Comment.Math.Univ.Carolin. 37,2 (1996)217–228

Remarks on the complementability of spaces of Bochner integrable functions in spaces of vector measures

G. Emmanuele[1]

Abstract. In the paper [5] L. Drewnowski and the author proved that if X is a Banach space containing a copy of c_0 then $L_1(\mu, X)$ is *not* complemented in $cabv(\mu, X)$ and conjectured that the same result is true if X is any Banach space without the Radon-Nikodym property. Recently, F. Freniche and L. Rodriguez-Piazza ([7]) disproved this conjecture, by showing that if μ is a finite measure and X is a Banach lattice not containing copies of c_0, then $L_1(\mu, X)$ is complemented in $cabv(\mu, X)$. Here, we show that the complementability of $L_1(\mu, X)$ in $cabv(\mu, X)$ together with that one of X in the bidual X^{**} is equivalent to the complementability of $L_1(\mu, X)$ in its bidual, so obtaining that for certain families of Banach spaces not containing c_0 complementability occurs (Section 2), thanks to the existence of general results stating that a space in one of those families is complemented in the bidual.

We shall also prove that certain quotient spaces inherit that property (Section 3).

Keywords: spaces of vector measures and vector functions, complementability, Banach lattices, preduals of W*-algebras, quotient spaces

Classification: 46B20, 46E27, 46E40, 46B30, 46L99

1. Introduction

In the paper [5] L. Drewnowski and the author proved that if X is a Banach space containing a copy of c_0 then $L_1(\mu, X)$ is *not* complemented in $cabv(\mu, X)$ and conjectured that the same result is true if X is any Banach space without the Radon-Nikodym property. Recently, F. Freniche and L. Rodriguez-Piazza ([7]) disproved this conjecture, by showing that if μ is a finite measure and X is a Banach lattice not containing copies of c_0, then $L_1(\mu, X)$ is complemented in $cabv(\mu, X)$, because such a $L_1(\mu, X)$ actually is a projection band in $cabv(\mu, X)$ (hence, from the results in [5] and [7] it follows that *if X is a Banach lattice, then $L_1(\mu, X)$ is complemented in $cabv(\mu, X)$ if and only if X does not contain c_0*); on the other hand, such a $L_1(\mu, X)$ is a projection band in its bidual, because it is a Banach lattice not containing copies of c_0. Motivated by this "similarity" we decided to study the relationships between the questions of the complementability of $L_1(\mu, X)$ in $cabv(\mu, X)$ and in the bidual $(L_1(\mu, X))^{**}$. It turned out that the complementability of $L_1(\mu, X)$ in $cabv(\mu, X)$ together with that one of X in its bidual is equivalent to the complementability of $L_1(\mu, X)$

[1]Work partially supported by M.U.R.S.T. of Italy (40%, 1992).

in the bidual $(L_1(\mu, X))^{**}$. This equivalence allows us to present some families of Banach spaces for which complementability occurs, thanks to the existence of general results about the complementability of a space in one of those families in the bidual. In particular, in some case, our results furnish improvements of the Lebesgue Decomposition Theorem.

Afterwards, we show that certain quotients of spaces for which complementability of $L_1(\mu, X)$ in $cabv(\mu, X)$ occurs still enjoy this property (Theorem 5). Such a result has several interesting applications to quite large families of Banach spaces; indeed we prove that for certain quotients of $L_1[0,1]$ complementability occurs as well as for quotients of either dual Banach spaces by w*-closed subspaces with the Radon-Nikodym property or arbitrary Banach spaces by reflexive subspaces. In passing we observe that one of the consequences of Theorem 5 can be applied to get some new information on certain well known isomorphic properties related to vector measures and on quotients of Banach lattices not containing copies of c_0. Theorem 5 can be, in some sense, reversed; indeed, we show that an assumption of surjectivity of a certain map considered in Theorem 5 is also necessary for the existence of the required projection. Other simple consequences of our results are also presented without proof; in particular we easily characterize complemented subspaces of Banach lattices for which complementability occurs.

Before starting with the statements of our results, we observe that throughout we shall consider just finite measure spaces (S, Σ, μ). Moreover, $cabv(\mu, X)$ will denote the Banach space of all X-valued countably additive measures ν that are absolutely continuous with respect to μ and have bounded variation (denoted by $|\nu|$), equipped with the variation norm. Sometimes, we shall also consider the space $cabv(\Sigma, X)$ of all countably additive X-valued measures with bounded variation on a σ-algebra of subsets of a set S and, when S is a Hausdorff compact space and Σ is the σ-algebra $Bo(S)$ of Borel sets, its closed subspace $rcabv(Bo(S), X)$ of all regular such measures; even these spaces are equipped with the variation norm.

2. Equivalence of two complementability problems

The first section of the paper contains results showing how the complementability problem studied in [5] and [7] is strictly related to another complementability problem considered, for instance, in [17]. In particular we show that the two considered questions are equivalent each other.

Here we explicitly observe that the main result of the paper, i.e. Theorem 4, is valid for both *real* and *complex* Banach spaces.

In order to prove our main result we need some well known facts.

Lemma 1 ([6, Theorem VI.7.1]). *For each compact Hausdorff space S and each Banach space X, there is an isometry ψ from $C(S, X^*)$ onto $K(X, C(S))$ so that $T_f(s)(x) = f(s)(x)$ for all $s \in S$, $x \in X$, where $\psi(f) = T_f$. Conversely, given $T \in K(X, C(S))$ we put $\psi^{-1}(T) = f_T$.*

Lemma 2 ([11, Lemma 1]). *For each compact Hausdorff space S and each Banach space X, $(K(X, C(S)))^*$ is isometric to a closed subspace of $(L(X, C(S)))^*$ by an isometry Λ where $\Lambda(\phi^*)(T) = \lim_i (K_i T)(\phi^*)$ for each $\phi^* \in (K(X, C(S)))^*$, (K_i) being a suitable net in the unit ball of $K(C(S))$ such that $\|K_i(f) - f\| \longrightarrow 0$ for each $f \in C(S)$. Moreover, if R denotes the restriction map, ΛR is a projection of $(L(X, C(S)))^*$ onto the copy of $(K(X, C(S)))^*$.*

Lemma 3 ([4, Theorem VIII.2.1]). *For each finite measure space (S, Σ, μ) and each Banach space X, $(L_1(\mu, X))^* = L(X, L_\infty(\mu))$.*

We shall use these lemmata in our Theorem 4 without any warning.

Now, in order to prove our main result we only need one more remark that follows the lines for the proof of the main theorem in [17]; as in [2], utilizing results to be found in [3], instead of a finite measure space (S, Σ, μ), in our main result we may consider an extremally disconnected compact Hausdorff space S and a regular Borel measure α on $Bo(S)$ such that

$$L_1(\mu, X) = L_1(\alpha, X), cabv(\mu, X) = cabv(\alpha, X)$$

where each equality means that the two involved spaces are isometrically isomorphic.

Theorem 4. *Let $(S, Bo(S), \alpha)$ be as above and X be a Banach space. Then the following statements are equivalent:*

1. *X is complemented in X^{**} by a projection $\tilde{P}$ and $L_1(\alpha, X)$ is complemented in $cabv(\alpha, X)$ by a projection Q;*
2. *$L_1(\alpha, X)$ is complemented in $(L_1(\alpha, X))^{**}$ by a projection F.*

Proof: For each $h^{**} \in (L_1(\alpha, X))^{**} = (L(X, C(S)))^*$, we may consider the element $(\Lambda R)(h^{**})$ belonging to $(K(X, C(S)))^*$. Hence $(\psi^* \Lambda R)(h^{**}) \in (C(S, X^*))^*$, a Banach space that is equal to $rcabv(Bo(S), X^{**})$. We can define in an obvious way a projection P from $rcabv(Bo(S), X^{**})$ onto $rcabv(Bo(S), X)$ determined by the projection $\tilde{P}$ of X^{**} onto X. So $(P\psi^* \Lambda R)(h^{**}) \in rcabv(Bo(S), X)$. We may also use the Lebesgue Decomposition Theorem ([4]) to get a further norm one projection H from $rcabv(Bo(S), X)$ onto $rcabv(\alpha, X)$, equal to $cabv(\alpha, X)$ since α is regular. Hence we have that $(HP\psi^* \Lambda R)(h^{**})$ belongs to $cabv(\alpha, X)$ and $(QHP\psi^* \Lambda R)(h^{**})$ belongs to $L_1(\alpha, X)$. To prove that $F = QHP\psi^* \Lambda R$ is the required projection it is enough to show that

$$(\psi^* \Lambda R)(f) = f \qquad \forall f \in L_1(\alpha, X)$$

since such any f clearly lives in $cabv(Bo(S), X^{**})$ too and $(QHP)(f) = f$. By the density of simple functions and by linearity, it is enough to suppose $f = \chi_E \otimes x$, for each $E \in Bo(S)$, $x \in X$. To reach our goal we calculate as it follows; for each

$g \in C(S, X^*)$ we get

$$(\psi^* \Lambda R)(\chi_E \otimes x)(g) = (\Lambda R)(\chi_E \otimes x)(\psi(g)) = (\Lambda R)(\chi_E \otimes x)(T_g) =$$
$$\lim_i R(\chi_E \otimes x)(K_i T_g) = \lim_i (\chi_E \otimes x)(K_i T_g) =$$
$$\lim_i \int_E \left[K_i(T_g(x))\right](s)\, d\alpha = \int_E T_g(x)(s) d\alpha = \int_E g(s)(x)\, d\alpha = (\chi_E \otimes x)(g).$$

The necessity of the Theorem is shown.

Let us now prove the sufficiency of our condition. It is clear that under (2) X is complemented in X^{**}. We observe that $rcabv(Bo(S), X^{**}) = (C(S, X^*))^*$ can be isometrically embedded into $(L(X, C(S)))^* = (L_1(\alpha, X))^{**}$ thanks to the map $\Lambda(\psi^{-1})^*$. So even $rcabv(\alpha, X) = cabv(\alpha, X)$ (since α is regular) embeds isometrically into $(L(X, C(S)))^*$; this means that F maps $cabv(\alpha, X)$ into $L_1(\alpha, X)$. To reach our goal it is now enough to show that

$$\Lambda(\psi^{-1})^*(\chi_E \otimes x) = \chi_E \otimes x$$

for each $E \in Bo(S)$, $x \in X$. To this aim, for each $T \in L(X, C(S))$ we have

$$\left[\Lambda(\psi^{-1})^*\right](\chi_E \otimes x)(T) = \lim_i (\psi^{-1})^*(\chi_E \otimes x)(K_i T) = \lim_i (\chi_E \otimes x)(f_{K_i T}) =$$
$$\lim_i \int_E f_{K_i T}(s)(x)\, d\alpha = \lim_i \int_E (K_i T)(x)(s)\, d\alpha = \int_E T(x)(s)\, d\alpha = (\chi_E \otimes x)(T),$$

which concludes our proof. □

Remark 1. The following facts are equivalent:

(1) $\|\tilde{P}\| = \|Q\| = 1$;
(2) $\|F\| = 1$.

This allows us to improve the main result of [17] obtained just for a Banach space H with the Radon-Nikodym property; it is enough to use the results in [7] and in the present paper.

Once we have Theorem 4 we can give the following list of families of Banach spaces for which the complementability of $L_1(\mu, X)$ in $cabv(\mu, X)$ occurs, thanks to the existence of general results stating that a space in one of the following families is complemented in the bidual:

(1) Banach lattices not containing copies of c_0 ([7]).

Indeed, since X is a Banach lattice, $L_1(\mu, X)$ is ([23]). Since X does not contain copies of c_0, $L_1(\mu, X)$ does ([12]). Hence $L_1(\mu, X)$ is complemented in its bidual ([13], [22]).

(2) Preduals of W*-algebras.

Indeed, in [24] it is shown that $L_1(\mu, X)$ is a predual of a W*-algebra, a space that is complemented in its bidual ([24]) (the separable case was obtained in [21]).

(3) Nicely placed subspaces of L_1-spaces (see [10]).

(4) Complemented subspaces of spaces for which complementability occurs (the proof of this is trivial).

Remark 2. We observe that under the assumptions (2) and (3) the projection from $cabv(\mu, X)$ onto $L_1(\mu, X)$ is an L-projection (see [10]), so that our result can be seen as an improvement of the Lebesgue Decomposition Theorem ([4]).

Remark 3. Under the assumptions (1), (2) and (3) above we have that $L_1(\mu, X)$ is norm one complemented in its bidual (as it happens in the case of X with the Radon-Nikodym property because in such a case $cabv(\mu, X) = L_1(\mu, X)$); it is easy to see that this fact, together with our main result, implies that also $cabv(\mu, X)$ is norm one complemented in its own bidual and so it has the FIP (for results about FIP in $L_1(\mu, X)$ compare [20]).

Remark 4. Under the assumptions (1), (2) and (3) above we actually got that $L_1(\mu, X)$ is norm one complemented in $cabv(\mu, X)$; so we have that if Y is a Banach space isometric to some L_1-space and X is a dual Banach space with the $R_{n,k}$ property (with $n > k \geq 3$) satisfying (1), (2) and (3) above, then the space $Y \otimes_\pi X$ has the $R_{n,k}$-property (use the same proof of Theorem 4.1 in [18]).

Remark 5. Of course, Remarks 3 and 4 are also applicable in the case (4) if we have that X is a norm one complemented subspace of a space for which complementability occurs (in this case we also improve the main result in [19]). Also Remark 2 holds true in the case 4 if we have that X is an L-summand in a space for which complementability occurs.

3. Quotient spaces

In this section we consider quotients of spaces for which complementability occurs and we show that sometimes they enjoy the same property.

The main result of the section is the following

Theorem 5. *Let (S, Σ, μ) be a finite measure space, X a Banach space such that $L_1(\mu, X)$ is complemented in $cabv(\mu, X)$ by a projection $\tilde{P}$, Z a closed subspace of X with the Radon-Nikodym property, $Y = X/Z$. Define $\tilde{Q} : cabv(\mu, X) \to cabv(\mu, Y)$ by putting $\big[\tilde{Q}(\tilde{\nu})\big](E) = Q\big[\tilde{\nu}(E)\big]$ (here Q denotes the quotient map of X onto Y), for all $E \in \mathcal{A}$ and $\tilde{\nu} \in cabv(\mu, X)$. If $\tilde{Q}$ is a quotient map, then $L_1(\mu, Y)$ is complemented in $cabv(\mu, Y)$.*

Proof: If $\tilde{Q}_0$ is the quotient map of $L_1(\mu, X)$ onto $L_1(\mu, Y)$ induced from Q ([4, Chapter VIII]) we clearly have that $\tilde{Q}_{|L_1(\mu,X)} = \tilde{Q}_0$; furthermore, $ker\tilde{Q} = cabv(\mu, Z) = L_1(\mu, Z) = ker\tilde{Q}_0$. Thanks to the above chain of equalities, we have that if $\tilde{\nu} \in ker\tilde{Q}$, a closed subspace of $L_1(\mu, X)$, then $\tilde{P}(\tilde{\nu}) = \tilde{\nu}$ and hence $\tilde{Q}_0\big[\tilde{P}(\tilde{\nu})\big] = \tilde{Q}_0(\tilde{\nu}) = \Theta$. This allows us to define an operator P on $cabv(\mu, Y)$ as it follows: for each $\nu \in cabv(\mu, Y)$ choose an arbitrary $\tilde{\nu} \in cabv(\mu, X)$ such that $\tilde{Q}(\tilde{\nu}) = \nu$ and put $P(\nu) = \tilde{Q}_0\big[\tilde{P}(\tilde{\nu})\big]$. It is not difficult to see that P is linear, thanks to the arbitrariness of the $\tilde{\nu}$ for which $\tilde{Q}(\tilde{\nu}) = \nu$, and that it is bounded, still thanks to the arbitrariness of the $\tilde{\nu}$ for which $\tilde{Q}(\tilde{\nu}) = \nu$ and the surjectivity of $\tilde{Q}$. Furthermore, if $\nu \in L_1(\mu, Y)$, we can choose $\tilde{\nu} \in L_1(\mu, X)$ so that $\tilde{Q}(\tilde{\nu}) = \tilde{Q}_0(\tilde{\nu}) = \nu$. Hence $P(\nu) = \nu$ and P is the required projection. We are done. □

Remark 6. We observe that if $\tilde{P}$ is a norm one projection, then the proof of Theorem 5 guarantees that even P in Theorem 5 is a norm one projection. Hence, Remarks 3, 4 and 5 are still applicable.

Remark 7. We observe that, maintaining the assumptions and the notations of Theorem 5, we have the equality $\tilde{Q}(ker\tilde{P}) = kerP$. To this end, let us first consider $\tilde{\nu} \in ker\tilde{P}$; hence $\tilde{P}(\tilde{\nu}) = \Theta$ and so $\tilde{Q}_0\big[\tilde{P}(\tilde{\nu})\big] = \Theta$. It follows that $P\big[\tilde{Q}(\tilde{\nu})\big] = \tilde{Q}_0\big[\tilde{P}(\tilde{\nu})\big] = \Theta$. This means that $\tilde{Q}(ker\tilde{P}) \subset kerP$. Now, let us choose $\nu \in kerP$. For some $\tilde{\nu} \in cabv(\mu, X)$ we have $\tilde{Q}(\tilde{\nu}) = \nu$; hence, by the very definition of P it follows that $P(\nu) = \tilde{Q}_0\big[\tilde{P}(\tilde{\nu})\big] = \Theta$, i.e. $\tilde{P}(\tilde{\nu}) \in ker\tilde{Q}_0 = ker\tilde{Q}$ since Z has the Radon-Nikodym property. But we also have $\tilde{\nu} = \tilde{P}(\tilde{\nu}) + \big[\tilde{\nu} - \tilde{P}(\tilde{\nu})\big]$; so we get

$$\tilde{Q}(\tilde{\nu}) = \tilde{Q}\left[\tilde{P}(\tilde{\nu})\right] + \tilde{Q}\left[\tilde{\nu} - \tilde{P}(\tilde{\nu})\right] =$$
$$\tilde{Q}_0\left[\tilde{P}(\tilde{\nu})\right] + \tilde{Q}\left[\tilde{\nu} - \tilde{P}(\tilde{\nu})\right] = \tilde{Q}\left[\tilde{\nu} - \tilde{P}(\tilde{\nu})\right].$$

Since $\tilde{Q}(\tilde{\nu}) = \nu$ and $\tilde{\nu} - \tilde{P}(\tilde{\nu}) \in ker\tilde{P}$ we are done: $\tilde{Q}(ker\tilde{P}) = kerP$.

This remark implies that the following facts are equivalent

(1) $\tilde{Q}$ is a quotient map;

(2) $L_1(\mu, Y)$ is complemented in $cabv(\mu, Y)$ and $\tilde{Q}(ker\tilde{P}) = kerP$.

Now, we present three concrete applications of Theorem 5 showing that quite large families of Banach spaces are good enough to satisfy the assumptions of Theorem 5.

Corollary 6. *Let, in Theorem* 5, $(S, \Sigma, \mu) = ([0,1], Bo[0,1], m)$ *be the usual Lebesgue measure space,* $X = L_1([0,1])$, Z *a closed subspace of* X *with the Radon-Nikodym property and verifying the following condition: for each operator* $T : X \to Y = X/Z$ *there is an operator* $R : X \to X$ *for which* $T = Q \circ R$, $Q : X \to Y$ *the quotient map* (*such subspaces exist thanks to results in* [8]). *Then* $L_1(\mu, Y)$ *is complemented in* $cabv(\mu, Y)$.

Proof: Since $X = L_1([0,1])$, the result from [7] gives that $L_1(\mu, X)$ is complemented in $cabv(\mu, X)$. Hence, among the assumptions of Theorem 5 just the surjectivity of $\tilde{Q}$ has to be verified. Let $\nu \in cabv(\mu, Y)$. Define $T : L_1(|\nu|) \to Y$ by

$$T(f) = \int_{[0,1]} f(s)\, d\nu \qquad \forall\, f \in L_1(|\nu|).$$

Since $\nu \ll m$ it follows that $([0,1], Bo[0,1], |\nu|)$ is separable and hence Boolean isomorphic with $([0,1], Bo[0,1], m)$ ([9]); so there is an isometry Ψ from $L_1([0,1])$ onto $L_1(|\nu|)$. The operator $T \circ \Psi : X \to Y$ factorizes as in our hypotheses, i.e.

there is $R : X \to X$ such that $T \circ \Psi = Q \circ R : X \to Y$. Hence $T = Q \circ R \circ \Psi^{-1}$ with $R \circ \Psi^{-1} : L_1(|\nu|) \to X$. Let $\tilde{\nu}$ be the vector measure defined by $\tilde{\nu}(E) = (R \circ \Psi^{-1})(\chi_E)$, $E \in Bo[0,1]$. We have

$$\nu(E) = T(\chi_E) = Q\left[(R \circ \Psi^{-1})(\chi_E)\right] = Q\left[\tilde{\nu}(E)\right] \qquad \forall\, E \in Bo[0,1].$$

Since it is well known that $\tilde{\nu} \in cabv(\mu, X)$ ([4]), we are done. □

Corollary 6 can be, for instance, applied with $Z = H_0^1$.

In the next corollary we shall also use the space $C(S,H)$, H a Banach space, of all H-valued continuous functions on a Hausdorff compact space S as well as the well known equality $(C(S,H))^* = rcabv(Bo(S), H^*)$ (see [4, p. 182]).

Corollary 7. *Let, in Theorem 5, (S,Σ,μ)=$(S, Bo(S), \alpha)$ be a regular Borel measure space over a Hausdorff compact space S, $X = F^*$ a dual Banach space, Z a w^*-closed subspace of X with the Radon-Nikodym property, $Y = X/Z$. If $L_1(\alpha, X)$ is complemented in $cabv(\alpha, X)$, then $L_1(\alpha, Y)$ is complemented in $cabv(\alpha, Y)$.*

Proof: Even in this case, we have just to prove that $\tilde{Q}$ is surjective, because then we can leave Theorem 5 works. We observe that, since α is regular, all elements are both in $cabv(\alpha, X)$ and $cabv(\alpha, Y)$ and, since Z is w^*-closed, Y is the dual of the closed subspace $Z^\perp = \{z^\perp \in F : z^\perp(z) = 0, \text{ for all } z \in Z\}$ of F. Hence $cabv(\alpha, X)$ (resp. $cabv(\alpha, Y)$) is a closed subspace of $(C(S,F))^* = rcabv(Bo(S), X)$ (resp. $(C(S), Z^\perp)^* = rcabv(Bo(S), Y)$). Let $i : Z^\perp \to F$ be the inclusion map; i^* is just the quotient map of X onto Y. Define $I : C(S, Z^\perp) \to C(S,F)$ by putting

$$[I(f)]\,(s) = i\,[f(s)] \qquad \forall\, s \in S, f \in C(S, Z^\perp).$$

$I^* : (C(S,F))^* \xrightarrow{onto} (C(S,Z^\perp))^*$ is a quotient map and it is not difficult to show that it verifies the following equality

$$[I^*(\tilde{\nu})]\,(E) = Q\,[\tilde{\nu}(E)] \qquad \forall\, E \in Bo(S), \tilde{\nu} \in (C(S,F))^*. \tag{1}$$

It is also clear that $I^*(cabv(\alpha, X)) \subseteq cabv(\alpha, Y)$. If we show that $I^*(cabv(\alpha, X)) = cabv(\alpha, Y)$ we get from (1) that $I^*_{|cabv(\alpha,X)} = \tilde{Q}$ that is so surjective. To this aim, let us consider $\nu \in cabv(\alpha, Y)$. There is $\tilde{\nu}_1 \in (C(S,F))^*$ such that $I^*(\tilde{\nu}_1) = \nu$. The Lebesgue Decomposition Theorem ([4]) implies that there are two elements $\tilde{\nu}, \tilde{\nu}_s$ of $cabv(Bo(S), X)$ for which

(i) $\tilde{\nu}_1 = \tilde{\nu} + \tilde{\nu}_s$;
(ii) $\tilde{\nu} \ll \alpha$;
(iii) $z\tilde{\nu}_s \perp \alpha$ for all $z \in F$.

From (i) it follows that

$$I^*(\tilde{\nu}_s) = I^*(\tilde{\nu}_1) - I^*(\tilde{\nu}) = \nu - I^*(\tilde{\nu}) \tag{2}$$

and so, thanks to (ii), we get $I^*(\tilde{\nu}_s) \ll \alpha$, from which $z^\perp I^*(\tilde{\nu}_s) \ll \alpha$ for all $z^\perp \in Z^\perp$. On the other hand, (iii) gives that $z^\perp I^*(\tilde{\nu}_s) \perp \alpha$ for all $z^\perp \in Z^\perp$. Hence, $z^\perp I^*(\tilde{\nu}_s) = \Theta$ for all $z^\perp \in Z^\perp$, i.e. $I^*(\tilde{\nu}_s) = \Theta$. This and (2) give $I^*(\tilde{\nu}) = \nu$. The proof is complete. □

As a consequence of Corollary 7 (and the results in [7]) we have that $L_1(\mu, A^*)$ (A the disk algebra) is complemented in $cabv(\mu, A^*)$ since A^* is isomorphic to $(C(\partial D))^*/H_0^1$ (see, for instance, [16]).

Remark 8. In Corollary 6 we could have considered any separable finite measure space, instead of $([0,1], Bo[0,1], m)$, because such a space is Boolean isomorphic to $([0,1], Bo[0,1], m)$. In Corollary 7 we could have considered any finite measure space, instead of a regular Borel measure space, because the Stone-Kakutani Theorem ([3]) could be used to reduce this general case to the one treated.

Using Corollary 7 we can prove the following result.

Corollary 8. *Let X be a Banach space such that $L_1(\mu, X)$ is complemented in $cabv(\mu, X)$, Z be a reflexive subspace of $X, Y = X/Z$. Then $L_1(\mu, Y)$ is complemented in $cabv(\mu, Y)$.*

Proof: It follows from Theorem 5 that it is enough to show that $\tilde{Q}$ is onto. If Q is like in Theorem 5, then Q^{**} is a quotient map from X^{**} onto $Y^{**} = X^{**}/Z$ (use the reflexivity of Z). Corollary 7 gives that for each $\nu \in cabv(\mu, Y)$, a closed subspace of $cabv(\mu, X^{**}/Z)$, there is $\tilde{\nu} \in cabv(\mu, X^{**})$ so that $\widetilde{(Q^{**})}(\tilde{\nu}) = \nu$. If we prove that $\tilde{\nu}$ takes its values into X we shall be done. Let $E \in \Sigma$; we have that $\nu(E) \in Y$ and so there is $x_E \in X$ for which $Q(x_E) = Q^{**}(x_E) = \nu(E)$. On the other hand, $Q^{**}\left[\tilde{\nu}(E)\right] = \nu(E)$ from which it follows that $\tilde{\nu}(E) - x_E \in ker Q^{**} = Z$. Hence $\tilde{\nu}(E) \in x_E + Z$ that is contained in X. We are done. □

Corollary 8 can be applied with $X = L_1(0,1)$ and Z the reflexive subspace spanned by the Rademacher functions.

We observe that the assumptions on Z in all of the above results cannot be dropped at all as one can see easily in the case $X = l_1$, $Y = c_0$. Indeed, in such a case $cabv(\mu, X) = L_1(\mu, X)$, but $L_1(\mu, Y)$ is not complemented in $cabv(\mu, Y)$ ([5]).

Remark 9. During the proofs of Corollaries 7 and 8 what actually we showed is that the map $\tilde{Q}$, defined in Theorem 5, is a quotient map, under any of the following assumptions:

(i) X is a dual Banach space and Z is a w*-closed subspace of X;
(ii) X is an arbitrary Banach space and Z is a reflexive subspace of X.

So we can obtain very easily that under (i) or (ii), if X has the Compact Range property or the Weak Radon-Nikodym property or the Weak** Radon-Nikodym property (we refer to [15] for these known definitions) or the Radon-Nikodym property, then $Y = X/Z$ has the same property(we observe that under (i) these facts are already known).

Another consequence of Corollaries 7 and 8 is the following result about the containment of c_0 by Banach lattices

Proposition 9. *Let X be a Banach lattice and Z a closed subspace of X both satisfying one of the two assumptions* (i) *and* (ii)*. If Z has the Radon-Nikodym property, the following facts are equivalent*

(1) *X does not contain c_0;*
(2) *$Y = X/Z$ does not contain c_0.*

Proof: Let us suppose X does not contain c_0. Hence the result in [7] implies that $L_1(\mu, X)$ is complemented in $cabv(\mu, X)$. From our previous results it follows that $L_1(\mu, Y)$ is complemented in $cabv(\mu, Y)$; hence c_0 is not allowed to live in Y ([5]). Conversely, if c_0 is not in Y, being also not contained in Z that has the Radon-Nikodym property, it follows from a (general) result in [1] that c_0 does not embed into X. Since there is no proof of this fact in [1] we shall give one here. So we prove that if $X/Z, Z$ do not contain c_0, then X itself does not contain c_0. By contradiction, let us suppose c_0 lives in X; hence, there is a weakly unconditionally converging series $\sum_i x_i$ in X that is not unconditionally converging. There are $\epsilon > 0$ and two sequences $(p_n), (q_n)$ of integers such that $p_n < q_n < p_{n+1}$ and $\|\sum_{i=p_n}^{q_n} x_i\| > \epsilon$ for all $n \in N$. If Q denotes the quotient map from X onto X/Z and $y_i = Q(x_i)$, $i \in N$, the series $\sum_n \big[\sum_{i=p_n}^{q_n} y_i\big]$ is (weakly and hence) unconditionally converging, since X/Z does not contain c_0. Choose $s_n \in X$ so that $Q(s_n) = \sum_{i=p_n}^{q_n} y_i$ and $\|s_n\| < 1/2^n$ for all $n \in N$ (passing to a subsequence if necessary). The series $\sum_n s_n$ is so absolutely summing and hence unconditionally converging in X. Put $z_n = s_n - \sum_{i=p_n}^{q_n} x_i$ for all $n \in N$. Clearly $(z_n) \subset Z$. If $x^* \in X^*$, we may calculate as follows

$$\sum_n |z_n(x^*)| \leq \sum_n |s_n(x^*)| + \sum_n |\left(\sum_{i=p_n}^{q_n} x_i\right)(x^*)| \leq$$
$$\|x^*\| \sum_n \frac{1}{2^n} + \sum_n |x_n(x^*)| < +\infty$$

since $\sum_n x_n$ was weakly unconditionally converging. It follows that $\sum_n z_n$ is (weakly and hence) unconditionally converging in Z, since Z does not contain c_0. So we have $z_n \to \Theta$. Since, $s_n \to \Theta$ too, we get $\sum_{i=p_n}^{q_n} x_i \to \Theta$, a contradiction. We are done. □

Actually, when X is a Banach lattice and (ii) is true, it is not difficult to show that (1) of Proposition 9 implies that Y is weakly sequentially complete; to reach such a result it is enough to use the Lohman's Lifting Theorem about weak Cauchy sequences ([14]).

At the end we give two more results, the first generalizing the main result from [7]; since the proofs are straightforward we do not give them.

Corollary 10. *Let Y be a quotient space of a complemented subspace X of a Banach lattice by a reflexive subspace. Then the following facts are equivalent*

(1) *X does not contain c_0;*
(2) *Y does not contain c_0;*
(3) *X is weakly sequentially complete;*
(4) *Y is weakly sequentially complete;*
(5) *X is complemented in its bidual and $L_1(\mu, X)$ is complemented in $cabv(\mu, X)$;*
(6) *Y is complemented in its bidual and $L_1(\mu, Y)$ is complemented in $cabv(\mu, Y)$;*
(7) *$L_1(\mu, X)$ is complemented in $(L_1(\mu, X))^{**}$;*
(8) *$L_1(\mu, Y)$ is complemented in $(L_1(\mu, Y))^{**}$.*

Corollary 11. *Let X be a dual Banach space complemented in a Banach lattice L, Z a w^*-closed subspace of X with the Radon-Nikodym property, $Y = X/Z$. Then the following facts are equivalent*

(1) *X does not contain c_0;*
(2) *Y does not contain c_0;*
(3) *X is complemented in its bidual and $L_1(\mu, X)$ is complemented in $cabv(\mu, X)$;*
(4) *Y is complemented in its bidual and $L_1(\mu, Y)$ is complemented in $cabv(\mu, Y)$;*
(5) *$L_1(\mu, X)$ is complemented in $(L_1(\mu, X))^{**}$;*
(6) *$L_1(\mu, Y)$ is complemented in $(L_1(\mu, Y))^{**}$.*

We finish the paper with the following natural questions

Question 1. *Is $L_1(\mu, X)$ complemented in $(L_1(\mu, X))^{**}$ if X is complemented in X^{**} and X does not contain copies of c_0?*

Question 2. *Is $L_1(\mu, X)$ complemented in $cabv(\mu, X)$ if X does not contain copies of c_0?*

References

[1] Bourgain J., Pisier G., *A construction of $\mathcal{L}_\infty$-spaces and related Banach spaces*, Bol. Soc. Bras. Mat. **14.2** (1983), 109–123.

[2] Cambern M., Greim P., *The dual of a space of vector measures*, Math. Z. **180** (1982), 373–378.

[3] Dinculeanu N., *Vector Measures*, Pergamon Press, New York, 1967.

[4] Diestel J., Uhl J.J., Jr., *Vector Measures*, Math. Surveys 15, Amer. Math. Soc., Providence, Rhode Island, 1977.

[5] Drewnowski L., Emmanuele G., *The problem of complementability for some spaces of vector measures of bounded variation with values in Banach spaces containing copies of* c_0, Studia Math. **104.2** (1993), 111–123.

[6] Dunford N., Schwartz J.T., *Linear Operators,* part I, Intersciences, New York and London, 1958.

[7] Freniche F., Rodriguez-Piazza L., *Linear projections from a space of measures onto its Bochner integrable functions subspace*, preprint, 1993.

[8] Ghoussoub N., Rosenthal H.P., *Martingales, G_δ-embeddings and quotients of L_1*, Math. Annalen **264** (1983), 321–332.

[9] Halmos P.R., *Measure Theory*, GTM 18, Springer Verlag, New York, Berlin, Heidelberg, 1974.

[10] Harmand P., Werner D., Werner W., *M-ideals in Banach spaces and Banach algebras*, LNM 1547, Springer Verlag, New York, Berlin, Heidelberg, 1994.

[11] Johnson J., *Remarks on Banach spaces of compact operators*, J. Funct. Analysis **32.3** (1979), 304–311.

[12] Kwapien S., *On Banach spaces containing* c_0, Studia Math. **52** (1974), 187–188.

[13] Lindenstrauss J., Tzafriri L., *Classical Banach Spaces, II, Function Spaces*, EMG 97, Springer Verlag, New York, Berlin, Heidelberg, 1979.

[14] Lohman R.H., *A note on Banach spaces containing l_1*, Canad. Math. Bull. **19** (1976), 365–367.

[15] Musial K., *Martingales of Pettis integrable functions*, in Measure Theory, Oberwolfach 1979, LNM 794, Springer Verlag, New York, Berlin, Heidelberg 1980.

[16] Pelczynski A., *Banach spaces of Analytic Functions and Absolutely Summing Operators*, CBMS 30, Amer. Math. Soc., Providence, Rhode Island, 1977.

[17] Rao T.S.S.R.K., *$L^1(\lambda, E)$ as a constrained subspace of its bidual*, Indian Statistical Institute, Tech. Report, 1988.

[18] Rao T.S.S.R.K., Roy A.K., Sundaresan K., *Intersection properties of balls in tensor products of some Banach spaces*, Math. Scand. **65** (1989), 103–118.

[19] Rao T.S.S.R.K., *A note on the $R_{n,k}$ property for $L_1(\mu, X)$*, Canad. Math. Bull. **32** (1989), 74–77.

[20] Rao T.S.S.R.K., *Intersection properties of balls in tensor products of some Banach spaces – II*, Indian J. Pure Appl. Math. **21** (1990), 275–284.

[21] Sakai S., *C^*-Algebras and W^*-Algebras*, EMG 60, Springer Verlag, 1971.

[22] Schaefer H.H., *Banach lattices and positive operators*, GMW 215, Springer Verlag, New York, Berlin, Heidelberg, 1974.

[23] Sundaresan K., *Banach lattices of Lebesgue-Bochner function spaces and conditional expectation operators, I*, Bull. Acad. Sinica **2.2** (1974), 165–184.

[24] Takesaki M., *Theory of operator algebras, I*, Springer Verlag, 1979.

Notes: While writing down the present version of the note we learnt that Prof. Rao (after reading a preliminary version of it) has also got Theorem 4 and Corollary 8, independently.

After submitting this revised version of the paper we learnt from Prof. Drewnowski that Prof. Wnuk realized that the main result of the paper [7] was already known; it is Lemma 3.7 in the paper

[25] Caselles V., *A characterization of weakly sequentially complete Banach lattices*, Math. Z. **190** (1985), 379–385.

Department of Mathematics, University of Catania, Viale A. Doria 6, 95125 Catania, Italy

E-mail: Emmanuele@Dipmat.Unict.It

(*Received* April 12, 1995)