A counter-example to some recent existence results on implicit variational inequalities

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Abstract. In this note we prove that some recent results on an implicit variational inequality problem for multivalued mappings, which seem to extend and improve some well-known and celebrated results, are not correct.

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1. Introduction

Very recently, in [1], J. Fu introduced the following implicit variational inequality problem for multivalued mappings; given two topological vector spaces X and Y, two nonempty subsets C and D of X and Y, respectively, two multivalued mappings $E : C \to 2^C$ and $F : C \to 2^D$, two real functions $f : C \times C \times D \to \mathbf{R}$ and $g : C \times C \to \mathbf{R}$, with $f(x, x, y) \geq 0$ for any $x \in C$ and any $y \in F(x)$, find $(v, u) \in C \times D$ such that

$$(1) \quad v \in E(v),\ u \in F(v) \ \text{ and } \ g(v, v) \leq f(v, w, u) + g(v, w) \ \text{ for all } \ w \in E(v).$$

Such problem extends an implicit variational problem studied by Mosco [2].

In [1], Fu stated the following assertion which he employed to obtain existence results for problem (1) and for some special cases as quasi-variational inequalities (for the basic definitions, we refer to [1]).

Assertion A (Theorem 1 of [1]). *Let X, Y be Hausdorff locally convex topological vector spaces, C be a nonempty compact convex set of X and D be a nonempty closed convex set of Y. Let $E : C \to 2^C$ be upper hemicontinuous with nonempty closed convex values and $F : C \to 2^D$ be a mapping with nonempty values. Suppose that $f : C \times C \times D \to \mathbf{R}$ satisfies the following conditions:*

(i) *for each $x \in C$ and $y \in F(x)$, $f(x, x, y) \geq 0$;*
(ii) *for any fixed $x \in C$ and $y \in D$, the function $f(x, u, y)$ of u is convex;*
(iii) *for any fixed $u \in C$, the function $\sup_{y \in F(x)} f(x, u, y)$ of x is upper semicontinuous.*

Then there exists $x^* \in C$ *such that*

$$x^* \in E(x^*) \ \ \textit{and} \ \ \sup_{y \in F(x^*)} f(x^*, u, y) \geq 0 \ \ \textit{for all} \ \ u \in E(x^*).$$

The aim of this note is to point out that Assertion A, in general, is false, together with several of its consequences obtained in [1]. We shall do this by means of a simple counter-example. We shall also illustrate in detail the gap in the original proof of Assertion A.

2. The counter-example

The following example shows that Assertion A, in general, is false.

Example 2.1. Let $X = Y = D = \mathbf{R}$, $C = [0, 1]$, $F(x) \equiv \{1\}$,

$$E(x) = \begin{cases} \left[\frac{3}{4}, 1\right] & \text{if } \ x \in \left[0, \frac{1}{2}\right[\\ \left[0, 1\right] & \text{if } \ x = \frac{1}{2} \\ \left[0, \frac{1}{4}\right] & \text{if } \ x \in \left]\frac{1}{2}, 1\right], \end{cases}$$

$f(x, u, y) = y(u - x)$. It is immediate to realize that all the assumptions of Assertion A are satisfied. In particular, we note that the graph of E is closed, hence, since C is compact, E is upper semicontinuous. Therefore, by Lemma 1 of [3], E is upper hemicontinuous. We note that the only fixed point of E is $x^* = \frac{1}{2}$. However, we have $\sup_{y \in F(x^*)} f(x^*, u, y) = u - x^* < 0$ for all $u \in [0, \frac{1}{2}[\subseteq E(x^*)$. Thus, Assertion A fails.

Remarks. (i) Example 2.1 also shows that Theorems 2, 3 and 9 of [1] are false.

(ii) We note that Theorem 2 of [1], if correct, would imply, in particular (taking into account Theorem 1.4.16 of [4]), that the celebrated Chan and Pang's existence theorem for generalized quasi-variational inequalities (see Theorem A in [5]) would be true without assuming the lower semicontinuity of the multifunction $E : C \to 2^C$. Example 2.1 shows that such improvement of Chan and Pang's result is not possible.

We note that the original proof of Assertion A (proof of Theorem 1 in [1]) is arranged as follows.

First step. By assuming that the conclusion is false, it is shown that there exists a finite set $\{p_1, \ldots, p_n\} \subseteq X^*$ (where X^* denotes the topological dual space of X) such that

$$C = V_0 \cup \bigcup_{i=1}^{n} V(p_i),$$

where we put

$$V(p_i) = \Big\{x \in C : \mathrm{Re}\langle p_i, x\rangle - \sup_{z \in E(x)} \mathrm{Re}\langle p_i, z\rangle > 0\Big\},$$

and

$$V_0 = \bigcup_{u \in A} \Big\{x \in C : \sup_{y \in F(x)} f(x,u,y) < 0\Big\},$$

with

$$A = \Big\{u \in C : \exists\, x \in C \text{ such that } u \in E(x) \text{ and } \sup_{y \in F(x)} f(x,u,y) < 0\Big\},$$

and $\langle \cdot, \cdot \rangle$ is the usual pairing between X^* and X.

Second step. The author considers a continuous partition of unity $\{\beta_0, \beta_1, \ldots, \beta_n\}$ subordinated to the open covering $\{V_0, V(p_1), \ldots, V(p_n)\}$ of C.

Third step. The author defines a function $\varphi : C \times C \to \mathbf{R}$ by setting

$$\varphi(x,u) = -\beta_0(x) \sup_{y \in F(x)} f(x,u,y) + \sum_{i=1}^{n} \beta_i(x)\mathrm{Re}\langle p_i, x-u\rangle$$

and observes that by the Ky Fan minimax principle (Theorem A of [3]) there exists $\hat{x} \in C$ such that

$$\varphi(\hat{x}, u) \le 0 \ \text{ for all } \ u \in C. \tag{2}$$

Fourth step. The author claims that inequality (2) is a contradiction. In particular, he claims that if $\beta_0(\hat{x}) > 0$, then there exists $u_0 \in A \subseteq C$ such that $\varphi(\hat{x}, u_0) > 0$.

But the gap is exactly here, since the inequality (2) does not imply, in general, any contradiction. To see this, take X, Y, D, C, F, f, and E, as in Example 2.1. The reader can easily check that in this case we have $A = [0, \frac{1}{2}[$, $V_0 =]0,1]$. Since $V(-1) = [0, \frac{1}{2}[$, we can take $n = 1$, $p_1 = -1$,

$$\beta_0(x) = \begin{cases} 2x & \text{if } x \in \left[0, \frac{1}{2}\right[, \\ 1 & \text{if } x \in \left[\frac{1}{2}, 1\right], \end{cases}$$

$$\beta_1(x) = \begin{cases} 1-2x & \text{if } x \in \left[0, \frac{1}{2}\right[, \\ 0 & \text{if } x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

According to Ky Fan minimax principle, there exists $\hat{x} \in [0,1]$ such that

$$\varphi(\hat{x}, u) = (\beta_0(\hat{x}) - \beta_1(\hat{x}))(\hat{x} - u) \le 0 \ \text{ for all } \ u \in [0,1].$$

In fact, one can take $\hat{x} = \frac{1}{4}$ since $\beta_0(\frac{1}{4}) = \beta_1(\frac{1}{4}) = \frac{1}{2}$, hence $\varphi(\frac{1}{4}, u) = 0$ for all $u \in [0,1]$. Thus, the contradiction claimed in the final part of the original proof of Assertion A does not hold.

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