^p-norm of the modulus of the matrix σ,

$$|\sigma| = (\sum_{ij} \sigma^2_{ij})^{1/2}.$$

In the case $p = 2$, it corresponds to the Hilbert norm

$$\|\sigma\|_{\mathcal{L}^2_{\mathrm{sym}}} = (\sum_{i,j=1}^{n} \int_\Omega \sigma^2_{ij}\, dx)^{1/2}.$$

It will usually be abbreviated to $\|\sigma\|$, when there is no risk of confusion. We shall use the notation (,) for the scalar product in $\mathcal{L}^2_{\mathrm{sym}}$.

It will be convenient to use the notation

$$\sigma.\tau = \sum_{i,j=1}^{n} \sigma_{ij}\tau_{ij}$$

to represent the scalar product of two matrices σ and τ, similar to the scalar product of vectors in R^n. We shall also use the notation

$$\operatorname{div} \sigma = \sum_{i=1}^{n} D_i\sigma_{ij},$$

which is a vector (we consider only symmetric matrices), and

$$\nu.\sigma = \sum_{i=1}^{n} \nu_i\sigma_{ij},$$

where ν is the outward unit normal on the boundary Γ.

We recall

$$\text{Deviator of } \sigma = \sigma_D = \sigma - \frac{1}{n}\,\mathrm{tr}\,\sigma\, I$$

which has trace 0. Also the strain of a (displacement) vector u is given by

$$\varepsilon(u) = \frac{1}{2}\,(Du + (Du)^T).$$

Since the paper is concerned with time dependent problems, we shall need functional spaces like $L^p(0,T;W^{1,p}(\Omega))$, $L^p(0,T;W^{1,p}(\Omega)^n)$ and $L^p(0,T;\mathcal{L}^p_{\mathrm{sym}})$. The notation for the norm of these spaces follows the standard one for Banach valued functions of time.

2.2 Setting of the model. We begin with the assumptions. We consider a tensor function

$$\begin{gathered} A_{i\,j,h\,k} \in L^\infty \text{ such that } A_{i\,j,h\,k} = A_{j\,i,h\,k} = A_{i\,j,k\,h} = A_{h\,k,i\,j} \\ \sum_{i,j;h,k=1}^{n} A_{i\,j,h\,k}\tau_{i\,j}\tau_{h\,k} \geq \alpha|\tau|^2 \;\; \forall\, \tau : \, \tau = \tau^T, \;\, \alpha > 0. \end{gathered} \tag{2.1}$$

In fact the previous tensor function could also depend on time. For simplicity, we omit this possibility.

We next consider functions

$$f \in L^2(0,T;L^2(\Omega)^n), \tag{2.2}$$

$$\phi \in L^2(0,T;L^2(\Gamma_1)^n), \tag{2.3}$$

$$\zeta \in L^2(0,T;H^1(\Omega)^n). \tag{2.4}$$

Let μ be a positive number. We assume further that there exists $\tau \in C([0,T];\mathcal{L}^2_{\mathrm{sym}})$ with the properties:

$$\begin{gathered} \dot\tau \in L^2(0,T;\mathcal{L}^2_{\mathrm{sym}}), \\ |\tau_D(t,x)| \leq \mu, \quad \forall\, t, \text{ a.e. } x \in \Omega, \\ \dot\tau_D \in L^\infty((0,T)\times\Omega) \\ \mathrm{div}\,\tau = f \text{ a.e. in } \Omega, \\ \nu.\tau = \phi \text{ a.e. on } \Gamma_1 \text{ and } u = 0 \text{ on } \Gamma_0. \end{gathered} \tag{2.5}$$

Let finally

$$\sigma_0 \in \mathcal{L}^2_{\mathrm{sym}}, \;\; |\sigma_{0,D}| \leq \mu \text{ a.e.} \tag{2.6}$$

The time dependent Norton-Hoff model is the following problem:

To find a pair $(\sigma^N(t),\ v^N(t))$ such that

$$\begin{aligned}
&\sigma^N \in H^1(0,T;\mathcal{L}^2_{\text{sym}}),\ \sigma^N_D \in L^N(0,T;\mathcal{L}^N_{\text{sym}}),\\
&v^N \in L^{\frac{N}{N-1}}(0,T;W^{1,\frac{N}{N-1}}_{\Gamma_0}(\Omega)^n),\ \operatorname{div} v^N \in L^2(0,T;L^2(\Omega)),\\
&A\dot{\sigma}^N + \frac{1}{\mu^{N-1}}|\sigma^N_D|^{N-2}\sigma^N_D = \varepsilon(v^N+\zeta),\ \operatorname{div}\sigma^N(t) = f(t),\\
&\nu.\sigma^N(t) = \phi(t) \text{ on } \Gamma_1,\ \sigma^N(0)=\sigma_0.
\end{aligned} \tag{2.7}$$

Remark 2.1. By v^N in (2.7) we mean the derivative of u^N with respect to time.

Our objective is to prove the following

Theorem 2.1. *Under the assumptions* (2.1) *to* (2.6) *there exists one and only one solution of* (2.7).

Remark 2.2. For fixed N the result is well known, see for example [9], but we shall emphasize the dependence of estimates with respect to N in order to obtain later further regularity results allowing us, to let N tend to ∞.

2.3 Proof of Theorem 2.1.

The uniqueness is easy and follows from standard monotonicity arguments. Let us set

$$\beta^N(x) = \frac{x^{N-2}}{\mu^{N-1}}\,.$$

In the proof we omit to write systematically the index N.

The existence will be derived from a discretization in time approximation model, where we shall use the results already obtained in the static case, see [1].

For that purpose, let L be an integer which will tend to ∞ and set $h=\frac{T}{L}$. We are going to consider step functions approximating $\zeta(t)$, $\tau(t)$, as follows

$$\tau^h(t) = \tau(h[\frac{t}{h}]), \quad \zeta^h(t) = \frac{1}{h}\int_{h[\frac{t}{h}]}^{h[\frac{t}{h}]+h} \zeta(s)\,ds$$

the difference of treatment stems from the fact that τ is continuous in t with respect to the norm of $\mathcal{L}^2_{\text{sym}}$ whereas ζ is not. We recall that $[x]$ denotes the integer part of x.

By definition, a step function satisfies

$$\sigma^h(t) = \sigma^h(h[\frac{t}{h}])$$

and it will be defined for $t\in[0,T+h[$, in order to incorporate the value at T. In case of $\zeta^h(t)$, where we need to have the values of ζ for $t\in(0,T+h)$, then we simply extend ζ by 0 outside $(0,T)$.

To a step function $\sigma^h(t)$ we associate the so-called Rothe function

$$\tilde{\sigma}^h(t) = \sigma^h(t+h)\frac{t - h[\frac{t}{h}]}{h} + \sigma^h(t)\frac{h([\frac{t}{h}]+1) - t}{h}\,.$$

It is a piecewise linear continuous function on $[0,T]$ such that

$$\tilde{\sigma}^h(h[\frac{t}{h}]) = \sigma^h(h[\frac{t}{h}])\ \forall\, t \in [0,T].$$

Unlike $\sigma^h(t)$, this function is not defined outside $[0,T]$. Its derivative is defined on $[0,T[$ by the formula

$$\dot{\tilde{\sigma}}^h(t) = \frac{\sigma^h(t+h) - \sigma^h(t)}{h}\,.$$

It is a step function.

It will be useful also to note the following formula,

$$\sigma^h(t) = \tilde{\sigma}^h(t-h) + \dot{\tilde{\sigma}}^h(t-h)(h([\frac{t}{h}]+1) - t), \quad t \in [h, T+h[\,. \tag{2.8}$$

We are now in a position to define our approximation model: To find a pair $(\sigma^h(t), v^h(t))$ of step functions (thus defined on $[0,T+h[$) such that

$$\left.\begin{aligned} A\dot{\tilde{\sigma}}^h(t-h) + \beta(|\sigma^h_D(t)|)\sigma^h_D(t) = \varepsilon(v^h(t) + \zeta^h(t)) \\ \operatorname{div}(\sigma^h - \tau^h)(t) = 0 \\ \nu.(\sigma^h - \tau^h)(t) = 0 \text{ on } \Gamma_1,\ \sigma^h(0) = \sigma_0,\ v^h(0) = 0 \end{aligned}\right\}\forall\, t \in [h, T+h[\,,$$

$$\left.\begin{aligned} \sigma^h(t)\, \in \mathcal{L}^2_{\text{sym}},\ \sigma^h_D(t) \in \mathcal{L}^N_{\text{sym}} \\ v^h(t) \in W^{1,\frac{N}{N-1}}_{\Gamma_0}(\Omega)^n,\ \operatorname{div} v^h \in L^2(\Omega) \end{aligned}\right\}\forall\, t \in [0, T+h[\,. \tag{2.9}$$

Equation $(2.9)_1$ is the Rothe approximation of the time dependent Norton-Hoff model. The existence and uniqueness of the solution to (2.9) follows from the static case, since once setting

$$\sigma^h_\ell = \sigma^h(\ell h), \quad v^h_\ell = v^h(\ell h), \quad \ell = 0, \ldots, L$$

then (2.9) amount to a sequence of static Norton-Hoff relations giving $(\sigma^h_\ell, v^h_\ell)$ in terms of $\sigma^h_{\ell-1}$.

We begin with a priori estimates. Let us emphasize that in the following the constants will be independent of N.

$$\|\sigma^h(t)\| \le C, \tag{2.10}$$

$$\forall\, t \in [0, T+h[\text{ and } \frac{1}{N\mu^{N-1}} \int_0^{T+h} \int_\Omega |\sigma^h_D(t,x)|^N \,dx\,dt \le C.$$

To prove (2.10) we test (2.9) with $(\sigma^h - \tau^h)(t)$. We have

$$(2.11)\quad (A\dot{\tilde{\sigma}}^h(t-h), (\sigma^h - \tau^h)(t)) + (\beta(|\sigma_D^h|)\sigma_D^h, \sigma_D^h - \tau_D^h) = (\varepsilon(\zeta^h), \sigma^h - \tau^h)$$

hence also by monotonicity properties

$$(2.12)\qquad (A\dot{\tilde{\sigma}}^h(t-h), (\sigma^h - \tau^h)(t)) + (\beta(|\tau_D^h|)\tau_D^h - \varepsilon(\zeta^h), \sigma^h - \tau^h) \le 0.$$

Using (2.8) in (2.12) we get

$$(2.13)\qquad \begin{aligned} &(A(\dot{\tilde{\sigma}}^h - \dot{\tilde{\tau}}^h)(t-h), (\tilde{\sigma}^h - \tilde{\tau}^h)(t-h)) \\ &+ (A(\dot{\tilde{\sigma}}^h - \dot{\tilde{\tau}}^h)(t-h), (\dot{\tilde{\sigma}}^h - \dot{\tilde{\tau}}^h)(t-h))(h([\frac{t}{h}]+1)-t) \\ &+ (A\dot{\tilde{\tau}}^h(t-h) + \beta(|\tau_D^h|)\tau_D^h(t) - \varepsilon(\zeta^h)(t), (\tilde{\sigma}^h - \tilde{\tau}^h)(t-h) \\ &+(\dot{\tilde{\sigma}}^h - \dot{\tilde{\tau}}^h)(t-h)(h([\frac{t}{h}]+1)-t)) \le 0. \end{aligned}$$

Integrating between h and t we obtain

$$(2.14)\qquad \begin{aligned} &\int_h^t (A(\dot{\tilde{\sigma}}^h - \dot{\tilde{\tau}}^h)(s-h), (\dot{\tilde{\sigma}}^h - \dot{\tilde{\tau}}^h)(s-h))(h([\frac{s}{h}]+1)-s)\,ds \\ &+ \int_h^t (\dot{\tilde{\tau}}^h(s-h) + \beta(|\tau_D^h|)\tau_D^h(s) - \varepsilon(\zeta^h)(s), (\tilde{\sigma}^h - \tilde{\tau}^h)(s-h) \\ &+ (\dot{\tilde{\sigma}}^h - \dot{\tilde{\tau}}^h)(s-h)(h([\frac{s}{h}]+1)-s))\,ds \\ &\le \frac{1}{2}(A\sigma_0, \sigma_0) - \frac{1}{2}(A(\tilde{\sigma}^h - \tilde{\tau}^h)(t-h), (\tilde{\sigma}^h - \tilde{\tau}^h)(t-h)). \end{aligned}$$

Note that from the assumptions one has

$$\int_h^{T+h} \|\dot{\tilde{\tau}}^h(s-h) + \beta(|\tau_D^h|)\tau_D^h(s) - \varepsilon(\zeta^h)(s)\|^2\,ds \le C.$$

Therefore one derives from (2.14) that

$$\frac{1}{2}(A(\tilde{\sigma}^h - \tilde{\tau}^h)(t-h), (\tilde{\sigma}^h - \tilde{\tau}^h)(t-h)) \le C\int_h^t \|\tilde{\sigma}^h - \tilde{\tau}^h\|^2(s-h)\,ds + C$$

and from Gronwall's inequality we get

$$\|\tilde{\sigma}^h - \tilde{\tau}^h\|^2(t-h) \le C,\ \forall\, t \in [h, T+h]\,,$$

which is the first part of (2.10). Moreover, going back to the previous calculation without using the monotonicity property, we deduce easily

$$\int_h^t (\beta(|\sigma_D^h|)\sigma_D^h, \sigma_D^h - \tau_D^h)(s)\,ds \le C.$$

By Young's inequality the second part of (2.10) follows easily.

Next we have the estimates

$$
\begin{aligned}
&\int_0^T \|\dot{\bar{\sigma}}^h(t)\|^2 \, dt \le CN, \\
&\frac{1}{N\mu^{N-1}} \int_\Omega |\sigma_D^h(t,x)|^N \, dx \le CN, \quad \forall\, t \in [0, T+h[\,.
\end{aligned}
\tag{2.15}
$$

To prove (2.15) we test (2.9) with $\dot{\bar{\sigma}}^h(t-h) - \dot{\bar{\tau}}^h(t-h)$ and get

$$
\begin{aligned}
&(A\dot{\bar{\sigma}}^h(t-h), \dot{\bar{\sigma}}^h(t-h) - \dot{\bar{\tau}}^h(t-h)) \\
&+(\beta(|\sigma_D^h|)\sigma_D^h(t), \dot{\bar{\sigma}}_D^h(t-h) - \dot{\bar{\tau}}_D^h(t-h)) \\
&= (\varepsilon(\zeta^h)(t), \dot{\bar{\sigma}}^h(t-h) - \dot{\bar{\tau}}^h(t-h)).
\end{aligned}
\tag{2.16}
$$

We integrate (2.16) between h and t. We note first that

$$
\begin{aligned}
&\int_h^t (A\dot{\bar{\sigma}}^h(s-h) - \varepsilon(\zeta^h)(s), \dot{\bar{\sigma}}^h(s-h) - \dot{\bar{\tau}}^h(s-h)) \, ds \\
&\qquad \ge \frac{\alpha}{2} \int_h^t \|\dot{\bar{\sigma}}^h(s-h)\|^2 \, ds - C.
\end{aligned}
$$

Next we have

$$
\begin{aligned}
&(\beta(\sigma_D^h)\sigma_D^h(t), \dot{\bar{\sigma}}_D^h(t-h)) = \frac{1}{h}(\beta(\sigma_D^h)\sigma_D^h(t), \sigma_D^h(t) - \sigma_D^h(t-h)) \\
&\ge \frac{1}{h\mu^{N-1}} \int_\Omega |\sigma_D^h(t,x)|^N \, dx - \frac{1}{h\mu^{N-1}} \int_\Omega |\sigma_D^h(t,x)|^{N-1} |\sigma_D^h(t-h,x)| \, dx \\
&\ge \frac{1}{hN\mu^{N-1}} \int_\Omega |\sigma_D^h(t,x)|^N \, dx - \frac{1}{hN\mu^{N-1}} \int_\Omega |\sigma_D^h(t-h,x)|^N \, dx.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\int_h^t (\beta(\sigma_D^h)\sigma_D^h(s), \dot{\bar{\sigma}}_D^h(s-h)) \, ds \ge \\
&\qquad \ge \frac{1}{hN\mu^{N-1}} \int_{t-h}^t \int_\Omega |\sigma_D^h(s,x)|^N \, dx \, ds - \frac{\mu \ \mathrm{Meas}(\Omega)}{N}.
\end{aligned}
$$

Furthermore,

$$
\left| \int_h^t (\beta(\sigma_D^h)\sigma_D^h(s), \dot{\bar{\tau}}_D^h(s-h)) \, ds \right| \le \frac{C}{\mu^{N-1}} \int_h^t \int_\Omega |\sigma_D^h(s,x)|^N \, dx \, ds + C
$$

and thus using the second estimate in (2.10) it follows

$$\left|\int_h^t (\beta(\sigma_D^h)\sigma_D^h(s), \dot{\tilde{\tau}}_D^h(s-h))\,ds\right| \le CN.$$

Collecting results we deduce

$$\int_h^t \|\dot{\tilde{\sigma}}^h(s-h)\|^2\,ds + \frac{1}{hN\mu^{N-1}}\int_{t-h}^t\int_\Omega |\sigma_D^h(s,x)|^N\,dx\,ds \le CN$$

and thus the estimates (2.15) are obtained.

We can now proceed with the proof of Theorem 2.1. We can consider a subsequence such that

$$\begin{aligned}
\sigma^h &\to \sigma \text{ in } L^\infty(0,T;\mathcal{L}^2_{\text{sym}}) \text{ weakly star},\\
\tilde{\sigma}^h &\to \sigma \text{ in } H^1(0,T;\mathcal{L}^2_{\text{sym}}) \text{ weakly},\\
\sigma_D^h &\to \sigma_D \text{ in } L^N(0,T;\mathcal{L}^N_{\text{sym}}) \text{ weakly},\\
v^h &\to u \text{ in } L^{\frac{N}{N-1}}(0,T;W^{1,\frac{N}{N-1}}_{\Gamma_0}(\Omega)^n) \text{ weakly}
\end{aligned}$$

and also

$$\beta(|\sigma_D^h|)\sigma_D^h \to \chi \text{ in } L^{\frac{N}{N-1}}(0,T;\mathcal{L}_{\text{sym}}^{\frac{N}{N-1}}) \text{ weakly}.$$

We shall identify χ by monotonicity arguments and then pass to the limit in (2.9). We first notice that σ satisfies the conditions

$$\operatorname{div}\sigma(t) = f(t) \text{ a.e. in } \Omega \qquad \text{and} \qquad \nu.\sigma(t) = \phi(t) \text{ a.e. on } \Gamma_1.$$

We define the function

$$\hat{\sigma}^h(t) = \sigma(h([\frac{t}{h}]+1))\frac{t-h[\frac{t}{h}]}{h} + \sigma(h[\frac{t}{h}])\frac{h([\frac{t}{h}]+1)-t}{h}.$$

The function $\hat{\sigma}^h(t)$ is defined from σ in the same way as $\tilde{\tau}^h(t)$ has been defined from τ. By construction

$$\begin{aligned}
\operatorname{div}(\hat{\sigma}^h - \tilde{\tau}^h)(t) &= 0 \text{ a.e. in } \Omega\,,\\
\nu.(\hat{\sigma}^h - \tilde{\tau}^h)(t) &= 0 \text{ a.e. on } \Gamma_1
\end{aligned}$$

and

$$\hat{\sigma}^h \to \sigma \in H^1(0,T;\mathcal{L}^2_{\text{sym}}).$$

We test (2.9) with $\sigma^h(t) - \hat{\sigma}^h(t)$ and integrate between h and T. We get

$$\int_h^T (A\dot{\tilde{\sigma}}^h(t-h), \sigma^h(t) - \hat{\sigma}^h(t))\,dt$$
$$+\int_h^T (\beta(|\sigma_D^h|)\sigma_D^h(t), \sigma_D^h(t) - \hat{\sigma}_D^h(t))\,dt = \int_h^T (\varepsilon(\zeta^h)(t), \sigma^h(t) - \hat{\sigma}^h(t))\,dt.$$

Hence

$$\int_h^T (A(\dot{\tilde{\sigma}}^h(t-h) - \dot{\hat{\sigma}}^h(t)), \tilde{\sigma}^h(t-h) - \hat{\sigma}^h(t))\,dt$$
$$+\int_h^T (\beta(|\sigma_D^h|)\sigma_D^h(t), \sigma_D^h(t) - \hat{\sigma}_D^h(t))\,dt = -\int_h^T (A\dot{\hat{\sigma}}^h(t), \tilde{\sigma}^h(t-h) - \hat{\sigma}^h(t))\,dt$$
$$+\int_h^T (A\dot{\tilde{\sigma}}^h(t-h), \tilde{\sigma}^h(t-h) - \sigma^h(t))\,dt + \int_h^T (\varepsilon(\zeta^h)(t), \sigma^h(t) - \hat{\sigma}^h(t))\,dt$$

and the right hand side of the previous relation tends to 0 as $h \to 0$. Note in particular that thanks to formula (2.8) and the first estimate of (2.15) we have

$$\int_h^T \|\tilde{\sigma}^h(t-h) - \sigma^h(t)\|^2\,dt \to 0.$$

Therefore we can assert that

$$\limsup_{h\to 0} \int_0^T (\beta(|\sigma_D^h|)\sigma_D^h(t), \sigma_D^h(t))\,dt \le \int_0^T (\chi(t), \sigma_D(t))\,dt.$$

Since

$$\int_0^T (\beta(|\sigma_D^h|)\sigma_D^h(t) - \beta(|\tau_D|)\tau_D(t), \sigma_D^h(t) - \tau_D(t))\,dt \ge 0$$

for any

$$\tau_D \in L^N(0,T;\mathcal{L}^N_{\mathrm{sym}})$$

we obtain that

$$\int_0^T (\chi(t) - \beta(|\tau_D|)\tau_D(t), \sigma_D(t) - \tau_D(t))\,dt \ge 0$$

and it follows that

$$\chi(t) = \beta(|\sigma_D|)\sigma_D(t).$$

We can then pass to the limit in (2.9) and obtain that (σ, u) is indeed a solution of (2.7). The proof of Theorem 2.1 has been completed. $\square$

3. Further estimates and H^1_{loc} regularity

We shall consider here assumptions similar to the static case (see [1]) and obtain estimates which are sharper than (2.10), (2.15) with respect to the dependence on N. In particular we shall derive H^1_{loc} estimates which are uniform with respect to N, like in the static case.

3.1 The $f_j(t)$ derive from a potential.

We assume here that

$$\begin{aligned} f(x,t) &= DF(x,t) \text{ in } \Omega, \\ \phi(t) &= F(t)\nu \text{ on } \Gamma_1 \end{aligned} \tag{3.1}$$

where $F, \dot{F} \in L^2(0,T;L^2(\Omega))$ and $F(t) \in W^{1,p}(\Omega)\ \forall\, p \in (1,\infty)$ and $\forall\, t \in [0,T]$.

We can state

Proposition 3.1. *Under the assumptions of Theorem* 2.1 *and* (3.1) *we have*

$$\begin{gathered} \frac{1}{\mu^{N-1}} \int_0^{T+h} \int_\Omega |\sigma_D^h(t,x)|^N \,dx\,dt \le C, \qquad \int_0^T \|\dot{\tilde{\sigma}}^h(t)\|^2\,dt \le C, \\ \frac{1}{N\mu^{N-1}} \int_\Omega |\sigma_D^h(t,x)|^N\,dx \le C,\ \forall\, t \in [0,T+h[\,. \end{gathered} \tag{3.2}$$

Proof: We consider

$$F^h(t) = F(h[\frac{t}{h}]) \text{ and } \tilde{F}^h(t) = F^h(t+h)\frac{t-h[\frac{t}{h}]}{h} + F^h(t)\frac{h([\frac{t}{h}]+1)-t}{h}\,.$$

Then we test (2.9) with $\sigma^h(t) - F^h(t)\mathrm{I}$ (I = identity on $\mathbb{R}^{n\times n}$) and obtain

$$(A\dot{\tilde{\sigma}}^h(t-h), \sigma^h(t) - F^h(t)\mathrm{I}) + (\beta(|\sigma_D^h|)\sigma_D^h, \sigma_D^h) = (\varepsilon(\zeta^h)(t), \sigma^h(t) - F^h(t)\mathrm{I}).$$

Let us use (see (2.8))

$$\sigma^h(t) - F^h(t)\mathrm{I} = \tilde{\sigma}^h(t-h) - \tilde{F}^h(t-h)\mathrm{I} + (\dot{\tilde{\sigma}}^h(t-h) - \dot{\tilde{F}}^h(t-h)\mathrm{I})(h([\frac{t}{h}]+1)-t)$$

then we can write

$$\begin{gathered} \frac{1}{2}\frac{d}{dt}(A\tilde{\sigma}^h(t-h), \tilde{\sigma}^h(t-h)) - \frac{d}{dt}(\operatorname{tr} A\tilde{\sigma}^h(t-h), \tilde{F}^h(t-h)) \\ + (A\dot{\tilde{\sigma}}^h(t-h), \dot{\tilde{\sigma}}^h(t-h))(h([\frac{t}{h}]+1)-t) \\ -(h([\frac{t}{h}]+1)-t)\dot{\tilde{\sigma}}^h(t-h).(A\mathrm{I}\dot{\tilde{F}}^h(t-h) + \varepsilon(\zeta^h)(t)) + (\beta(|\sigma_D^h|)\sigma_D^h, \sigma_D^h) \\ = \tilde{\sigma}^h(t-h).(-A\mathrm{I}\dot{\tilde{F}}^h(t-h) + \varepsilon(\zeta^h)(t)) - \operatorname{div}\zeta^h(t).F^h(t). \end{gathered}$$

Using the fact that $\|\tilde{\sigma}^h(t-h)\|$ is bounded for any $t \in [h, T+h]$, the right hand side in the previous relation is bounded. Thus we have

$$\int_h^{T+h} (\beta(|\sigma_D^h|)\sigma_D^h, \sigma_D^h)\, dt \leq C.$$

Recalling the value of $\sigma^h(t)$ on the interval $[0,h]$ the first part of (3.2) follows. The proof of the two other results of (3.2) is then done as for the corresponding ones of (2.15), except we can use the better estimate just obtained and the proof is finished. □

3.2 Safe load condition.

Alternatively to (3.1) we can assume the following safe load condition: There exists

$$\begin{aligned} &\tau \in H^1(0,T;\mathcal{L}^2_{\text{sym}}) \text{ with} \\ &|\tau_D| - \mu \leq -\delta \text{ a.e. in } \Omega, \text{ for some } \delta > 0 \text{ such that} \\ &\operatorname{div}\tau(t) = f(t) \text{ a.e. in } \Omega \text{ and } \nu.\tau(t) = \phi(t) \text{ a.e. on } \Gamma_1 \end{aligned} \tag{3.3}$$

then we have

Proposition 3.2. *Under the same hypotheses as in Theorem* 2.1 *and* (3.3) *the same conclusions as those of Proposition* 3.1 *hold.*

PROOF: From (2.11), with τ as in (3.3) we get

$$\int_0^{T+h} (\beta(|\sigma_D^h|)\sigma_D^h - \beta(|\tau_D^h|)\tau_D^h, \sigma_D^h - \tau_D^h)\, dt \leq C$$

hence

$$\frac{1}{\mu^{N-1}} \int_0^{T+h} \int_\Omega (|\sigma_D^h|^{N-2}\sigma_D^h - |\tau_D^h|^{N-2}\tau_D^h).(\sigma_D^h - \tau_D^h)\, dx\, dt \leq C.$$

From the positivity of the integrand, it follows also

$$\frac{1}{\mu^{N-1}} \int_0^{T+h} \int_E (|\sigma_D^h|^{N-2}\sigma_D^h - |\tau_D^h|^{N-2}\tau_D^h).(\sigma_D^h - \tau_D^h)\, dx\, dt \leq C,$$

where $E = |\sigma_D^h| \geq \mu$, and the constant C being independent from E, h, N. We deduce from this estimate

$$\frac{1}{\mu^{N-1}} \int_0^{T+h} \int_E (|\sigma_D^h|^{N-1} - |\tau_D^h|^{N-1}).(|\sigma_D^h| - |\tau_D^h|)\, dx\, dt \leq C$$

thus also

$$\frac{1}{\mu^{N-1}}\int_0^{T+h}\int_E |\sigma_D^h|^{N-1}(|\sigma_D^h|-|\tau_D^h|)\,dx\,dt \le C \tag{3.4}$$

and the assumption (3.3) yields

$$\frac{\delta}{\mu^{N-1}}\int_0^{T+h}\int_E |\sigma_D^h|^{N-1}\,dx\,dt \le C \tag{3.5}$$

and using again (3.4), we obtain

$$\frac{1}{\mu^{N-1}}\int_0^{T+h}\int_E |\sigma_D^h|^{N}\,dx\,dt \le C$$

and

$$\frac{1}{\mu^{N-1}}\int_0^{T+h}\int_\Omega |\sigma_D^h|^{N}\,dx\,dt \le C.$$

This means that we have obtained the same basic estimate as in Proposition 3.1 and thus the same conclusions hold. □

3.3 H^1_{loc} estimates.

From the first and second estimates of equation (3.2) it follows that

$$\varepsilon(v^h) \text{ is bounded in } L^1(0,T;\mathcal{L}^1_{\mathrm{sym}}) \tag{3.6}$$

and from Korn's inequality (see [8] for example) we obtain

$$v^h \text{ is bounded in } L^1(0,T;L^{\frac{n}{n-1}}(\Omega)^n). \tag{3.7}$$

Remark 3.1. We recall that all constants are not only independent of h but also of N. More precisely the dependence with respect to N is expressed explicitly.

We also assume for the function τ in (2.5)

$$|\operatorname{div}\tau|,\ |D\operatorname{div}\tau|,\ |\Delta\operatorname{div}\tau| \in L^\infty(0,T;L^n_{\mathrm{loc}}(\Omega)) \tag{3.8}$$

and

$$\begin{gathered}\zeta \in L^2(0,T;H^2(\Omega)^n),\\ \sigma_{0,ij} \in H^1_{\mathrm{loc}}(\Omega).\end{gathered} \tag{3.9}$$

Note that from the static theory, we can assert that thanks to the assumptions of Theorem 2.1, (3.1) or (3.3) and (3.8), (3.9), the sequence σ^h_ℓ belongs to H^1_{loc}. We test (2.9) with $-D_k^{-r}(\theta^2 D_k^r\sigma^h)$, where θ is scalar and has compact support, and

D_k^r, D_k^{-r} denote the usual forward (backward) difference operator with respect to the k-th coordinate direction. We perform partial summation (i.e. we move D_k^{-r} onto the other factor), and taking the definiteness properties of the penalty term into account we may pass to the limit $r \to 0$:

$$
\begin{aligned}
&\int \theta^2 A D_k \dot{\tilde{\sigma}}^h(t-h).D_k\sigma^h(t)\,dx + \int \theta^2 \beta(|\sigma_D^h|) D_k\sigma_D^h.D_k\sigma_D^h\,dx \\
&\leq \int v_k^h [D_j\theta^2 D_j D_i \tau_{ik}^h + \theta^2 \Delta D_i \tau_{ik}^h + D_i\tau_{ij}^h D_j D_k \theta^2 + \sigma_{ij}^h D_i D_j D_k \theta^2]\,dx \\
&\quad + \int (\operatorname{tr} A\dot{\tilde{\sigma}}^h(t-h) - \operatorname{div}\zeta^h)[D_i\tau_{ij}^h D_j\theta^2 + \sigma_{ij}^h D_i D_j \theta^2]\,dx \\
&\quad -2\int (A\dot{\tilde{\sigma}}^h(t-h))_{jk} D_k\sigma_{ij}^h D_i\theta^2 + 2\varepsilon_{jk}(\zeta^h) D_k \sigma_{ij}^h D_i\theta^2\,dx \\
&\quad -2\int \beta(|\sigma_D^h|)\sigma_{D,jk}^h D_k \sigma_{D,ij}^h D_i\theta^2\,dx - \frac{2}{n}\int \beta(|\sigma_D^h|)\sigma_{D,jk}^h D_k \operatorname{tr}\sigma^h D_j\theta^2\,dx \\
&\quad + \int \theta^2 D_k\varepsilon(\zeta^h).D_k\sigma^h\,dx.
\end{aligned}
\tag{3.10}
$$

Using again (2.8) we get after easy transformations

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \theta^2 A D_k\tilde{\sigma}^h(t-h).D_k\tilde{\sigma}^h(t-h)\,dx \\
&\quad + \int \theta^2 A D_k \dot{\tilde{\sigma}}^h(t-h).D_k\dot{\tilde{\sigma}}^h(t-h)(h([\frac{t}{h}]+1)-t)\,dx \\
&\quad + \int \theta^2 \beta(|\sigma_D^h|) D_k\sigma_D^h.D_k\sigma_D^h\,dx \\
&\leq \int v_k^h [D_j\theta^2 D_j D_i \tau_{ik}^h + \theta^2 \Delta D_i \tau_{ik}^h + D_i\tau_{ij}^h D_j D_k \theta^2 + \sigma_{ij}^h D_i D_j D_k \theta^2]\,dx \\
&\quad + \int (\operatorname{tr} A\dot{\tilde{\sigma}}^h(t-h) - \operatorname{div}\zeta^h)[D_i\tau_{ij}^h D_j\theta^2 + \sigma_{ij}^h D_i D_j \theta^2]\,dx \\
&\quad -2\int A\dot{\tilde{\sigma}}^h(t-h)_{jk} D_k\tilde{\sigma}^h(t-h)_{ij} D_i\theta^2\,dx \\
&\quad -2\int A\dot{\tilde{\sigma}}^h(t-h)_{jk} D_k\dot{\tilde{\sigma}}^h(t-h)_{ij} D_i\theta^2 (h([\frac{t}{h}]+1)-t)\,dx \\
&\quad +2\int \varepsilon_{jk}(\zeta^h) D_k\tilde{\sigma}^h(t-h)_{ij} D_i\theta^2\,dx \\
&\quad +2\int \varepsilon_{jk}(\zeta^h) D_k\dot{\tilde{\sigma}}^h(t-h)_{ij} D_i\theta^2 (h([\frac{t}{h}]+1)-t)\,dx \\
&\quad -2\int \beta(|\sigma_D^h|)\sigma_{D,jk}^h D_k \sigma_{D,ij}^h D_i\theta^2\,dx - \frac{2}{n}\int \beta(|\sigma_D^h|)\sigma_{D,jk}^h D_k \operatorname{tr}\sigma^h D_j\theta^2\,dx \\
&\quad + \int \theta^2 D_k\varepsilon(\zeta^h).D_k\tilde{\sigma}^h(t-h)\,dx + \int \theta^2 D_k\varepsilon(\zeta^h).D_k\dot{\tilde{\sigma}}^h(t-h)(h([\frac{t}{h}]+1)-t)\,dx.
\end{aligned}
\tag{3.11}
$$

We also use the following inequality identical to the static case

$$\int_\Omega \theta^2\beta(|\sigma_D^h|)|D\operatorname{tr}\sigma^h|^2\,dx \le 2n^2\int_\Omega \theta^2\beta(|\sigma_D^h|)D_k\sigma_D^h.D_k\sigma_D^h\,dx + 2n\int_\Omega \theta^2\beta(|\sigma_D^h|)|\operatorname{div}\tau^h|^2\,dx. \tag{3.12}$$

Using the third estimate in (3.2) we deduce in particular that for $N > n$ one has

$$|\sigma_D^h| \text{ is bounded in } L^\infty(0,T;L^n(\Omega)).$$

Using next the relation

$$\operatorname{div}\sigma_D^h(t) + D\operatorname{tr}\sigma^h(t) = \operatorname{div}\tau^h(t)$$

and the first assumption (3.9) we deduce also that

$$|\sigma^h| \text{ is bounded in } L^\infty(0,T;L^n(\Omega)).$$

Collecting results, already obtained estimates and using Gronwall's inequality after integrating (3.12) between h and t we obtain

$$\|\tilde\sigma^h(t)\|_{H^1_{\mathrm{loc}}} \le C \tag{3.13}$$

and also

$$\frac{1}{\mu^{N-1}}\int_h^T \|\,|D\sigma_D^h|^2|\sigma_D^h|^{N-2}\,\|_{L^1_{\mathrm{loc}}} \le C. \tag{3.14}$$

3.4 Main result.
We can now state the following

Theorem 3.1. *We assume* (2.1) *to* (2.6), (3.1) *or* (3.3) *and* (3.8), (3.9). *Then the solution of* (2.7) *verifies the following estimates*

$$\begin{gathered}\frac{1}{\mu^{N-1}}\int_0^T\int_\Omega |\sigma_D^N(t,x)|^N\,dx\,dt \le C, \quad \int_0^T \|\dot\sigma^N(t)\|^2\,dt \le C\\ \frac{1}{N\mu^{N-1}}\int_\Omega |\sigma_D^N(t,x)|^N\,dx \le C, \text{ for a.e. } t, \quad \|\sigma^N(t)\|_{H^1_{\mathrm{loc}}} \le C,\\ \frac{1}{\mu^{N-1}}\int_0^T \|\,|D\sigma_D^N|^2|\sigma_D^N|^{N-2}\,\|_{L^1_{\mathrm{loc}}} \le C.\end{gathered} \tag{3.15}$$

4. Prandtl-Reuss model

4.1 Statement of the result.

We are going to let N tend to ∞. We introduce the Prandtl-Reuss model as follows:

To find $\quad \sigma \in H^1(0,T;\mathcal{L}^2_{\rm sym})$ with $|\sigma_D(t,x)| \le \mu,\ \sigma \in L^\infty(0,T;H^1_{\rm loc})$
such that $\quad \operatorname{div}\sigma(t) = f(t), \nu.\sigma(t) = \phi(t)$ on Γ_1 for a.e. $t,\ \sigma(0) = \sigma_0$
and such that

$$(4.1)\qquad \begin{gathered} (A\dot\sigma - \varepsilon(\zeta), \psi(t) - \sigma(t)) \ge 0 \ \ \forall\,\psi \text{ with} \\ \psi \in L^2(0,T;\mathcal{L}^2_{\rm sym}), |\psi_D(t,x)| \le \mu, \\ \operatorname{div}\psi(t) = f(t)\ \ \nu.\psi(t) = \phi(t) \text{ on } \Gamma_1,\ \text{ for a.e. } t. \end{gathered}$$

We shall need an additional assumption which completes slightly (3.8) namely

$$(4.2)\qquad |\operatorname{div}\tau| \in L^\infty(0,T;L^p(\Omega)),\ p > 2.$$

Our objective is to prove the following result.

Theorem 4.1. *Under the assumptions of Theorem* 3.1 *and* (4.2) *there exists one and only one solution of* (4.1).

Remark 4.2. Note that the $H^1_{\rm loc}$ regularity result is contained in the formulation that $\sigma \in L^\infty(0,T;H^1_{\rm loc})$.

Proof: We notice that thanks to the third estimate of (3.15) we have for all fixed $p < N$

$$|\sigma^N_D(t)| \text{ bounded in } L^\infty(0,T;L^p(\Omega)).$$

Using the relation

$$\operatorname{div}\sigma^N_D(t) + D\operatorname{tr}\sigma^N(t) = \operatorname{div}\tau(t)$$

as well as the assumption (4.2) we have also

$$(4.3)\qquad |\sigma^N(t)| \text{ bounded in } L^\infty(0,T;L^p(\Omega)).$$

If θ is any smooth function with compact support in Ω and $0 \le \theta \le 1$ we have

$$(4.4)\qquad \begin{gathered} \theta\sigma^N \text{ bounded in } L^\infty(0,T;H^1(\Omega)^{n\times n}), \\ \sigma^N \text{ bounded in } H^1(0,T;\mathcal{L}^2_{\rm sym}). \end{gathered}$$

We can extract a subsequence also called σ^N such that

$$\begin{gathered} \sigma^N \to \sigma \text{ weakly in } H^1(0,T;\mathcal{L}^2_{\rm sym}), \\ \sigma^N \to \sigma \text{ weakly star in } L^\infty(0,T;L^p(\Omega)). \end{gathered}$$

Moreover

$$\forall\,\theta\,,\theta\sigma^N \to \theta\sigma \text{ strongly in } L^2(0,T;\mathcal{L}^2_{\text{sym}}).$$

From this and the bound in $L^\infty(0,T;L^p(\Omega))$ we deduce that

$$\sigma^N \to \sigma \text{ strongly in } L^2(0,T;\mathcal{L}^2_{\text{sym}}).$$

Let ψ be as in the statement of the theorem, we can write testing (2.7) with $\psi-\sigma^N$

$$(A\dot\sigma^N + \frac{1}{\mu^{N-1}}|\sigma_D^N|^{N-2}\sigma_D^N - \varepsilon(\zeta),\psi-\sigma^N) = 0. \tag{4.5}$$

For a.e. t when (4.5) holds, we can interpret it as an optimality condition. Therefore we can write

$$\begin{aligned}(A\dot\sigma^N - \varepsilon(\zeta),\sigma^N) + \frac{1}{N\mu^{N-1}}\int_\Omega |\sigma_D^N|^N\,dx \\ \le (A\dot\sigma^N - \varepsilon(\zeta),\chi) + \frac{1}{N\mu^{N-1}}\int_\Omega |\chi_D|^N\,dx \\ \forall\,\chi \text{ such that } \operatorname{div}\chi = f(t) \text{ in } \Omega \ \ ,\nu.\chi = \phi(t) \text{ on } \Gamma_1\,.\end{aligned} \tag{4.6}$$

In particular we can take $\chi = \psi(t)$ in (4.6) where ψ is as the statement of the theorem. It follows

$$\begin{aligned}(A\dot\sigma^N - \varepsilon(\zeta),\sigma^N) + \frac{1}{N\mu^{N-1}}\int_\Omega |\sigma_D^N(t,x)|^N\,dx \\ \le (A\dot\sigma^N - \varepsilon(\zeta),\psi(t)) + \frac{1}{N\mu^{N-1}}\int_\Omega |\psi_D(t,x)|^N\,dx.\end{aligned} \tag{4.7}$$

Since (4.7) holds for a.e. t we can integrate this inequality between t and $T_h := \min(t+h,T)$ and obtain

$$\begin{aligned}\int_t^{T_h}(A\dot\sigma^N - \varepsilon(\zeta),\sigma^N)\,ds + \frac{1}{N\mu^{N-1}}\int_t^{T_h}\int_\Omega |\sigma_D^N(s,x)|^N\,dx\,ds \\ \le \int_t^{T_h}(A\dot\sigma^N - \varepsilon(\zeta),\psi(s))\,ds + \frac{1}{N\mu^{N-1}}\int_t^{T_h}\int_\Omega |\psi_D(s,x)|^N\,dx\,ds.\end{aligned} \tag{4.8}$$

Letting $N\to\infty$, we get thanks to the first estimate (3.15) and the convergence properties of σ^N that

$$\int_t^{T_h}(A\dot\sigma - \varepsilon(\zeta),\psi(s)-\sigma(s))\,ds \ge 0.$$

Since h is arbitrary the equation (4.1) is established. All other properties of σ are easily established, in particular the uniform $L^\infty(0,T;H^1_{\text{loc}})$ estimate follows from Theorem 3.1. The existence part has been proved. The uniqueness part is immediate. This concludes the proof of Theorem 4.1. $\square$

5. Additional regularity result for the time dependent Norton-Hoff model

5.1 Presentation of the result.

We shall present here a regularity result, which has interesting features. It concerns only the time dependent Norton-Hoff model: N is fixed here and there will be no uniformity, so we shall omit to make explicit reference to it. It states that whenever the function $|\sigma_D(t,x)|$ is bounded then H^2_{loc} regularity is available. Curiously, the corresponding result for the static case is not available, and it would be remarkable to get it.

In practice, the L^∞ bound is not available easily, which reduces the impact of the result, and makes it rather a curiosity than a usable result. This also explains why we shall only give a formal proof, although it might be made rigorous using discretization in space.

We state the result as follows:

Theorem 5.1. *With the same assumptions as in Theorem* 3.1, *additional smoothness hypotheses on the data*[1] *and*

$$|\sigma_D(t,x)| \leq C \tag{5.1}$$

the solution of (2.7) *satisfies the following estimates*

$$\int_0^T \|\dot{\sigma}(t)\|^2_{H^1_{\text{loc}}}\, dt \leq C, \quad \|\sigma(t)\|_{H^2_{\text{loc}}} \leq C. \tag{5.2}$$

Remark 5.1. In fact the method shows that the solution is as smooth as the data permit.

Remark 5.2. Since the additional regularity results are local, only a local bound is necessary in (5.1).

5.2 Formal proof.

(a) Proof of the first estimate:
We write (2.7) as follows

$$\begin{aligned} &A\dot{\sigma} + \beta(|\sigma_D|)\sigma_D = \varepsilon(v+\zeta) \\ &\operatorname{div}\sigma(t) = f(t) \text{ in } \Omega, \qquad \nu.\sigma(t) = \phi(t) \text{ on } \Gamma_1 \\ &u = 0 \text{ on } \Gamma_0, \qquad\qquad\qquad \sigma(0) = \sigma_0 \end{aligned} \tag{5.3}$$

where β has been already defined in the proof of Theorem 2.1. We shall use the following derivation formula

$$D_k(\beta(|\sigma_D|)\sigma_D) = \beta(|\sigma_D|)D_k\sigma_D + \frac{\beta'(|\sigma_D|)}{|\sigma_D|}(\sigma_D.D_k\sigma_D)\sigma_D\,. \tag{5.4}$$

[1] What is necessary will follow from the estimates derived in the proof.

Let us test equation (5.3) by $-D_k(\theta^2 D_k\dot\sigma)$, then we get

$$(5.5)\qquad \begin{aligned}&(\theta^2 A D_k\dot\sigma, D_k\dot\sigma) + (\beta(|\sigma_D|)D_k\sigma_D, \theta^2 D_k\dot\sigma_D)\\ &+(\frac{\beta'(|\sigma_D|)}{|\sigma_D|}(\sigma_D.D_k\sigma_D), \theta^2(\sigma_D.D_k\dot\sigma_D))\\ &= (D_k\varepsilon(v+\zeta), \theta^2 D_k\dot\sigma_D).\end{aligned}$$

The main point is to compute the term $(D_k\varepsilon(v), \theta^2 D_k\dot\sigma_D)$, performing integration by parts and using the equation. The calculation is quite similar to the static case. Eventually, we get

$$(5.6)\qquad (D_k\varepsilon(v), \theta^2 D_k\dot\sigma) = -2(D_k v_j, \theta D_i\theta D_k\dot\sigma_{ij}) - (D_k v_j, \theta^2 D_k \dot f_j)$$

and thus, using (5.6) in (5.5) yields

$$(5.7)\qquad \begin{aligned}&(\theta^2 A D_k\dot\sigma, D_k\dot\sigma) + (\beta(|\sigma_D|)D_k\sigma_D, \theta^2 D_k\dot\sigma_D)\\ &+(\frac{\beta'(|\sigma_D|)}{|\sigma_D|}(\sigma_D.D_k\sigma_D), \theta^2(\sigma_D.D_k\dot\sigma_D))\\ &= -2(D_k v_j, \theta D_i\theta D_k\dot\sigma_{ij}) - (D_k v_j, \theta^2 D_k \dot f_j).\end{aligned}$$

Since $|\sigma_D|$ is bounded, $\beta(|\sigma_D|)$ is also bounded, hence from the equation (5.3) it follows that

$$\varepsilon(v) \in L^2(0,T;L^2(\Omega)^{n\times n})$$

hence

$$v \in L^2(0,T;H^1_{\Gamma_0}(\Omega)^n)$$

because of Korn's inequality. Assuming the necessary regularity on the data arising in formula (5.7) the first estimate in (5.2) follows easily from this equation.

(b) Proof of the second estimate:

We shall need the following second order derivative formula:

$$(5.8)\qquad \begin{aligned}D_kD_l(\beta(|\sigma_D|)\sigma_D) &= \beta(|\sigma_D|)D_kD_l\sigma_D + \frac{\beta''(|\sigma_D|)}{|\sigma_D|^2}(\sigma_D.D_l\sigma_D)(\sigma_D.D_k\sigma_D)\sigma_D\\ &+\frac{\beta'(|\sigma_D|)}{|\sigma_D|}[(\sigma_D.D_k\sigma_D)D_l\sigma_D + (\sigma_D.D_l\sigma_D)D_k\sigma_D]\\ &+\frac{\beta'(|\sigma_D|)}{|\sigma_D|}[D_l\sigma_D.D_k\sigma_D + \sigma_D.D_kD_l\sigma_D - \frac{1}{|\sigma_D|^2}(\sigma_D.D_l\sigma_D)(\sigma_D.D_k\sigma_D)]\sigma_D\,.\end{aligned}$$

We then test (5.3) with $D_kD_l(\theta^2 D_kD_l\sigma)$ and obtain using the symmetry of A

$$(5.9)\qquad \begin{aligned}&\frac{d}{dt}(AD_kD_l\sigma, \theta^2 D_kD_l\sigma) + (D_kD_l(\beta(|\sigma_D|)\sigma_D), \theta^2 D_kD_l\sigma_D)\\ &= (D_kD_l\varepsilon(v+\zeta), \theta^2 D_kD_l\sigma).\end{aligned}$$

We compute

$$(D_kD_l\varepsilon(v), \theta^2 D_kD_l\sigma) = -2(D_kD_lv_j, \theta D_i\theta D_kD_l\sigma_{ij}) - (D_kD_lv_j, \theta^2 D_kD_lf_j). \tag{5.10}$$

So we get from (5.9)

$$\begin{aligned}
&\frac{d}{dt}(AD_kD_l\sigma, \theta^2 D_kD_l\sigma) + \beta(|\sigma_D|)(D_kD_l\sigma_D, \theta^2 D_kD_l\sigma_D) \\
&\quad +2\left(\frac{\beta'(|\sigma_D|)}{|\sigma_D|}(\sigma_D.D_k\sigma_D)D_l\sigma_D, \theta^2 D_kD_l\sigma_D\right) \\
&+\left(\frac{\beta'(|\sigma_D|)}{|\sigma_D|}[D_l\sigma_D.D_k\sigma_D + \sigma_D.D_kD_l\sigma_D]\sigma_D, \theta^2 D_kD_l\sigma_D\right) \\
&-\left(\frac{\beta'(|\sigma_D|)}{|\sigma_D|}\frac{1}{|\sigma_D|^2}(\sigma_D.D_l\sigma_D)(\sigma_D.D_k\sigma_D)\sigma_D, \theta^2 D_kD_l\sigma_D\right) \\
&+\left(\frac{\beta''(|\sigma_D|)}{|\sigma_D|^2}(\sigma_D.D_l\sigma_D)(\sigma_D.D_k\sigma_D)\sigma_D, \theta^2 D_kD_l\sigma_D\right) \\
&= -2(D_kD_lv_j, \theta D_i\theta D_kD_l\sigma_{ij}) - (D_kD_lv_j, \theta^2 D_kD_lf_j).
\end{aligned} \tag{5.11}$$

From (5.11) we want to make use of Gronwall's inequality. Using previous estimates, and among them part (a) of this proof, we see that all terms in (5.11) are fine. The only terms we have to worry about are of the type

$$\int \theta^2 |D_k\sigma_D||D_l\sigma_D||D_kD_l\sigma_D|\,dx$$

so in fact, by Young's inequality, we introduce terms to be estimated of the type

$$\int \theta^2 |D_k\sigma_D|^4\,dx.$$

Since $|\sigma_D|$ is bounded we can use an inequality of Gagliardo-Nirenberg type and this term is again estimated by

$$\sum_{k\,l}\int \theta^2 |D_kD_l\sigma_D|^2\,dx$$

which is fine for applying Gronwall's inequality.

The proof has been completed. □

Acknowledgement: The authors wish to thank J. Málek and M. Steinhauer for correcting an earlier version of the manuscript and pointing out some simplifications.

References

[1] Bensoussan A., Frehse J., *Asymptotic Behaviour of Norton-Hoff's Law in Plasticity theory and H^1 Regularity*, Collection: Boundary Value Problems for Partial Differential Equations and Applications, RMA Res. Notes Appl. Math. (Vol. in honor of E. Magenes), vol. 29, Masson, Paris, 1993, pp. 3–25.

[2] Duvaut G., Lions J.L., *Inequalities in Mechanics and Physics*, Springer-Verlag, Berlin, 1976.

[3] Lions J.L., *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Gauthier-Villars, Paris, 1969.

[4] Seregin G.A., *Differentiabilty properties of the stress tensor in perfect elastic-plastic theory*, Differentsial'nye Uravneniya **23** (1987), 1981–1991, English translation in Differential Equations **23** (1987), 1349–1358.

[5] Seregin G.A., *Differentiability of solutions of certain variational inequalities describing the quasi-static equilibrium of an elastic-plastic body*, Pomi, Preprints E-1-92, Steklov Mathematical Institute, Sankt Petersburg, 1992.

[6] Seregin G.A., *Differentiability properties of the stress-tensor in perfect elastic-plastic theory*, Preprint UTM321-Settembre, Universita degli Studi di Trento, 1990.

[7] Le Tallec P., *Numerical Analysis of Viscoelastic problems*, Masson, Paris, 1990.

[8] Temam R., *Mathematical Problems in Plasticity*, Gauthier Villars, Paris, 1985.

[9] Temam R., *A Generalized Norton-Hoff-Model and the Prandtl-Reuss-Law of Plasticity*, Arch. Rat. Mech. Anal. 95 (1986), 137–181.

Université Paris Dauphine and INRIA, Domaine de Voluceau–Rocquencourt–B.P. 105, 78153 Le Chesnay Cedex, France

E-mail: alain.bensoussan@inria.fr

Institut für Angewandte Mathematik, Universität Bonn, Beringstrasse 4-6, D–53115 Bonn, Germany

E-mail: erdbeere@fraise.iam.uni-bonn.de

(*Received* June 5, 1995)